2 Review of Consistent Deformation and Slope Deflection Methods

The consistent deformation and slope deflection methods are the basis for the development of flexibility and stiffness matrix methods. These methods are taught in basic courses on structural analysis and the author has covered them in detail in his books on Structural Analysis – I and II. A brief review of these methods are presented in this chapter.

2.1 CONSISTENT DEFORMATION METHOD

In this method of analysis, excess restraints on the structure are removed to get **basic determinate structure**. Such structure is also known as **released structure**. The released structure is analysed for given loading to get displacements in the direction of released restraint. Then redundant forces which are unknown are applied in the direction of restraints removed and the displacements in each direction of restraint removed, are obtained separately for each redundant force. These displacements are in terms of redundant forces. Then considering all these displacements of released structure total displacement due to loading and due to redundant forces in each restraint removed is found. Considering the displacement compatibility of original structure equations are assembled. These conditions result into as many equations as there are number of redundant forces. Solution of these simultaneous equations gives the values of redundant forces. Knowing these values moments and forces at any point in the structure can be found. The method is illustrated below by solving few typical cases:

Example 2.1: A propped cantilever of span L is fixed at A and is on roller at B. Analyse it when it is subjected to a concentrated load P at midspan. Assume uniform cross-section throughout.

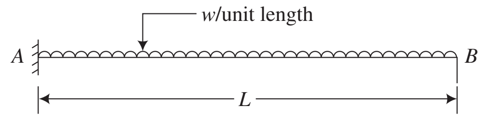
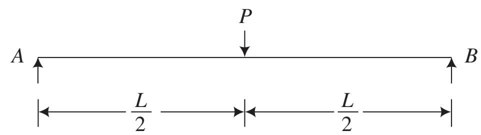
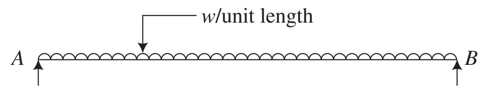
Solution: Total number of reactions = 3 + 1 = 4

Number of equilibrium equations available = 3.

$\therefore$ Degree of static indeterminacy = 4 – 3 = 1.

By releasing support B restraint to vertical deflection is removed and we get a cantilever as basic determinate structure. This released structure is analysed for the given load and the redundant force R_B to get vertical displacements at B. Figure 2.1(a) shows the original structure whereas Fig. 2.1(b) and (c) show released structure subjected to given loads and redundant force respectively.

Contd.

	$\frac{wL^3}{6EI}$	$\frac{wL^3}{8EI}$
	$\frac{PL^2}{16EI}$	0
	$\frac{wL^3}{24EI}$	0

2.2.2 Conjugate Beam Method

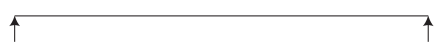
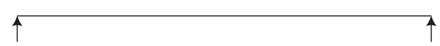
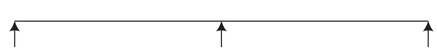
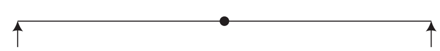
A conjugate beam of a beam is the one which when loaded with $\frac{M}{EI}$ diagram of the beam satisfies the following two conditions:

1. Rotation of the beam with respect to the axis of beam = shear in conjugate beam.
2. Deflection of the beam

= Moment in conjugate beam.

Hence, from moment area theorem, the conjugate beams for various beams can be obtained. Table 2.2 shows conjugate beams for commonly used beams.

Table 2.2 Beams and corresponding conjugate beams

Beam	*Conjugate beam*
	
	
	

Example 2.2: Determine the end rotation in a simply supported beam of span L subjected to uniformly distributed load w per unit length. Use conjugate beam method.

Solution: Figure 2.3(a) shows the beam. In this bending moment diagram is a parabolic curve with maximum ordinate equal to $\frac{wL^2}{8}$ at centre. Its conjugate beam is the simply supported beam with $\frac{M}{EI}$ diagram as load on it as shown in Fig. 2.3(b).

$$\theta_A = \text{shear in conjugate beam at } A$$

$$= \frac{1}{2} \times \text{total load on conjugate beam}$$

$$= \frac{1}{2} \times \frac{2}{3} \times L \times \frac{wL^2}{8EI}$$

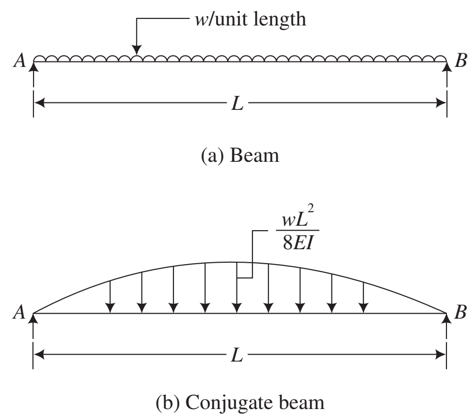


Fig. 2.3

$$= \frac{wL^3}{24EI}$$

Example 2.3: Determine the rotation and displacement of free end in the cantilever beam shown in Fig. 2.4(a) by conjugate beam method.

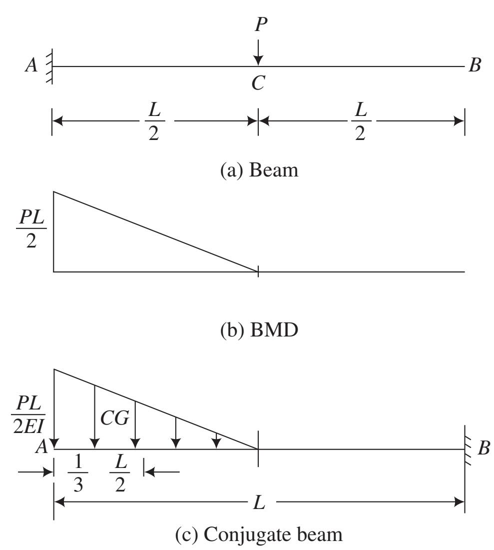


Fig. 2.4

Solution: Figure 2.4(b) shows the bending moment diagram of the beam and Fig. 2.4(c) shows the conjugate beam with $\frac{M}{EI}$ diagram as load on it.

$$\theta_B = \text{shear force in conjugate beam at } B$$

$$= \text{Total load on the conjugate beam}$$

$$= \frac{1}{2} \times \frac{PL}{2EI} \times \frac{L}{2}$$

$$= \frac{PL^2}{8EI} \qquad \textbf{\textit{Ans.}}$$

Δ_B = Moment in conjugate beam at B

= Total load on conjugate beam × distance of its centroid from B

$$= \frac{PL^2}{8EI}\left(L - \frac{1}{3}\frac{L}{2}\right)$$

$$= \frac{5PL^3}{48EI} \qquad \textbf{\textit{Ans.}}$$

2.2.3 Unit Load Method for Finding Rotations and Deflection of Beams and Frames

In this method to find rotations and deflections in a determinate beam, first moment M due to loadings is found. Then a unit load is applied in the direction of required displacement and resulting moment 'm' is found. The required displacement is $\int \frac{Mmdx}{EI}$, where integration is over the entire structure. The following examples illustrate the method.

Example 2.4: Determine the rotation and deflection at the free end of a cantilever beam of span L subject to uniformly distributed load over its entire span.

Solution: Figure 2.5(a) shows the beam under consideration. Now due to given load

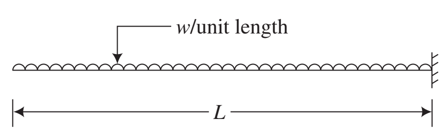
$$M = \frac{wx^2}{2}$$

To find rotation at free end a unit moment is applied (Ref Fig. 2.5(b)). Due to this moment

$$m = 1, \text{ throughout.}$$

$$\therefore \qquad \theta = \int_0^L \frac{Mmdx}{EI} = \int_0^L \frac{wx^2}{2} \times 1 \times \frac{1}{EI}\, dx$$

$$= \frac{w}{2EI}\left[\frac{x^3}{3}\right]_0^L = \frac{wL^3}{6EI} \qquad \textbf{\textit{Ans.}}$$



(a) Beam

(b) Unit load for rotation at free end

(c) Unit load for deflection at free end.

Fig. 2.5

To find the deflection at free end a unit vertical load is applied as shown in Fig. 2.5(c) In this case,

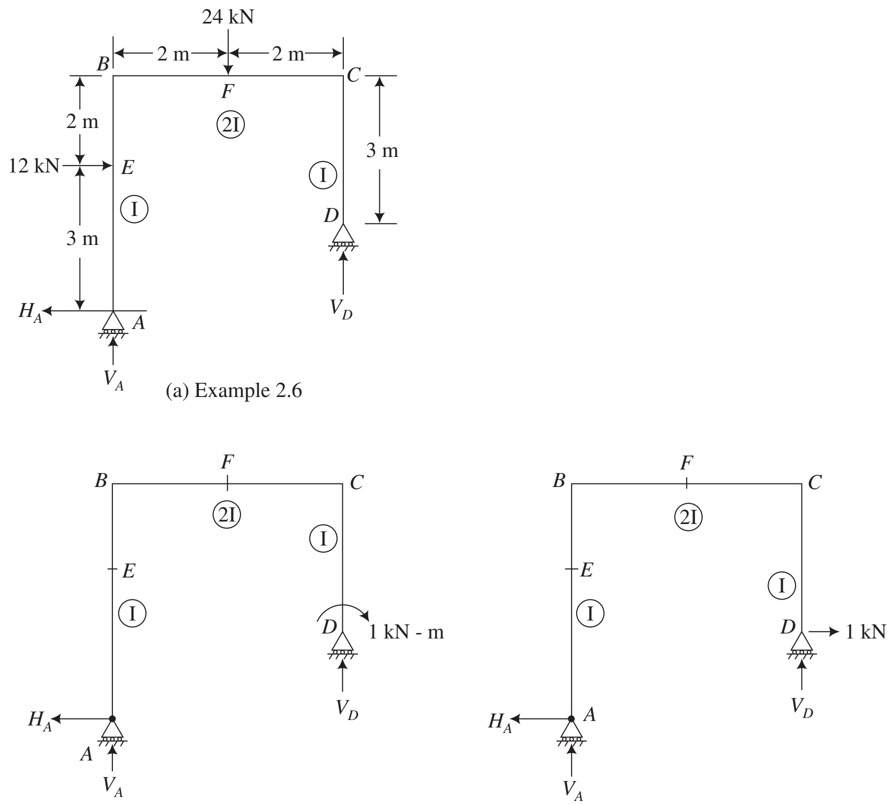
$$m = 1 \times x = x$$

$$\therefore \quad \Delta = \int_0^L \frac{Mxdx}{EI} = \int_0^L \frac{wx^2}{2} x \frac{1}{EI} dx$$

$$= \frac{w}{2EI}\left[\frac{x^4}{4}\right]_0^L$$

$$= \frac{wL^4}{8EI}$$

Example 2.5: Determine the rotation and horizontal displacement at *D* in the frame shown in Fig. 2.6(a)



(a) Example 2.6

(b) Frame with unit moment at *D*

(c) Frame with unit horizontal force at *D*

Fig. 2.6

Solution: For given loads,

$$V_D \times 4 = 24 \times 2 + 12 \times 3$$

$$\therefore \quad V_D = 21 \text{ kN}$$

∴ $V_A = 24 - 21 = 3$ kN

and $H_A = 12$ kN

Hence, moment in various portion of frame are as shown in Table 2.3.

Due to unit moment applied at D [Refer Fig. 2.6(b)]

$$V_D = \frac{1}{4}\uparrow \quad \therefore\ V_A = \frac{1}{4}\downarrow \quad \text{and} \quad H_A = 0$$

Expression for moment are tabulated as m_1 as shown in Table 2.3

When unit horizontal force is applied at D, as shown Fig. 2.6(c),

$$V_D = \frac{1 \times 2}{4} = \frac{1}{2}\text{ kN} \quad \therefore\ V_A = \frac{1}{2}\text{ kN}\uparrow$$

and $H_A = 0$

Expressions for moment due to this unit load are tabulated as m_2.

Table 2.3 Calculation table for example 2.5

Portion	*AE*	*EB*	*BF*	*FC*	*CD*
Origin	A	E	F	C	D
Limits	0 – 3	0 – 2	0 – 2	0 – 2	0 – 3
Moment of Inertia	I	I	2I	2I	I
M	$12x$	36	$21(2 + x) - 24x = 42 - 3x$	$21\ x$	0
m_1	0	0	$-1 + \frac{1}{4}(x + 2) = -\frac{1}{2} + x/4$	$-1 + \frac{x}{4}$	–1
m_2	x	$x + 3$	$\frac{1}{2}(x + 2) + 1 \times 3 = \frac{x}{2} + 4$	$\frac{1}{2}x + 3$	x

Note: Moment is taken +ve if it creates tension on inner side of the frame.

$$\theta_D = \int_0^F \frac{Mm_1 dx}{EI}$$

$$\therefore\ EI\,\theta_D = \int_0^3 0\,dx + \int_0^2 0\,dx + \frac{1}{2}\int_0^2 (42 - 3x)\left(x/4 - \frac{1}{2}\right)dx + \int_0^2 21x\left(-1 + \frac{x}{4}\right)dx + \int_0^3 0\,dx$$

$$= 0 + 0 + \frac{1}{4}\int_0^2 (24x - 21 - 1.5\,x^2)\,dx + \int_0^2 \left(-21x + \frac{21}{4}x^2\right)dx$$

$$= \frac{1}{4}[12x^2 - 21x - 0.5x^3]_0^2 + \left[\frac{-21}{2}x^2 + \frac{7}{4}x^3\right]_0^2$$

$$= \frac{1}{4}[48 - 42 - 4] + [-42 + 14]$$

$$= -27.5$$

$$\therefore\quad \theta_D = \frac{-27.5}{EI} = \frac{27.5}{EI}\text{ radians, clockwise.}$$

Solution: Due to applied load, vertical reactions at L_0 and L_3 are 400 kN. Using method of joint, the forces P in all members are found and noted in Table 2.4. Then the unit load is applied in the horizontal direction at roller support as shown in Fig. 2.7(b) and k-forces are found and noted in Table 2.4. Other calculation details are shown in Table 2.4.

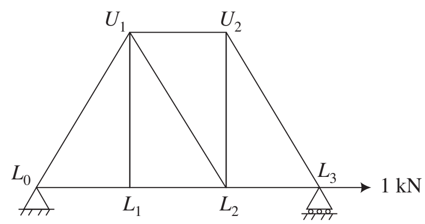


Fig. 2.7(b) Truss with unit horizontal force at L_3.

Horizontal displacement of the roller support

$$L_3 = \Sigma \frac{PkL}{AE} \frac{1}{E} \Sigma PkL = \frac{1350}{E} \qquad \textbf{\textit{Ans.}}$$

Table 2.4 Calculate Table for example 2.6

Member	*L*	*A*	*P*	*K*	$\frac{PKL}{A}$
L_0U_1	7500	6000	–500	0	0
U_1U_2	4500	6000	–300	0	0
U_2L_3	7500	6000	–500	0	0
L_0L_1	4500	3000	300	1	450
L_1L_2	4500	3000	300	1	450
L_2L_3	4500	3000	300	1	450
L_1U_1	6000	3000	400	0	0
L_2U_2	6000	3000	400	0	0
U_1L_2	7500	3000	0	0	0
					Σ 1350

2.3 SLOPE DEFLECTION METHOD

In this method analysis of indeterminate structure, slopes and deflections of end points of members are considered basic unknowns. Expressions for moments are assembled in terms of these displacements and joint equilibrium equations are assembled. Solution of these equations give the values of displacements. These values are substituted in the expressions for moment to get end moments in the members. Then shear force and moment can be found at any required point.

Figure 2.8 shows a typical member *AB* with end moments and displacements. Let θ_A and θ_B be the end rotations and Δ be the difference of levels between *A* and *B*. Let *EI* be flexural rigidity. The following sign convention is used to find the expressions for moments:

Solution: *Fixed End Moment*

$$M_{FAB} = \frac{-60 \times 4 \times 2^2}{6^2} = -26.67 \text{ kN-m}$$

$$M_{FBA} = \frac{-60 \times 2 \times 4^2}{6^2} = 53.33 \text{ kN-m}$$

$$M_{FBC} = \frac{-30 \times 6^2}{12} = -90 \text{ kN-m}$$

$$M_{FCB} = 90 \text{ kN-m.}$$

Slope Deflection Equations

$$M_{AB} = -26.67 + \frac{2EI}{6}(2\theta_A + \theta_B - 0)$$

$$= -26.67 + \frac{1}{3}EI\theta_B, \quad \text{since } \theta_A = 0$$

$$M_{BA} = 53.33 + \frac{2EI}{6}(\theta_A + 2\theta_B - 0)$$

$$= 53.33 + \frac{2}{3}EI\theta_B, \quad \text{since } \theta_A = 0$$

$$M_{BC} = -90 + \frac{2EI}{6}(2\theta_B + \theta_C - 0)$$

$$= -90 + \frac{2}{3}EI\theta_B + \frac{1}{3}EI\theta_C$$

$$M_{CB} = 90 + \frac{2EI}{6}(\theta_B + 2\theta_C - 0)$$

$$= 90 + \frac{1}{3}EI\theta_B + \frac{2}{3}EI\theta_C$$

Equilibrium Equations

$$\Sigma M_B = 0, \quad \text{gives}$$

$$M_{BA} + M_{BC} = 0$$

$$53.33 + \frac{2}{3}EI\theta_B - 90 + \frac{2}{3}EI\theta_B + \frac{1}{3}EI\theta_C = 0$$

i.e. $$4EI\theta_B + EI\theta_C = 110 \qquad (1)$$

$$\Sigma M_C = 0, \quad \text{gives}$$

$$90 + \frac{1}{3}EI\theta_B + \frac{2}{3}EI\theta_C = 0$$

$$EI\theta_B + 2\,EI\theta_C = -270 \qquad (2)$$

Subtracting equation (2) from twice equation (1), we get

$$7EI\theta_B = 490$$

i.e. $EI\theta_B = 70$

$\therefore$ From equation (1), $EI\theta_C = 110 - 4 \times 70 = -170.$

End Moments: Substituting the values of $EI\theta_B$ and $EI\theta_C$ in slope deflection equations, we get

$$M_{AB} = -26.67 + \frac{1}{3} \times 70 = -3.33 \text{ (kN-m)}$$

$$M_{BA} = 53.33 + \frac{2}{3} \times 70 = 100 \text{ kN-m}$$

$$M_{BC} = -90 + \frac{2}{3} \times 70 - \frac{1}{3} \times 170 = -100 \text{ kN-m}$$

$$M_{CB} = 90 + \frac{1}{3} \times 70 - \frac{2}{3} \times 170 = 0$$

Figure 2.9(b) shows the bending moment diagram.

Example 2.8: Analyse the frame shown in Fig. 2.10(a) by slope deflection method.

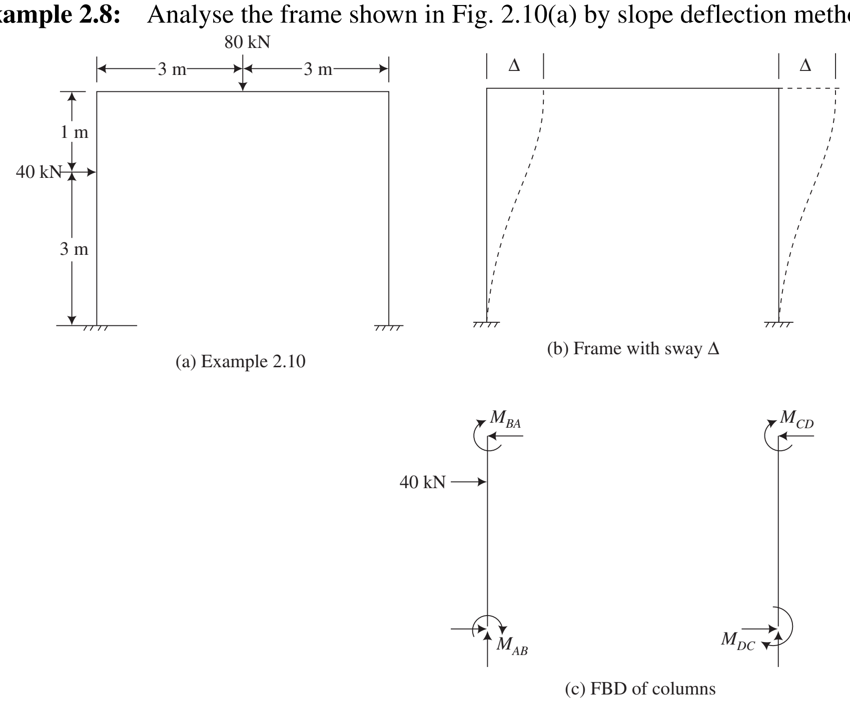


Fig. 2.10

Fixed End Moments

$$M_{FAB} = \frac{-40 \times 3 \times 1^2}{4^2} = -7.5 \text{ kN-m}$$

$$M_{FBA} = \frac{-40 \times 3^2 \times 1}{4^2} = -22.5 \text{ kN-m}$$

$$M_{FBC} = \frac{-80 \times 6}{8} = -60 \text{ kN-m}$$

$$M_{FCB} = 60 \text{ kN-m}$$

$$M_{FCD} = M_{FDC} = 0$$

Slope Deflection Equations

$$M_{AB} = -7.5 + \frac{2EI}{4}\left(2\theta_A + \theta_B - \frac{3\Delta}{4}\right)$$

$$= -7.5 + 0.5EI\theta_B - 0.375EI\Delta, \quad \text{since } \theta_A = 0.$$

$$M_{BA} = 22.5 + \frac{2EI}{4}\left(\theta_A + 2\theta_B - \frac{3\Delta}{4}\right)$$

$$= 22.5 + EI\theta_B - 0.375EI\Delta, \quad \text{since } \theta_A = 0$$

$$M_{BC} = -60 + \frac{2E(2I)}{6}(2\theta_B + \theta_C - 0)$$

$$= -60 + 1.333EI\theta_B + 0.667\, EI\theta_c$$

$$M_{CB} = 60 + \frac{2E(2I)}{6}(\theta_B + 2\theta_C - 0)$$

$$= 60 + 0.667EI\theta_B + 1.333EI\theta_C$$

$$M_{CD} = \frac{2EI}{4}\left(2\theta_C + \theta_D - \frac{3\Delta}{4}\right)$$

$$= EI\theta_C - 0.375EI\Delta, \quad \text{since } \theta_D = 0$$

$$M_{DC} = \frac{2EI}{4}\left(\theta_C + 2\theta_D - \frac{3\Delta}{4}\right)$$

$$= 0.5EI\theta_C - 0.375EI\Delta, \quad \text{since } \theta_D = 0.$$

Equilibrium Equations

$$\sum M_B = 0, \quad \text{gives}$$

$$M_{BA} + M_{BC} = 0$$

$$22.5 + EI\theta_B - 0.375EI\Delta - 60 + 1.333EI\theta_B + 0.667EI\theta_C = 0$$

i.e. $$2.333EI\theta_B + 0.667EI\theta_C - 0.375EI\Delta = 37.5 \qquad (1)$$

$$\sum M_C = 0, \quad \text{gives}$$

$$M_{CB} + M_{CD} = 0$$

$$60 + 0.667EI\theta_B + 1.333EI\theta_C + EI\theta_C - 0.375EI\Delta = 0$$

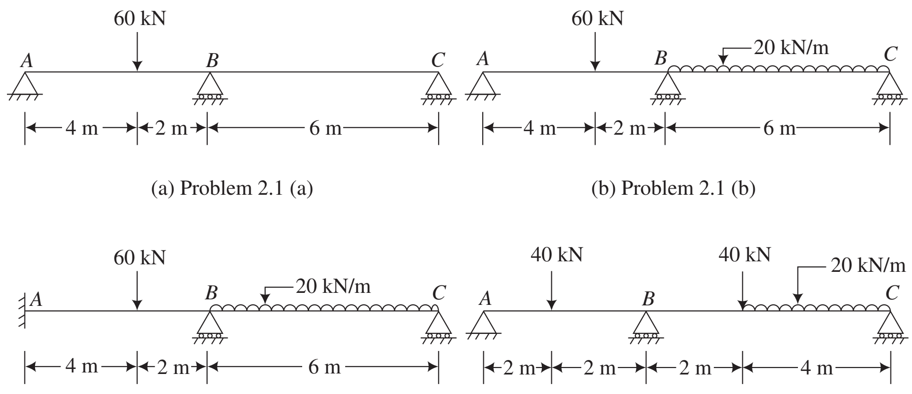
i.e. $$0.667EI\theta_B + 2.333EI\theta_C - 0.375EI\Delta = 0 \qquad (2)$$

To determine shear equilibrium equation, consider the free body diagram of columns shown in Fig. 2.10(c)

$$\sum M_B = 0, \quad \text{gives}$$

$$H_A \times 4 + 40 \times 1 = M_{AB} + M_{BA}$$

i.e. $$4H_A = M_{AB} + M_{BA} - 40$$



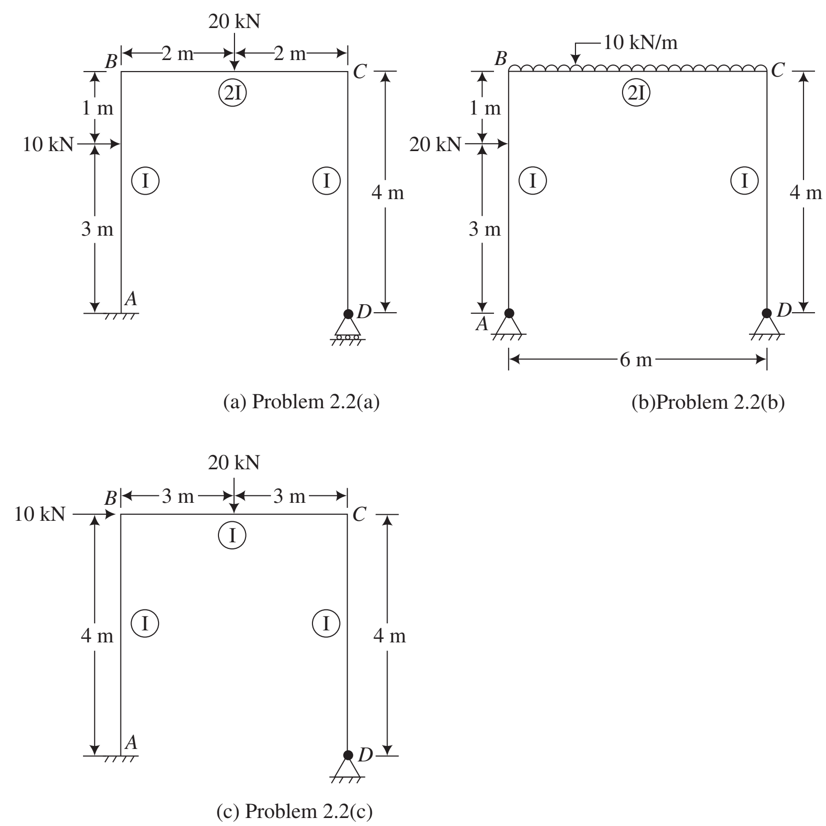
(a) Problem 2.1 (a)

(b) Problem 2.1 (b)

(c) Problem 2.1 (c)

(d) Problem 2.1 (d)

Fig. 2.11



(a) Problem 2.2(a)

(b)Problem 2.2(b)

(c) Problem 2.2(c)

Fig. 2.12

Ans. (a) $H_A = 10$ kN, $V_A = 8.125$ kN $M_A = -22.5$ kN-m $V_D = 11.875$ kN

(b) $H_A = 7.76$ kN, $V_A = 20$ kN, $H_D = 12.24$ kN, $V_D = 40$ kN,