2

Envelopes and Developables

2.1 FAMILY OF SURFACES (ONE PARAMETER)

An equation of the form

$$F(x, y, z, a) = 0 \tag{1}$$

represents on infinitely many surfaces, each surface being determined by a value of the parameter a. We call such a system one-parameter family of surfaces.

Characteristic of a Family of Surfaces

Let $F(x, y, z, \alpha) = 0$ be the one parameter family of surfaces. The curve of intersection of two surfaces of the family corresponding to the values of α and $\alpha + \delta\alpha$, is given by

$$F(x,y,z,\alpha) = 0, \quad F(x,y,z,\alpha+\delta\alpha) = 0$$

or
$$F(x,y,z) = 0, \quad \frac{F(x,y,z,\alpha+\delta\alpha) - F(x,y,z,\alpha)}{\delta\alpha} = 0$$

Now as $\delta\alpha \to 0$, the curves tend to limiting position given by

$$F(x,y,z,\alpha) = 0, \quad \frac{\partial}{\partial\alpha} F(x,y,z,\alpha) = 0 \tag{2}$$

The curve given by (2) is thus a characteristic of the family of surfaces $F(x, y, z, a) = 0$ for $a = \alpha$. Clearly each value of α of a determines a characteristic.

2.2 EDGE OF REGRESSION

Let the family of surfaces be given by $f(x, y, z, a) = 0$. Let us consider the equations to the two neighbouring characteristics for the values $a = \alpha$ and $a = \alpha + \delta\alpha$ as

$$f(x,y,z,\alpha)=0, \quad f_\alpha(x,y,z,\alpha)=0 \tag{1}$$

and

$$f(x,y,z,\alpha+\delta\alpha)=0, \quad f_\alpha(x,y,z,\alpha+\delta\alpha)=0 \tag{2}$$

The above two characteristics (1) and (2), in general, intersect in a point. The position of the point of intersection of two neighbouring characteristics, as $\delta\alpha \to 0$, is called a characteristic point.

The locus of all characteristic points of a one-parameter family of surfaces is called the edge of regression.

Thus, equations (1) and (2) together, as $\delta\alpha \to 0$, on eliminating α yield the edge of regression or cuspidal edge of the envelope.

When $\delta\alpha \to 0$, equations (1) and (2) become

$$f(x,y,z,\alpha)=0, \quad f_\alpha(x,y,z,\alpha)=0 \quad \text{and} \quad f_{\alpha\alpha}=0 \\ \left(f_{\alpha\alpha}=\frac{\partial^2 f}{\partial\alpha^2}\right) \tag{3}$$

Equation (3) determines a characteristic point. The equation of edge of regression is obtained by eliminating α from equation (3).

Theorem 1

Each characteristic touches the edge of regression.

Proof:

The edge of regression is given by

$$f(x,y,z,\alpha)=0 \quad \text{and} \quad f_\alpha(x,y,z,\alpha)=0, \tag{1}$$

provided $f_{\alpha\alpha}(x, y, z, \alpha) = 0$, so that α is a function of x, y, z.

The tangent line to a characteristic curve $f = 0$, $f_\alpha = 0$ is $\perp$ to the normals to the surfaces

$$f(x, y, z, \alpha) = 0 \quad \text{and} \quad f_\alpha(x, y, z, \alpha) = 0$$

at any point (x, y, z) on them.

If $\delta b = \lambda \delta a$, then we get

$$f(a,b) = 0, \quad \frac{\partial f}{\partial a} + \lambda \frac{\partial f}{\partial b} + \cdots = 0$$

and when $\delta a, \delta b \to 0$,

$$f(a,b) = 0, \quad \frac{\partial f}{\partial a} + \lambda \frac{\partial f}{\partial b} = 0$$

As δa and δb are independent, thus λ can take any value. Thus, limiting position of the curve depends on λ and will be different for different values of λ. However, for all values of λ, the limiting positions will pass through the point or points given by

$$f(a,b) = 0, \quad f_a = 0, \quad f_b = 0 \quad \left(f_a \equiv \frac{\partial f}{\partial a} \text{ etc.} \right) \tag{2}$$

These are called characteristic points and the locus of these points is the envelope of the family of surfaces. We get the equation of envelope by eliminating a and b from equation (2).

Let $f(x, y, z, a, b) = 0$, where a and b are functions of x, y, z given by

$$\frac{\partial f}{\partial a} = 0, \quad \frac{\partial f}{\partial b} = 0 \tag{3}$$

as the equation of the envelope. The normal to the envelope is parallel to the vector

$$\left(f_x + f_a \frac{\partial a}{\partial x} + f_b \frac{\partial b}{\partial x}, \quad f_y + f_a \frac{\partial a}{\partial y} + f_b \frac{\partial b}{\partial y}, \quad f_z + f_a \frac{\partial a}{\partial z} + f_b \frac{\partial b}{\partial z} \right);$$
$$\left(\text{here } f_a = \frac{\partial f}{\partial a}, \quad f_b = \frac{\partial f}{\partial b} \right),$$

which in view of (3) is

$$(f_x, f_y, f_z)$$

But this vector is parallel to the normal to a surface of the family at the point (x, y, z). Therefore, the envelope has the same normal and so the same tangent plane to a surface of the family at a characteristic point. Thus, we have the following: The envelope touches each surface of the system at the corresponding characteristic points.

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SOLVED EXAMPLE 1

Q1: Find the envelope of the plane $3xt^2 - 3yt + z = t^3$, and show that its edge of regression is the curve of intersection of the surfaces $y^2 = xz$, $xy = z$.

Solution:

Now family of planes is given by

$$f \equiv 3xt^2 - 3yt + z - t^3 = 0 \tag{1}$$

$$\therefore \quad \frac{\partial f}{\partial t} = 0 = 6xt - 3y - 3t^2 = 0 \tag{2}$$

Let us multiply (2) by t and subtracting from (1), we get

$$xt^2 - 2yt + z = 0 \tag{3}$$

Now in view of (2) and (3), we have

$$\frac{t^2}{-2xz + 2y^2} = \frac{1}{xy - z} = \frac{1}{-2y + 2x^2}$$

which on eliminating t, gives equation of envelope as

$$(xy - z)^2 = 4(x^2 - y)(y^2 - zx) \tag{4}$$

Now from (2) on differentiation, we get

$$f_{tt} \equiv 2x - 2t = 0 \Rightarrow t = x$$

Putting value of $t = x$ in (1) and (2), we get,

$$2t^2 - 3yt + z = 0 \quad \text{and} \quad y = t^2;$$

i.e.,

$$x = t, \quad y = t^2, \quad z = t^2$$

On eliminating t, we get

$$y^2 = xz \quad \text{and} \quad xy = z$$

which represents the edge of regression.

Q2: The envelope of osculating plane of a curve is a ruled surface generated by the tangents to the curve and has the curve for the edge of regression.

Solution:

Let the curve be given by $r = r(s)$ and osculating plane at any point on it is given by

$$(R - r) \cdot b = 0; \tag{1}$$

where r and b are functions of s.

In view of (1), on differentiation w.r.t. s, we get,

$$(R - r) \cdot b' - r' \cdot b = 0$$

$$\Rightarrow \quad (R - r) \cdot n = 0 \tag{2}$$

as $r' \cdot b = t \cdot b = 0$ by Frenet's formula.

Equations (1) and (2) simultaneously determines a characteristic. Equation (2) gives rectifying plane and intersection of (1) and (2), therefore represents the tangent line to the curve.

Thus, the envelope is the locus of tangent line, that is, it is a ruled surface generated by the tangents to the given curve. This proves the first part.

Now, differentiating (2) w.r.t. s, and in view of Frenet's formula, we get

$$(R - r) \cdot (\tau b - kt) - r' \cdot n = 0$$

$$\Rightarrow \quad (R - r) \cdot (\tau b - kt) = 0 \qquad (\because \quad r' \cdot n = t \cdot n = 0)$$

$$\Rightarrow \quad (R - r) \cdot t = 0 \quad (\text{by (1)}) \tag{3}$$

Now, equations (1), (2) and (3) simultaneously represent the edge of regression. But (1), (2) and (3) are planes whose intersection is point $p(r)$. Hence, the points on the edge of regression coincide with the points on the curve. Hence, the curve itself is the edge of regression.

Q3: Find the envelope of the planes $lx + my + nz = p$ when $a^2l^2 + b^2m^2 + 2np = 0$.

Solution:

$$\text{Let} \quad \frac{l}{p} = \lambda, \quad \frac{m}{p} = \mu \quad \text{and} \quad \frac{n}{p} = \nu,$$

the equation of plane becomes

$$f \equiv \lambda x + \mu y + \nu z - 1 = 0 \tag{1}$$

when
$$a^2\lambda^2 + b^2\mu^2 + 2\nu = 0 \tag{2}$$

Differentiation of (1) and (2) gives

$$x d\lambda + y d\mu + z d\nu = 0$$

and
$$a^2\lambda d\lambda + b^2\mu^2 d\mu + d\nu = 0$$

Comparing the above two equations, we have

$$\frac{a^2\lambda}{x} = \frac{b^2\mu}{y} = \frac{1}{z} = \frac{a^2\lambda^2 + b^2\mu^2 + \nu}{\partial x + \mu y + \nu z} = -\frac{\nu}{1}$$

$$\Rightarrow \quad \lambda = \frac{x}{a^2 z}, \quad \mu = \frac{y}{b^2 z}, \quad \nu = -\frac{1}{z}$$

Putting λ, μ, ν in (1), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z.$$

UNSOLVED EXERCISE 1

Q1. The envelope of the surfaces $f(x, y, z, a, b, c) = 0$, where a, b, c are the parameters connected by the equation $\phi(a, b, c) = 0$ and f and ϕ are homogeneous with respect to a, b, c is obtained by eliminating a, b, c between the equations $f = 0, \quad \phi = 0, \quad \frac{f_a}{\phi_a} = \frac{f_b}{\phi_b} = \frac{f_c}{\phi_c}$.

Q2. Show that the edge of regression of the envelope of plane $\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$ is the cubic curve given by $x = \frac{(a+\lambda)^2}{(c-a)(b-a)}, \quad y = \frac{(b+\lambda)^2}{(c-b)(a-b)}, \quad z = \frac{(c+\lambda)^2}{(a-c)(b-c)}$.

Q3. Prove that the envelope of plane $\frac{x}{a}\cos\theta\sin\phi + \frac{y}{b}\sin\theta\sin\phi + \frac{z}{c}\cos\phi = 1$ is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Q4. (a) A plane makes intercepts a, b, c on the rectangular axes, so that $a^{-2} + b^{-2} + c^{-2} = k^{-2}$, show that it envelopes a conicoid which has the axes as equal conjugate diameters.

(b) If $a^2 + b^2 + c^2 =$ constant, prove that envelope is $x^{2/3} + y^{2/3} + z^{2/3} =$ constant.

Q5. Find the envelope of the cones $(x - a)^2 + y^2 = z^2\tan^2\alpha$, where α is same for all cones.

Q6. Prove that the envelope of the family of paraboloids is the circular cone $x^2 + y^2 = z^2$.

Q7. Find the envelope of the sphere $(x - a\cos\theta)^2 + (y - a\sin\theta)^2 + z^2 = b^2$.

Q8. Find the envelope of the plane $lx + my + nz = 0$, where $al^2 + bm^2 + cn^2 = 0$.

Q9. A tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the axes in A, B, C. Show that the envelope of the sphere $OABC$ is $(ax)^{2/3} + (by)^{2/3} + (cz)^{2/3} = (x^2 + y^2 + z^2)^{2/3}$.

ANSWERS 1

Q5. $y^2 = z^2 \tan^2 \alpha$

Q7. $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$

Q8. $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

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2.4 RULED SURFACES

A surface which is generated by the motion of one-parameter family of straight lines is called a ruled surface.

The straight line itself is called the generating line, generator, or ruling. Examples of ruled surfaces are cylinders, cones, conicoid, etc.

Types of Ruled Surfaces

(1) ***Developable surface:*** If consecutive generators intersect, the ruled surface is called developable.

Example: cones and cylinders.

(2) ***Skew surfaces:*** If consecutive generators do not intersect, the ruled surface is called skew-surface or scroll.

Hyperboloid of one sheet and hyperbolic paraboloids are skew surfaces as shortest distance between their consecutive generators do not vanish.

where r is position vector on the base curve and g is unit vector along the generator at that point. Let r, g be functions of single parameter u. Then R is a function of two parameters u, v. Thus, equation (1) defines a surface

$$R_1 = r_1 + vg_1 \quad \text{and} \quad R_2 = g$$

Hence, equation of tangent plane to the ruled surface is

$$[R - r, r_1 + vg_1, g] = 0.$$

2.6 DEVELOPABLE SURFACE

A surface generated by one parameter family of planes is called developable surface or simply developable.

If n is unit normal to the plane, then equation of a family of plane is

$$r \cdot n = p,$$

where n and p are functions of single parameter t, and p is length of perpendicular from the origin to the plane.

Envelope of a Plane Involving One Parameter

Here we shall prove that the envelope of a plane whose equation involves one parameter is a developable surface.

In order to prove this, let a plane be given by

$$r \cdot n = p \quad \text{or} \quad V \equiv r \cdot n - p = 0, \tag{1}$$

where n and p are functions of one-parameter t. In view of (1), we get, on differentiating w.r.t. t,

$$\dot{V} \equiv r \cdot \dot{n} - \dot{p} = 0 \tag{2}$$

Equations (1) and (2) together determine a characteristic of the family (1). Since (1) and (2) are planes, therefore characteristic is a straight line. Hence, the locus of characteristic, that is, envelope of family of planes is a ruled surface.

Now, we consider two consecutive characteristics, given by

$$V = 0 = \dot{V}; \quad V + \delta V = 0 = \dot{V} + \delta\dot{V}$$

i.e., $$V = 0 = \dot{V}; \quad V + \dot{V}\delta t = 0 = \dot{V} + \ddot{V}\delta t$$

Clearly the two consecutive lines lie in the plane $V + \dot{V}\delta t = 0$. Thus, the consecutive characteristics intersect. Hence, ruled surfaces, i.e., the envelope of one-parameter family of planes is developable.

Edge of Regression

The edge of regression of the envelope, of one parameter family of planes $V = 0$ is given by

$$V = 0, \quad \dot{V} = 0, \quad \ddot{V} = 0,$$

i.e., (i) $r \cdot N - p = 0$ (ii) $r \cdot \dot{N} - \dot{p} = 0$ (iii) $r \cdot \ddot{N} - \ddot{p} = 0$

where r is regarded as function of t.

In view of (i) , we get on differentiation w.r.t., t

$$\dot{r} \cdot N + r \cdot \dot{N} = \dot{p}$$

or $$\dot{r} \cdot N = 0 \quad (\text{by ii}) \tag{iv}$$

Again from (ii),

$$\dot{r} \cdot \dot{N} + r \cdot \ddot{N} = \ddot{p}$$

or $$\dot{r} \cdot \dot{N} = 0 \quad (\text{by (iii)}) \tag{v}$$

Again from (iv)

$$\ddot{r} \cdot N + \dot{r} \cdot \dot{N} = 0$$

or $$\ddot{r} \cdot N = 0 \quad (\text{by use of (v)}) \tag{vi}$$

From equations (iv) and (vi), we notice that both $\dot{r}$ and $\ddot{r}$ are $\perp$ to N, it follows that $\dot{r} \times \ddot{r}$ is parallel to N; but $\dot{r} \times \ddot{r}$ is perpendicular to the osculating plane and N being perpendicular to the plane $r \cdot N - p = 0$ and hence the osculating plane of the edge of regression at any point is the tangent plane to the developable at the same point.

Note: Here in place of n, we have taken N but the meaning is same.

Condition for the Surface *z* = *f*(*x*, *y*) to be Developable

The equation of the tangent plane at the point (x, y, z) to the surface $z = f(x, y)$ is

$$(X - x)f_x + (Y - y)f_y + (Z - z)f_z = 0$$

or $$pX + qY - Z = px + qy - z = \phi \quad (\text{say}) \tag{1}$$

where $p = \dfrac{\partial f}{\partial x}, \quad q = \dfrac{\partial f}{\partial y}.$

If $z = f(x, y)$ is developable surface, then the tangent plane to it should involve only one parameter t (say),

$$p = f_1(t), \quad q = f_2(t), \quad \phi = f_3(t) \tag{2}$$

By eliminating t between (2), we can express p and ϕ as functions of q. But when p is a function of q,

$$\frac{\partial(p,q)}{\partial(x,y)}=0, \quad p=p(x,y), \quad q=q(x,y) \tag{3}$$

Similarly,
$$\frac{\partial(\phi,q)}{\partial(x,y)}=0; \quad \phi \text{ is a function of } q. \tag{4}$$

In view of (3) and (4), we obtain

$$\begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial q}{\partial x} \\ \frac{\partial p}{\partial y} & \frac{\partial q}{\partial y} \end{vmatrix}=0 \quad \text{and} \quad \begin{vmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial q}{\partial x} \\ \frac{\partial \phi}{\partial y} & \frac{\partial q}{\partial y} \end{vmatrix}=0 \qquad \left(\because \; r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial p}{\partial x}, \text{ etc.}\right)$$

i.e.,
$$\begin{vmatrix} r & s \\ s & t \end{vmatrix}=0 \quad \text{and} \quad \begin{vmatrix} rs+sy & s \\ sx+ty & t \end{vmatrix}=0$$

i.e., $rt-s^2=0$ and $x(rt-s^2)=0$

$\Rightarrow \quad rt-s^2=0$

Thus, $rt-s^2=0$ is necessary as well as sufficient, for $z=f(x, y)$ to be developable surface.

2.7 DEVELOPABLE ASSOCIATED WITH SPACE CURVE

Since the equation to the three principal planes, namely, osculating plane, normal plane and rectifying plane contain only a single parameter which is usually taken to be the arc length s, hence their envelopes are developable surfaces and they are called osculating developable or tangential developable, polar developable and rectifying developable respectively. Also the generators of polar developable and rectifying developable are known as polar lines and rectifying lines respectively.

Let us prove the following:

1. The curve itself is the edge of regression of the osculating developable.

Proof:

Let the curve be given by $r = r(s)$ and at any point r on the curve, the equation of the osculating plane is

$$(R-r)\cdot b=0, \quad \text{where} \quad r=r(s), \quad b=b(s) \tag{i}$$

In order to get the edge of regression, let us differentiate (3) and use (2), we get

$$(R-r)\cdot(\tau' b - k' t) + k = 0 \tag{5}$$

Since the rectifying line is parallel to $\tau t + kb$, the point R on the edge of regression is such that,

$$(R-r) = l(\tau t + kb), \quad l \text{ some number} \tag{6}$$

Thus, the point on the edge of regression corresponding to the point r on the curve is

$$R = r + \frac{k(\tau t + kb)}{k'\tau - k\tau'} \tag{7}$$

This gives the edge of regression.

Thus, we obtained by putting the value of $l = k / k'\tau - k\tau$, which we get from (5) and (6).

The reason for the term, rectifying, applied to this developable lies in the fact that, when the surface is developed into a plane by unfolding about consecutive generators, the original curve becomes a straight line.

We here notice in passing that, if the given curve is a helix k/τ is constant, and then angle ϕ will be constant and then the curve in space will be helix. Thus, rectifying lines are the generators of the cylinder on which the helix is drawn, and rectifying developable is the cylinder, itself.

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SOLVED EXAMPLE 2

Q1: Prove that surface $xy = (z-c)^2$ is developable.

Solution:

Given $\qquad (z-c)^2 = xy = z - c = \sqrt{xy}$

or $\qquad z = c + \sqrt{xy}$

$\therefore \qquad p = \dfrac{\partial z}{\partial x} = \dfrac{1}{2}\sqrt{\dfrac{y}{x}}, \quad q = \dfrac{\partial z}{\partial y} = \dfrac{1}{2}\sqrt{\dfrac{x}{y}}$

$$r = \frac{\partial^2 z}{\partial x^2} = -\frac{1}{4} y^{1/2} x^{-3/2}, \quad t = \frac{\partial^2 z}{\partial y^2} = -\frac{1}{4} x^{1/2} y^{-3/2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{4} x^{-1/2} y^{-1/2}$$

$\therefore \quad rt - s^2 = 0$

$\Rightarrow$ Surface is developable.

Q2: For $z = a^2/xy$, show that the surface is developable (use $rt - s^2 = 0$ for this).

Q3: Show that the line given by $y = tx - t^3$, $z = t^3y - t^6$, generates a developable surface.

Solution:

The equation of line can be written as

$$\frac{x - t^2}{t} = \frac{y}{t} = \frac{z + t^6}{t^4}$$

Here $\quad r = \left(t^2, 0, -t^6\right) \quad$ and $\quad g = \left(1, t, t^4\right)$

$$\dot{r} = \left(2t, 0, -6t^5\right), \dot{g} = \left(0, 1, 4t^3\right)$$

$$g \times \dot{g} = \left(3t^4, -4t^3, 1\right)$$

$$\left[\dot{r}, g, \dot{g}\right] = \left(2t, 0, -6t^5\right) \cdot \left(3t^4, -4t^3, 1\right)$$

$$= \left(6t^5, -0, -6t^5\right)$$

$$= 0$$

$\Rightarrow$ The given line generates a developable surface.

Q4: Show that tangent to a curve generates a developable surface and principal normal generates skew-surface.

Solution:

Let the space curve be given by $r = r(s)$ and let t be the unit tangent vector at any point on it whose position vector is r. Thus, tangent line is given by

$R = r + ut, \quad u$ is a scalar

$\therefore \quad R = R(u, s)$ gives the surface generated by the tangent.

Here $\; g = t, \quad g' = t' = kn$

$\therefore \quad \left[t, g, g'\right] = \left[t, t, kn\right] = 0$

$\Rightarrow$ Surface is developable.

Equation of the principal normal is given by

$$R = r + un$$

here $$g = n, \quad g' = n' = (\tau b - kt)$$

$\therefore$ $$[t, g, g'] = [t, n, \tau b - kt]$$
$$= \tau[t, n, b]$$
$$= 0, \qquad (\because \quad \tau \neq 0)$$

$\Rightarrow$ Surface generated by the principal normal is skew.

Q5: Show that the edge of regression of the developable that passes through the parabola $x = 0$, $z^2 = 4ay$; $x = a$, $y^2 = 4az$, is given by $\dfrac{3x}{y} = \dfrac{y}{z} = \dfrac{z}{3(a-x)}$.

Solution:

A straight line touching the parabola $x = 0$, $z^2 = 4ay$ is given by

$$x = 0, \quad z = my + \frac{a}{m}$$

Hence, any tangent plane to the parabola is

$$z - my - \frac{a}{m} + \lambda x = 0 \quad \text{for all } \lambda$$

(plane through the tangent line)

or, $$y = \frac{z}{m} - \frac{a}{m^2} + \frac{\lambda x}{m} \tag{1}$$

Now its section by the plane $x = a$ is the line given by

$$x = a, \quad y = \frac{z}{m} - \frac{a}{m^2} + \frac{\lambda a}{m} \tag{2}$$

Clearly if this line is tangent to the parabola $x = a$, $y^2 = 4az$ then, it must be of the form

$$x = a, \quad y = Mz + \frac{a}{M} \tag{3}$$

Comparing (2) and (3), we get

$$M = \frac{1}{m}, \quad \frac{1}{M} = -\frac{1}{m^2} + \frac{\lambda}{m}$$

The rectifying developable of c is the envelope of the rectifying planes, given by

$$(R-r)\cdot n=0 \tag{2}$$

Again the polar developable being the envelope of normal planes the polar developable of c_1 is the envelope of family of planes

$$(R-r_1)\cdot t_1=0 \tag{3}$$

Now, we shall prove that (2) and (3) are same. Differentiate (1) w.r.t. s, we get,

$$t_1\frac{ds_1}{ds}=t+(c-s)kn+(-1)t=(c-s)kn \tag{4}$$

On squaring (4), we obtain

$$\left(\frac{ds_1}{ds}\right)^2=(c-s)^2k^2$$

Let us put $\frac{ds_1}{ds}$ in (4), we get

$$t_1=n \tag{5}$$

Putting r_1 from (1) in (3) and using $t_1 = n$, provided

$$\Rightarrow \quad \begin{aligned}&\left[R-\left(r+(c-s)t\right)\right]\cdot n=0,\\ &(R-r)\cdot n=0\end{aligned} \tag{6}$$

which is same as (2). Conversely putting

$n = t_1$ and $r = r_1 - (c - s)t$ in (2)

$\{R-r_1+(c-s)t\}\cdot t_1=0$ (In view of (5) and (1))

or $$(R-r_1)\cdot t_1=0 \qquad (\because\ t\cdot t_1=t\cdot n=0) \tag{7}$$

which is same as (3). Hence, proved.

Q8: Find the equation of developable surface which has the curve $x = 6t$, $y = 3t^2$, $z = 2t^3$ for its edge of regression.

Solution:

The equation of the tangent lines to the edge of regression are

$$\frac{x-6t}{\dot{x}}=\frac{y-3t^2}{\dot{y}}=\frac{z-2t^3}{\dot{z}}$$

or
$$\frac{x-6t}{6}=\frac{y-3t^2}{6t}=\frac{z-3t^3}{6t^2}$$

⇒
$$\frac{x-6t}{1}=\frac{y-3t^2}{t}=\frac{z-3t^3}{t^2}$$

or
$$ty-3t^3=z-2t^3, \quad \text{i.e.,} \quad ty=t^3+z \tag{1}$$

and
$$tx-6t^2=y-3t^2, \quad \text{i.e.,} \quad tx-y=3t^2 \tag{2}$$

The developable surface is generated by the tangent to its edge of regression, i.e., it is the locus of the tangent to the edge of regression. Consequently, for the required developable surface, we require to eliminate t between (1) and (2). This gives

$$3t^2-tx+y=0 \tag{3}$$

and
$$t(tx-y)=3t^3=3(ty-z)$$

or
$$xt^2-4y+3z=0 \tag{4}$$

∴ From (3) and (4),

$$\frac{t^2}{3zx+4y^2}=\frac{t}{xy-yz}=\frac{1}{-12y+x^2}$$

or
$$\frac{4y^2-3zx}{x^2-12y}=t^2=\frac{(xy-9z)^2}{(x^2-12y)^2}$$

∴ required developable surface is

$$(xy-9z)^2=(x^2-12y)(4y^2-2zx).$$

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UNSOLVED EXERCISE 2

Q1. Prove that the surface $xyz=a^2$ is not developable.

Q2. Show that the developable which passes through the curve $z=0, y^2=4ax, x=0, y^2=4bz$ is the cylinder $y^2=4ax+4bz$.

Q3. Show that the line $x=3t^2z+2t(1-3t^4), y=-2tz+t^2(3+4t^2)$ generate a skew surface.

Q4. Prove that the generator of the rectifying developable of the skew curve makes with the tangent to the curve an angle θ given by $\tan\theta=k/\tau$.

Q5. Prove that the rectifying developable of a curve is the polar developable of its involute, and conversely.

Parameter of Distribution

The parameter of distribution p of generator PG_1 is defined as

$$p = \lim_{QG_2 \to PG_1} \frac{\text{shortest distance (S.D.)}}{\delta\psi} \tag{1}$$

where $\delta\psi$ is the angle between consecutive generators PG_1 and QG_2. Consequently

$$P(u) = \frac{[r', g, g']}{g'^2}, \quad \text{where } r' = \frac{dr}{du}$$

and ruling of the surface is

$$R = r + vg.$$

Determination of Central Point

Let g be a unit vector along the direction of PG_1. Now as we know that S.D. of PG_1 and a consecutive generator, in the limiting position, is in surface. Hence, if d be limiting direction of S.D., then

$$d \cdot N = 0, \quad \text{and as } d \text{ is} \perp \text{to } g, \quad \text{therefore } d \cdot g = 0.$$

Thus, d is parallel to $g \times N$. Hence, let $d = \lambda g \times N$. Also d is perpendicular to vector $g + \delta g$ along the second generator through Q.

$$\therefore \quad d \cdot (g + \delta g) = 0 \quad \Rightarrow \quad d \cdot \delta g = 0 \quad \Rightarrow \quad d \cdot g' = 0$$

in the limit when $Q \to p$ and $g' = \dfrac{\partial g}{\partial u}$, putting $d = \lambda g \times N$ in it, we get

$$\lambda g' \cdot (g \times N) = 0 \quad \Rightarrow \quad (g' \times g) \cdot N = 0 \tag{1}$$

If generator PG_1 is given by $R = r + vg$, then R is a function of u and v, where r, g are functions of u.

Hence, $$R_1 \times R_2 = HN = (r' + vg') \times (g)$$

Let us multiply this by $g' \times g$ scalarly, we obtain, by (1)

$$(g' \times g) \cdot \left[(r' + vg') \times g \right] = H(g' \times g) \cdot N = 0$$

$$\Rightarrow \quad (g' \times g) \cdot \left[r' \times g + vg' \times g \right] = 0$$

$$\Rightarrow \quad g' \cdot r' + vg'^2 = 0 \tag{2}$$

since

$$g^2 = 1 \Rightarrow g \cdot g' = 0$$

Equation (2) determines v uniquely provided $g'^2 \neq 0$ and for this value of ν, we have the central point given by

$$R = r + \nu g.$$

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SOLVED EXAMPLE 3

Q1: Find the line of striction of hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

Solution:

The two consecutive generators are

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{c}$$

and

$$\frac{x - a\cos(\theta + \partial\theta)}{a\sin(\theta + \partial\theta)} = \frac{y - b\sin(\theta + \partial\theta)}{-b\cos(\theta + \partial\theta)} = \frac{z}{c'}$$

If λ, μ, ν are direction cosines of the shortest distance between these two,

$$\lambda \cdot a\sin\theta + \mu(-b\cos\theta) + vc = 0$$

and

$$\lambda \cdot a\sin(\theta + \partial\theta) + \mu(-b\cos(\theta + \partial\theta)) + cv = 0$$

$$\therefore \quad \frac{\lambda}{-bc\cos\theta + bc\cos(\theta + \partial\theta)} = \frac{\mu}{ac\sin(\theta + \partial\theta) - ac\sin\theta}$$

$$= \frac{\nu}{-ab\sin\theta\cos(\theta + \partial\theta) + ab\cos\theta\sin(\theta + \partial\theta)}$$

or
$$\frac{\lambda}{-bc\sin\theta\partial\theta}=\frac{\mu}{ac\cos\theta\partial\theta}=\frac{\nu}{ab\partial\theta}$$

or
$$\frac{\lambda}{-bc\sin\theta}=\frac{\mu}{ac\cos\theta}=\frac{\nu}{ab}$$

Now, if we take (x, y, z) as a point where S.D. (shortest distance) meets the consecutive generators, the normal at (x, y, z) must be perpendicular to the given generator and also to the shortest distance.

As direction cosines of the normal at (x, y, z) are proportional to $\frac{x}{a^2}, \frac{y}{b^2}, -\frac{z}{c^2}$, we have

$$\frac{x\cdot a\sin\theta}{a^2}+\frac{y(-b\cos\theta)}{b^2}+\left(-\frac{z}{c^2}\right)\frac{c}{1}=0$$

and
$$\frac{-x\sin\theta\cdot bc}{a^2}+\frac{y}{b^2}(ac\cos\theta)-\frac{z}{c^2}\cdot ab=0$$

or
$$\frac{x\sin\theta}{a}-\frac{y\cos\theta}{b}-\frac{z}{c}=0$$

where
$$\frac{\sin\theta}{-\left(yz/c^3b^3\right)\left(b^2+c^2\right)}=\frac{\cos\theta}{-\left(zx/a^2c^2\right)\left(a^2+c^2\right)}=\frac{1}{-\left(xy/b^3c^3\right)\left(b^2+a^2\right)}$$

Elimination of θ and by simplification, we get for the lines of striction the intersection of the surface and the following curve.

$$\frac{a^2}{x^2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)^2+\frac{b^2}{y^2}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)=\frac{c^2}{z^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

Q2: If the line $x=az+\alpha, y=bz+\beta$, where a, b, α, β are functions of t, generate a skew-surface, the parameter of distribution for the generator is

$$\frac{(\alpha'b'-a'\beta')\left(1+a^2+b^2\right)}{a'^2+b'^2+(ab'-a'b)^2}$$

Solution:

Line is given as

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z}{1}$$

or $$\lambda[b,\partial b,t]=\partial s[t,b+\partial b,t] \qquad (\because \ [\alpha,\beta,\gamma]=0)$$

or $$-\lambda\tau\partial s[b,n,t]=0$$

or $$\lambda\tau\partial s=0$$

but $$\tau\neq 0, \quad \partial s\neq 0 \qquad \therefore \quad \lambda=0$$

That is the central point lies on the curve and hence the curve is line of stricion.