UNIT 2

FOURIER TRANSFORMS

2.1 DEFINITION

The integral transform of a function $f(x)$ denoted by $I[f(x)]$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x)\, k(s, x)\, dx$$

where $k(s, x)$ is called the KERNEL of the transform and is a known function of s and x. The function $f(x)$ is called the inverse transform of $\bar{f}(s)$.

The examples of a kernel are

i) When $k(s, x) = e^{-sx}$, it leads to the Laplace transform of $f(x)$ i.e.,

$$\bar{f}(s) = \int_{0}^{\infty} e^{-sx} f(x)\, dx$$

ii) When $k(s, x) = e^{isx}$ we have the Fourier transform of $f(x)$ i.e.,

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

iii) When $k(s, x) = x^{s-1}$, it gives the Mellin transform of $f(x)$ i.e.,

$$M(s) = \int_{0}^{\infty} f(x)\, x^{s-1} dx$$

iv) Hankel transform $K(s, x) = x\, J_n(sx)$

2.3 INFINITE FOURIER TRANSFORMS

Consider,

$$f(x) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t)\, e^{iu(t-x)} dt\, du$$

$$= \frac{1}{2\pi} \int_{u=-\infty}^{\infty} \left(\int_{t=-\infty}^{\infty} f(t) e^{iut} dt \right) e^{-ixu} du$$

We define

$$F(u) = \int_{t=-\infty}^{\infty} f(t)\, e^{iut} dt \quad \text{as the complex.}$$

Fourier transform or simply Fourier transform of $f(t)$ or $f(x)$.

Further,

$$f(x) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} F(u) e^{-ixu} du$$ is called the inverse Fourier transform of $F(u)$.

Note: We may define the Fourier integral as follows also

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{iut} dt \right] e^{-ixu}\; du$$

and the transformation pair will be

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} f(t)\, e^{iut} dt$$

and $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{u-\infty}^{\infty} F(u)\, e^{-ixu} du$$

2.4 FOURIER COSINE AND SINE TRANSFORMS

1) We define

$$F_c(u) = \int_{t=0}^{\infty} f(t) \cos ut\; dt$$ as a Fourier cosine transform of $f(t)$

and

$$f(x) = \frac{2}{\pi} \int_{u=0}^{\infty} F_c(u) \cos xu \, du, \text{ as the inverse.}$$

Fourier cosine transform

ii) We define

$$F_s(u) = \int_{t=0}^{\infty} f(t) \sin ut \, dt \text{ as the Fourier sine transform of } f(t)$$

and

$$f(x) = \frac{2}{\pi} \int_{u=0}^{\infty} F_s(u) \sin xu \, du \text{, as the inverse}$$

Fourier sine transform.

Note: Also we may define the above formulae as follows

$$F_c(u) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos ut \, dt$$

$$F_s(u) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin ut \, dt$$

and,
$$f(x) = \sqrt{\frac{2}{\pi}} \int_{u=0}^{\infty} F_c(u) \cos xu \, du$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{u=0}^{\infty} F_s(u) \sin xu \, du.$$

Example 1: Express the function $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as a Fourier integral. Hence evaluate $\displaystyle\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$

Solution: The Fourier integral is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(x-t) \, dt \, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(xu)\,\frac{2\sin u}{u}\,du + 0 \qquad (\because\ I_2 = 0)$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos ux \sin u}{u}\,du$$

Changing u to λ

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos \lambda x \sin \lambda}{\lambda}\,d\lambda$$

$$\therefore \quad \int_{0}^{\infty} \frac{\cos \lambda x \sin \lambda}{\lambda}\,d\lambda = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2}\begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

put $x = 0$

$$\int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\,d\lambda = \frac{\pi}{2}, \text{ or } \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Example 2: Find the Fourier transform of

$$f(x) = 1 - x^2 \quad \text{for } |x| \le 1$$
$$= 0 \quad \text{for } |x| > 1$$

Solution:

We have, Fourier transform of $f(x)$ is

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx}\,dx = F(s)$$

$$= \int_{-\infty}^{-1} (0)\, e^{isx}\,dx + \int_{-1}^{1} (1 - x^2)\, e^{isx}\,dx + \int_{1}^{\infty} (0) e^{isx}\,dx$$

$$= \int_{-1}^{1} (1 - x^2)\, e^{isx}\,dx$$

$$= \left[(1-x^2)\,\frac{e^{isx}}{is} - (-2x)\,\frac{e^{isx}}{(is)^2} + 2\,\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= 2\,\frac{(e^{is}+e^{-is})}{-s^2} - 2\,\frac{(e^{is}-e^{-is})}{-is^3}$$

$$= -\frac{4}{s^3}(s\cos s - \sin s)$$

Now, by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}ds$$

$$\therefore \quad -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)\,e^{-isx}ds = \begin{cases} 1-x^2 & |x|\le 1 \\ 0 & |x|>0 \end{cases}$$

Putting $x = \frac{1}{2}$ we get

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{5^3}(s\cos s - \sin s)e^{-\frac{is}{2}}ds = 1-\frac{1}{4} = \frac{3}{4}$$

$$\therefore \quad \int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = \frac{-3\pi}{8}$$

$$\therefore \quad \int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}\,ds = -\frac{3\pi}{8} \qquad \text{(Since the integral is even)}$$

Changing the dummy variable s to x we get

$$\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx = -\frac{3\pi}{16}$$

Example 3: Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

Solution: Let $f(x) = \frac{e^{-ax}}{x}$ then its Fourier sine transform is

$$F_s\{f(x)\} = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\, dx = F(s) \quad \text{(say)}.$$

Differentiating both sides w.r.t. s underintegral sign, we get

$$\frac{d}{ds}F(s) = \int_0^\infty \frac{xe^{-ax}\cos sx}{x}dx$$

$$= \int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \left[e^{-ax}\frac{(-a\cos sx + s\sin sx)}{a^2+s^2}\right]_0^\infty$$

$$= \frac{a}{s^2+a^2}$$

Integrating w.r.t. s we get

$$F(s) = \int \frac{a}{s^2+a^2}ds = \tan^{-1}\frac{s}{a} + c$$

But $F(s) = 0$ when $s = 0$ $\therefore$ $c = 0$

$$\therefore \quad F(s) = \tan^{-1}\frac{s}{a}$$

Example 4: Find the Fourier transform of $f(t) = e^{-|t|}$

Solution: Fourier transform of $f(t)$ is given by

$$F(u) = \int_{-\infty}^{\infty} f(t)e^{iut}\, dt$$

Here,
$$f(t) = \begin{cases} e^{-t} & \text{for } t \geq 0 \\ e^{t} & \text{for } t \leq 0 \end{cases}$$

$$\therefore \quad F(u) = \int_{-\infty}^{0} e^{t}\, e^{iut}dt + \int_0^\infty e^{-t}e^{iut}\, dt$$

$$= \int_{-\infty}^{0} e^{(1+iu)t}dt + \int_0^\infty e^{-(1-iu)t}dt$$

Example 6: Find the Fourier sine transform of $f(x) = e^{-|x|}$ and hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx,\ m > 0.$

Solution:

Fourier sine transform is given by

$$F_s(u) = \int_0^\infty f(t) \sin ut\ dt$$

$$= \int_0^\infty e^{-|t|} \sin ut\ dt \qquad \because\ e^{-|t|} = e^{-t}$$

$$= \int_0^\infty e^{-t} \sin ut\ dt$$

$$= e^{-t} \left[\frac{(-1 \sin ut - u \cos ut)}{1^2 + u^2} \right]_0^\infty$$

$$F_s(u) = \frac{u}{1+u^2}$$

By inverse Fourier sine transform we have

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(u) \sin xu\ du$$

$$= \frac{2}{\pi} \int_0^\infty \frac{u}{1+u^2} \sin\ xu\ du$$

Putting $x = m,\quad m > 0$ we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{u \sin mu}{1+u^2} du$$

$$\therefore \int_0^\infty \frac{u \sin mu\ du}{1+u^2} = \frac{\pi}{2} e^{-m} \qquad \because\ f(x) = e^{-x},\ f(m) = e^{-m}$$

Changing the dummy variable u to x we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$$

Example 7: Find the Fourier cosine transform of

$$f(x) = \begin{cases} 4x & 0 < x < 1 \\ 4 - x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

Solution: The Fourier cosine transform is

$$F_c\{f(x)\} = \int_0^\infty f(x) \cos ux\, dx$$

$$= \int_0^1 4x \cos ux\, dx + \int_1^4 (4 - x) \cos ux\, dx + 0$$

$$= 4\left[\frac{x \sin ux}{u} + \frac{1 \cos ux}{u^2}\right]_0^1 + \left[(4-x)\frac{\sin ux}{u} - \frac{(-1)(-\cos ux)}{u^2}\right]_1^4$$

$$= \frac{4 \sin u}{u} + \frac{4 \cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2} - \frac{3 \sin u}{u} + \frac{\cos u}{u^2}$$

$$= \frac{\sin u}{u} + \frac{5 \cos u}{u^2} - \frac{\cos 4u}{u^2} - \frac{4}{u^2}$$

Example 8: Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x > 1 \end{cases}$$

and hence evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$

Solution: The Fourier cosine transform is

$$F_c\{f(x)\} = \int_0^\infty f(x) \cos ux\, dx$$

$$= \int_0^1 1\cos ux\, dx + 0$$

$$= \left[\frac{\sin ux}{u}\right]_0^1 = \frac{\sin u}{u} = \varphi(u)$$

Hence from inverse Fourier cosine transform

We have, $f(x) = \frac{2}{\pi}\int_0^\infty \varphi(u)\cos xu\, du$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin u}{u}\cos xu\, du$$

Put $x = 0$

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin u}{u}\, du$$

$$\therefore \int_0^\infty \frac{\sin u}{u}\, du = \frac{\pi}{2} f(0)$$

$$\therefore \int_0^\infty \frac{\sin u}{u}\, du = \frac{\pi}{2} \qquad \because f(0) = 1 \qquad 0 \le x \le 1$$

Changing the dummy variable u to x we get

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Example 9: Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$. Hence derive Fourier cosine transform of $\varphi(x) = \frac{x}{1+x^2}$

Solution: Let the Fourier cosine transform be

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I \quad \text{(say)} \qquad ....(1)$$

$$\therefore \quad \frac{dI}{ds} = -\int_0^\infty \frac{x\sin sx}{1+x^2}\,dx$$

$$= -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)}\,dx \qquad ....(2)$$

$$= -\int_0^\infty \frac{(1+x^2)-1}{x\,(1+x^2)}\sin sx\; dx$$

$$= -\int_0^\infty \frac{\sin sx}{x}\,dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$\therefore \quad \frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \qquad ....(3)$$

$$\therefore \quad \frac{d^2I}{ds^2} = 0 + \int_0^\infty \frac{x\cos sx}{x(1+x^2)}\,dx = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I$$

$$\therefore \quad \frac{d^2I}{ds^2} - I = 0 \qquad \text{or} \qquad (D^2-1)\,I = 0 \qquad \text{where } D = \frac{d}{ds}$$

It is solution is

$$I = c_1\, e^{s} + c_2\, e^{-s} \qquad ....(4)$$

$$\frac{dI}{ds} = c_1\, e^{s} - c_2\, e^{-s} \qquad ....(5)$$

When $s = 0$ (1) and (4) gives

$$c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

$$= \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha\theta\, d\alpha \qquad \because F(\alpha)=0 \text{ for } \alpha>0$$

$$= \frac{2}{\pi}\left[(1-\alpha)\frac{\sin\alpha\theta}{\theta}-(-1)\frac{(-\cos\alpha\theta)}{\theta^2}\right]_0^1$$

$$= \frac{2}{\pi}\,\frac{(1-\cos\theta)}{\theta^2} = \frac{2\,.\,2\sin^2\theta/2}{\theta^2} = \frac{4\sin^2\theta/2}{\theta^2}$$

Thus we have

$$\int_0^\infty \frac{4\sin^2\theta/2}{\pi\,\theta^2}\cos\alpha\theta\, d\theta = F(\alpha)$$

$$\int_0^\infty \frac{\sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2}\cos\alpha\theta\, d\theta = \pi\, F(\alpha)$$

Put $\frac{\theta}{2}=t \qquad \therefore \quad d\theta = 2dt \quad t\to 0 \text{ to } \infty$

$$\therefore \quad \int_0^\infty \frac{\sin^2 t}{t^2}\cos(2\alpha t)\, 2dt = \pi F(\alpha)$$

Put $\quad \alpha = 0 \qquad F(\alpha) = 1 - 0 = 1$

$$\therefore \quad \int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$$

Example 11: Find the Fourier cosine transform of e^{-x^2}

Solution:

Fourier cosine transform of e^{-x^2} is given by

$$F_c\left\{e^{-x^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\, dx = I \quad \text{(say)} \qquad ....(1)$$

Differentiating w.r.t. s under integral sign

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} x \sin sx \, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty (\sin sx)(-2xe^{-x^2})\, dx$$

Integrating by parts

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\left(\sin sx\, e^{-x^2}\right)_0^\infty - s\int_0^\infty \cos sx\, e^{-x^2}\, dx\right]$$

$$= 0 - \frac{s}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2} \cos x\, dx = -\frac{s}{2} I$$

$$\therefore \quad \frac{dI}{ds} = -\frac{sI}{2} \quad \text{or} \quad \frac{dI}{I} = -\frac{s}{2} ds$$

Integrating we get

$$\therefore \quad \log I = -\frac{s^2}{4} + \log k$$

$$\therefore \quad I = K e^{-\frac{s^2}{4}} \qquad ....(2)$$

Now, when $s = 0$, from (1)

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$$

From (2) $\frac{1}{\sqrt{2}} = k$

$$\therefore \quad I = F_c\{f(x)\} = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

6. Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 0, & \text{other wise} \end{cases}$$

7. Find the finite Fourier sine transform of the function $f(x) = \cos kx$, where k is a non-integer over $(0, \pi)$.

8. Find the inverse Fourier sine transform of $\dfrac{s}{1+s^2}$

9. Find the Fourier transform of the function

$$f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$