CHAPTER 2

Functions and Their Applications

Chapter Outline

- Introduction
- The Concept of a Function
- Types of Functions
- Roots (Zeros) of a Function
- Some Useful Functions in Business and Economics
- Equilibrium of an Economic System
- Break-Even Analysis

Learning Objectives

After reading this chapter, you should be able to

Define

- types of functions
- useful functions in business and economics
- roots (zeros) of a function
- equillibrium of an economic system

Explain

- concept of a function
- the relationship between roots and coefficients of a quadratic equation
- break-even point and its explanation.

INTRODUCTION

For the use of mathematical models in decision-making the first requirement is to identify relevant factors (also called *variables*) involved in the problem and then defining their interrelationships. Such relationships are expressed in the form of an equation or set of equations/inequalities. These equations or inequalities with or without an objective function help the decision-maker in better understanding of the problem and arriving at an optimal decision. For example, total inventory cost is expressed in terms of total purchase cost, ordering cost, holding cost and shortage cost. The differential calculus method is used to calculate economic order quantity to achieve minimum total inventory cost.

The aim of this chapter is to explain some fundamental concepts about functions, their classification and application in the context of business and economic problems.

have a specific relationship among the selected variables. For example, for the purpose of finding total inventory incremental cost (TIC), the specific relationship between T and Q is stated as :

$$\text{TIC} = \frac{D}{Q^*}C_p + \frac{Q^*}{2}C_h \quad ...(i)$$

where $Q^* = \sqrt{2\,DC_p/C_h}$ is the optimal order size and D, C_p and C_h are total annual demand, procurement cost per order and holding cost per unit per time period respectively. It may be noted here that equation (*i*) indicates *rule of correspondence* between the dependent variable TIC and independent variable (Q). That is, as soon as different values to Q are assigned in the set of real numbers, the corresponding *unique* value of TIC is determined by the given relationship and that relationship is called a *real function*. The various values of Q form a set called the *domain* and the corresponding values of TIC form another set called the *range* of the function. It can also be expressed as $f : Q \rightarrow \text{TIC}$.

Based on the above discussion, we can now define the function as a correspondence among variables of two non-empty sets A and B as follows :

Definition : If A and B are two non-empty sets and there exists a rule of correspondence by which each element x of set A is related to unique element y of set B, then such correspondence is called the function from A to B. It is represented as :

$$f : A \rightarrow B, \text{ such that } y \in B \text{ and } x \in A.$$

where $y = f(x)$

Remarks

1. *Value of function :* The element y in B that is associated to x by f is denoted by $f(x)$ and is called the value of f at x.
2. *Domain of f :* The set A is called the domain of function f.
3. *Co-domain of f :* The set B is called the co-domain of function f.
4. *Range of f :* The set $\{f(x) : x \in A, f(x) \in B\}$ of all values taken by f is called the range of f. It is obviously a subset of B.

Intervals

If a and b are two real numbers such that $a > b$, then a set of real numbers can be enumerated between a and b.

The set of all real numbers between a and b without these end points is called the *open interval* and is written as :

$$(a, b) = \{x \in R : a < x < b\}$$

However, if end points a and b are included in the set, then it is called a *closed interval* and is written as :

$$[a , b] = \{x \in R : a \le x \le b]$$

There are also intervals which are closed at only end point. For example,

$$(a, b] = \{x \in R : a < x \le b\}$$

and

$$[a, b) = \{x \in R : a \le x < b\}$$

Every mathematical relationship may not define the function. For example, the equation $y = \sqrt{x}$ does not define a function, since we find that there exist twovalues ± 2 of y corresponding to the given value $x = 4$.

The dimension of a function is determined by the number of independent variables. For example,

(*a*) $y = f(x)$ is a single-variable (or one-dimensional) function.

(*b*) $y = f(x, z)$ is a two-variable (or two-dimensional) function.

(*c*) $y = f(x, z, r)$ is a three-variable (or three-dimensional) function.

Example 1 : Let $p + 3q/2 = 27$ be an equation involving two variables p (price) and q (quantity). Indicate the meaningful domain and range of this function when (*a*) the price (*b*) the quality are considered independent variables.

Solution : (*a*) When price (p) is taken as independent variable, we have

$$q = 18 - \frac{2}{3}p$$

Domain : $0 \leq p \leq 27$

Range : $0 \leq q \leq 18$

(*b*) When quantity (q) is taken as independent variable, we have

$$p = 27 - \frac{3}{2}q$$

Domain : $0 \leq q \leq 18$

Range : $0 \leq p \leq 27$

Example 2 : Find the domain and range of the following functions :

(*a*) $y = -|x|$ (*b*) $y = \sqrt{3 - 2x}$.

Solution : (*a*) Domain = set of values of x for which y is meaningful

= set of real numbers (R) because for all $x \in$ R, y has a unique value, i.e., $|x| \geq 0$ for all $x \in$ R.

Thus, $-|x| \leq 0$ or $y \leq 0$, for all $x \in$ R

Range : $\{y : y = -|x| \text{ and } x \in R\} = \{y : y \leq 0\}$

(*b*) The value of y will be real for those values of x for which the expression under the radical sign is non-negative (≥ 0). Thus, the domain of y is given by $3 - 2x \geq 0$ or $3 \geq 2x$ or $x \leq 3/2$, i.e., the set of all real numbers in the interval $(\infty, 3/2)]$. For range, we have $y = \sqrt{3 - 2x}$ or $y^2 = 3 - 2x$ which gives $x = (3 - y^2)/2$.

Since $x \leq 3/2$, therefore, $(3 - y^2)/2 \leq (3/2)$ or $3/2 - y^2/2 \leq 3/2$ or $-y^2/2 \leq 0$ or $y^2/2 \geq 0$ which implies either $y \leq 0$ or $y \geq 0$. Also y is the positive square root, therefore rejecting $y \leq 0$, we have $y \geq 0$. Thus, the range of y is the set of all non-negative real numbers.

Example 3 : Find the domain and range of the following:

(*a*) $\frac{|x|}{x}$ (*b*) $[x]$ (*c*) $[x] - x$ (*d*) $|x - 1|$

Solution. (*a*) For $f(x) = \frac{|x|}{x}$, where f is defined for all x except at $x = 0$. Therefore, domain $(f) = R - \{0\}$ when $x > 0$, $f(x) = 1$ and when $x < 0$, $f(x) = -1$. Thus, range $(f) = \{1, -1\}$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$$

The graph of this function is shown in Fig. 2.2.

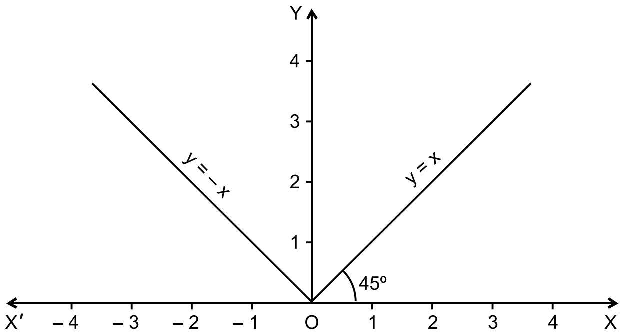


Fig. 2.2 : Absolute Value of a Function

Step-function : If a function is defined on a closed interval $[a, b]$ and assume a constant value in the interior of each sub-interval say $[a, x_1]$, $[x_2, x_3]$,..., $[x_n, b]$ of $[a, b]$ where $a < x_1 < x_2 < ... < x_n < b$, then such function is called a step function. Symbolically, it may be expressed as :

$$y \text{ or } f(x) = k_i$$

for all values of x in the i^{th} sub-interval.

The graph of this function is given in Fig. 2.3, $y_1 < y_2 < y_3$.

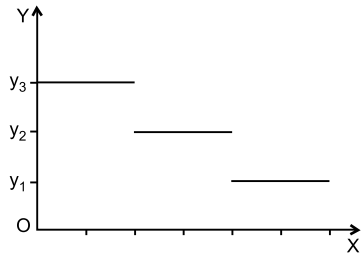


Fig. 2.3 : Step Function

Convex Set and Convex Function : A set S of points in the two-dimensional plane is said to be *convex* if for any two points (x_1, y_1) and (x_2, y_2) in the set the line segment joining these points is also in the set.

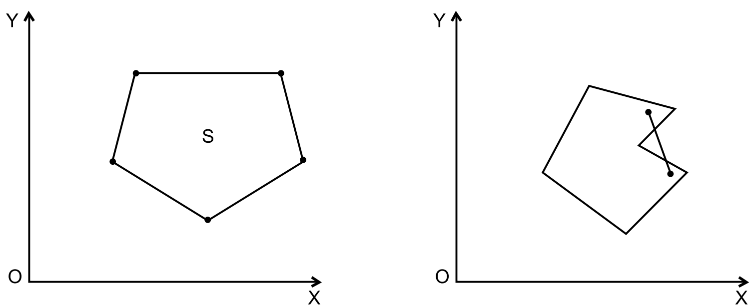
Mathematically, this definition implies (x_1, y_1) and (x_2, y_2) are two different points in S, Then the point whose coordinates are given by

$$\{\lambda x_1 + (1 - \lambda) x_2; \quad \lambda y_1 + (1 - \lambda) y_2\} \; ; \; 0 \leq \lambda \leq 1$$

must also be in the set S.

If $\lambda = 0$, then we get the coordinates (x_2, y_2) of the given point. But, if $\lambda = 1/2$, the corresponding point on the line segment is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. This point lies in the centre of the line segment joining two points (x_1, y_1) and (x_2, y_2).

The typical examples of a convex set are a circle and triangle. Figure 2.4 (*a*) and (*b*) illustrates the example of convex and non-convex sets.



(*a*) Convex Set (*b*) Non-Convex Set

Fig. 2.4 : Convex and non-Convex Set

A function $f(x)$ defined over a convex set S is said to be *convex function* if for any two distinct points x_1 and x_2 lying in S for any $0 \le \lambda \le 1$,

$$f\{\lambda x_1 + (1-\lambda)x_2\} \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

The graph of this function is given in Fig. 2.5.

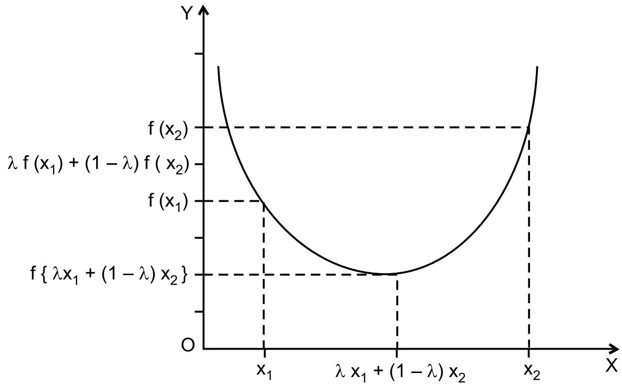


Fig. 2.5 : Convex Functions

Inverse Function : If variables x and y are inter-dependent such that (*i*) y is the function of x; $y = f(x)$

(*ii*) If x is the function of y, $x = g(x)$, then f is known as the inverse of g and vice versa. For example, if $y = x^2 + 2x + 6$, then the inverse function is: $x = -1 \pm \sqrt{y-6}$.

These functions are related as follows:

$$y = f(x) = f[g(y)] \text{ or } fog(y)$$

and

$$x = g(y) = g[f(x)] \text{ or } gof(x)$$

Consequently, $fog = gof$. Functions fog and gof are known as *composite functions*.

Rational Function : A rational function is defined as the quotient of two polynomial functions and is of the form :

$$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0} = \frac{P(x)}{Q(x)}, \; Q(x) \neq 0$$

where P (x) is a polynomial function of degree n and Q (x) is a non-zero polynomial function of degree m. The function y is defined for all values of x provided the denominator does not become zero. For example,

$$y = \frac{3x+5}{2x^2+3x+7} \text{ is a rational function.}$$

An expression which involves root extraction on terms involving x is called an *irrational function*. The functions such as $\sqrt{x}$, $\sqrt{4x^2+5x+8}$ are examples of irrational function.

Algebraic Function : A function consisting of a finite number of terms involving powers and roots of the variable x and the four basic mathematical operations (addition, subtraction, multiplication and division) is called an algebraic function. In general, it can be expressed as

$$y^n + A_1 y^{n-1} + \ldots + A_n = 0$$

where $A_1, A_2, \ldots, A_n$ are rational functions of x.

There are two categories of algebraic functions namely : *explicit* and *implicit* algebraic functions. For example, $y = \sqrt{x} + 3x^3$ is an explicit algebraic function, whereas $xy^2 + xy + x^2 = 0$ is an implicit function.

Transcendental Function : All functions which are not algebraic are called transcendental functions. These functions include

1. *Trigonometric Functions.* The trigonometric functions of an angle θ (θ be any real number) are given by :

$$\sin\theta = \sin\theta^c, \quad \cos\theta = \cos\theta^c, \quad \tan\theta = \tan\theta^c$$
$$\operatorname{cosec}\theta = \operatorname{cosec}\theta^c, \quad \sec\theta = \sec\theta^c, \quad \cot\theta = \cot\theta^c$$

where θ denotes the angle whose radian measure is in θ.

The *sin* and *cosec* are said to be *co-functions,* as are the *cos* and *sec* and *tan* and *cot*.

The trigonometric functions are defined similarly for negative angles. An angle θ may be measured in degrees or radians. However, in calculus and its applications to business and economics radian measure is usually more convenient.

The trigonometric functions are very useful in the study of business cycles, seasonal or other cyclic variations are described by *sine* or *cosine* functions.

2. *Exponential Functions :* A function having a constant base and a variable exponent is called an exponent function, such as

(*i*) $y = a^x$, $a \neq 1$, $a > 0$ (*ii*) $y = k\, a^x$, $a \neq 1$, $a > 0$

(*iii*) $y = k\, a^{bx}$, $a \neq 1$, $a > 0$ (*iv*) $y = k\, e^x$

where a, b, e, and k are constants and x is an exponent.

In calculus and its applications to business and economics, such functions are useful for describing sharp increase and decrease in the value of dependent variable. For example, the graph of exponential function $y = k\,a^x$ indicates rise to the right in the value of y for $a > 1$ and $k > 0$ whereas indicates fall to the left for $a < 1$ and $k > 0$, as shown in Fig. 2.6 (*a*) and (*b*).

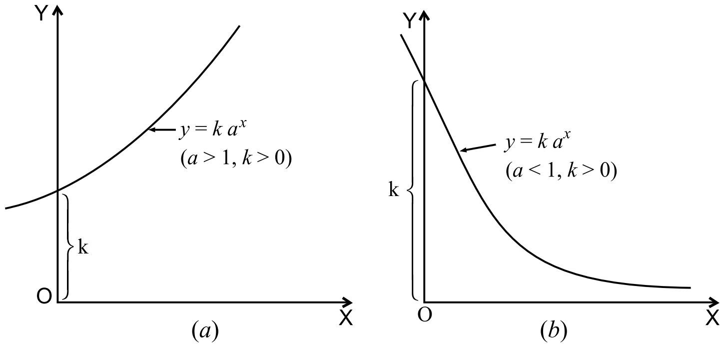


Fig. 2.6 : Exponential Function

The rules governing the exponents are as under:

(*i*) $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$ (*ii*) $a^{x_1} / a^{x_2} = a^{x_1-x_2}$ (*iii*) $(a^{x_1})^{x_2} = a^{x_1 x_2}$

(*iv*) $(a \cdot b)^{x_1} = a^{x_1} \cdot b^{x_1}$ (*v*) $\left(a/b\right)^{x_1} = a^{x_1} \cdot b^{-x_1}$ (*vi*) $a^{o} = 1$.

3. *Logarithmic Functions :* A logarithmic function is expressed as : $y = \log_a x$ where $a > 0$, $a \neq 1$ is the base. It is read as "y is the log to the base a of x". This relationship may also be expressed by the equation $x = a^y$. It is an exponential function. Thus, logarithmic and exponential functions are inverse functions, i.e., if x is an exponential function of y, then y is the logarithmic function of x.

Although the base of logarithm can be any positive number other than 1, but most widely used bases are either 10 (common or Briggsian logarithms) or $e = 2.718$ (natural or Naperian logarithms).

By convention, log x denotes the common logarithm of x and *ln* x denotes natural logarithm of x. If any other base is meant, it is specified.

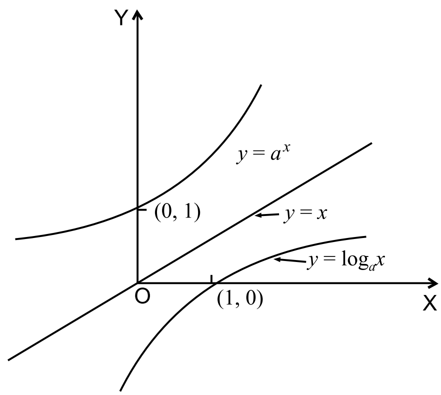


Fig. 2.7 : Logarithmic Function

Some important properties of the logarithms are as follows. If x and y are positive real numbers, then

(*i*) $\log_a (x\,y) = \log_a x + \log_a y$ (*ii*) $\log_a (x/y) = \log_a x - \log_a y$

(*iii*) $\log_a x^n = n \log_a x$ (*iv*) $\log_a (x)^{1/n} = (1/n) \log_a x$

(*v*) $\log_a x = \log_a b \times \log_b x$

(*vi*) logarithm of zero and negative number is not defined.

Since exponential function $x = a^y$ and logarithmic function $y = \log_a x$ are universe functions, therefore graph of these curves for a particular value of '*a*' can be obtained from the graph of each other by taking reflection about the line $y = x$, as shown in Fig. 2.7.

4. *Incommensurable Power Functions :* A function having a variable base and a constant exponent is called and incommensurable power function, such as $y = x^{\sqrt{5}}$ or $x^{3/2}$ or x^a, etc.

Even and Odd Functions : If a function does not change its sign when the sign of its independent variable is changed, then it is said to be an *even* function, i.e., $f(-x) = f(x)$. Examples of even function are x^6, cos *x*, etc. It follows that the graph of an even function is symmetrical about the *y*-axis.

On the other hand, $f(x)$ is said to be *odd* if $f(-x) = -f(x)$. The examples of odd function are: x^7, sin *x*, cot *x*, etc. It follows that the graph of an odd function is symmetrical about the origin.

Periodic Function : If $f(x + T) = f(x)$, where T is a real number, then $f(x)$ is called a periodic function. The real number T is called a *period* of $f(x)$.

The least positive period of a periodic function is called the *principal period* of that function. Since for all real numbers *x*,

$$\sin(2\pi + x) = \sin x \text{ and } \cos(2\pi + x) = \cos x,$$

therefore, the function sin *x* and cos *x* are periodic functions with period 2 π. Other than this 4 π, – 2 π, 6 π, etc., are also periods of sin *x* and cos *x*.

The graph of function sin *x* and cos *x* is shown below in Figs. 2.8 and 2.9.

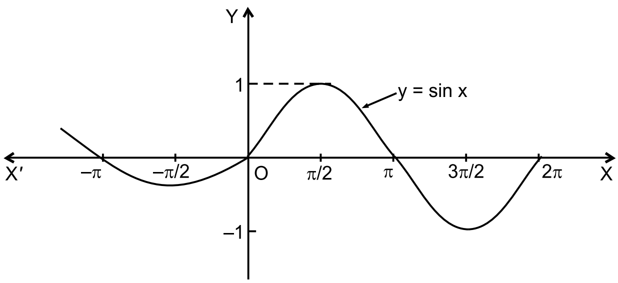


Fig. 2.8 : Sine Function

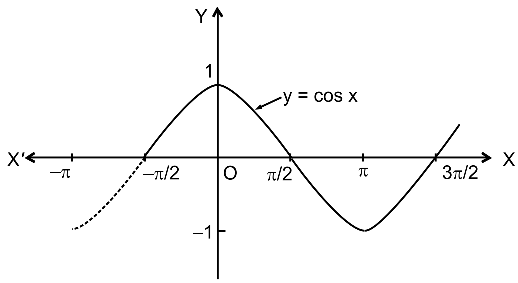


Fig. 2.9 : Cosine Function

SELF PRACTICE PROBLEMS B

1. Sketch the graph of the functions given below :

(*i*) $f(x) = |x|$, at $x = 0$

(*ii*) $f(x) = \begin{cases} x, & \text{if } x \le 0 \\ x^2, & \text{if } 0 < x \le 2 \end{cases}$

(*iii*) $f(x) = \begin{cases} 1 & ; x > 0 \\ 0 & ; x = 0 \\ -1 & ; x < 0 \end{cases}$

(*iv*) $f(x) = \begin{cases} x^2 \text{ in } (-\infty, 0) \\ x \text{ in } (0, 1) \\ \dfrac{1}{x} \text{ in } (1, \infty) \end{cases}$

(*v*) $f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$

(*vi*) $f(x) = |x| + |x - 1|$

(*vii*) $f(x) = \begin{cases} \frac{1}{2} - x\ , & 0 < x < \frac{1}{2} \\ \frac{3}{2} - x\ , & \frac{1}{2} < x < 1 \\ \frac{1}{2} & ,\ x = \frac{1}{2} \end{cases}$ (*viii*) $f(x) = \frac{x}{|x|}$

2. If $f(x) = 1 - x^2 + x^4$, then prove that $f(-x) = f(x)$.

3. Examine whether the function below is even or odd

(*a*) $y = \frac{e^x + e^{-x}}{2}$, (*b*) $y = \frac{e^x - e^{-x}}{2}$

4. If $f(x) = \log x$, then proved that $f(x\,y) = f(x) = f(y)$ and $f(x^n) = n\,f(x)$.

5. Find a quadratic function, $y = ax^2 + bx + c$, that fits the data points (1, 4), (– 1, – 2) and (2, 13). Estimate the value of y when $x = 3$.

6. Graph the function $y = -x^2 + 4x - 2$ with the set of values $-5 \leq x \leq 5$ as the domain.

7. On the same graph paper, draw the graphs of function : $y = x^2$, $y = x^2 + x$ (plot at least 10 points for each graph).

ROOTS (ZEROS) OF A FUNCTION

The value (or values) of x at which the given function $f(x)$ becomes equal to zero are called zeros of the function $f(x)$. The zeros of the function are also called the roots of the given function $f(x)$.

For the linear function $y = ax + b$ the roots are given by

$$ax + b = 0\ ,\ \text{i.e., } x = -b/a$$

Thus, if $x = -b/a$, then substituting in the given equation, $ax + b = 0$, the left hand side of its becomes equal to its right hand side.

For the quadratic function

$$y = ax^2 + bx + c\ ,\ a \neq 0$$

the quadratic equation $ax^2 + bx + c = 0$; $a \neq 0$ has to be solved to find the roots of the function y. The general value of x which satisfy the given quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This shows that, in general, there are two values of x (also called roots) for which $ax^2 + bx + c$ becomes zero. These two values are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Remark : The number of roots of the given function is always equal to the highest power of the independent variable.

Special Cases

The expression $b^2 - 4ac$ in the value of x is known as *discriminant* which determines the nature of the roots of the quadratic equation as discussed below:

(*i*) If $b^2 - 4ac > 0$, then the two roots are *real and distinct.*

(*ii*) If $b^2 - 4ac = 0$, then the two roots are equal and are equal to $-b/2a$.

(*iii*) If $b^2 - 4ac < 0$, then the two roots are imaginary (not-real) because of the square root of negative number.

The roots of a polynomial : $y = (x - a)(x - b)(x - c) \ldots$ are $a, b, c \ldots..$

Remark : From this discussion, it is clear that we need to find out the actual roots to determine their nature. The value of $(b^2 - 4ac)$ is sufficient to determine the nature of the roots.

Relationship Between the Roots and the Coefficients of a Quadratic Equation

Consider the equation : $ax^2 + bx + c = 0$; $(a \neq 0)$. Suppose

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are the roots of this equation. Then the *sum* and *product* of these roots is given by

$$\text{Sum of roots} = \alpha + \beta = \frac{-b}{a} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of roots} = \alpha . \beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

The above general equation can be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \text{ or } x^2 - (\text{Sum of roots})\,x + (\text{Product of roots}) = 0.$$

This is the formula to find an equation whose roots are given. Alternatively, an equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$.

SELF PRACTICE PROBLEMS C

1. Consider the general quadratic expressions $ax^2 + bx + c$, $a \neq 0$ and $c > 0$. Show that

(*i*) $ax^2 + bx + c \geq 0$ for all x if and only if its discriminant $b^2 - 4ac \leq 0$.

(*ii*) of the two roots of the equation : $ax^2 + bx + c = 0$, it is not possible that one is imaginary and the other is real.

(*iii*) $ax^2 + bx + c$ can be written as $(kx + d)^2$ for some k and d if $b^2 = 4ac$.

2. Consider the quadratic equation $2x^2 - 8x + c = 0$. For what value of c the equation has (*i*) real roots (*ii*) equal roots, and (*iii*) imaginary roots ?

3. Determine the quadratic equation whose roots are :

(*i*) $-1, -2$ (*ii*) $a/b, b/a$ (*iii*) $-3, -2/3$ (*iv*) α^2, β^2

4. Consider the general quadratic equation $ax^2 + bx + c = 0$. If α and β are the roots of the equation, then show that, $\alpha + \beta = -b/a$ and $\alpha \cdot \beta = c/a$.

5. If a and b are the roots of the equation $x^2 - (1 + n^2)x + \frac{(1 + n^2 + n^4)}{2} = 0$, then prove that $a^2 + b^2 = n^2$.

SOME USEFUL FUNCTIONS IN BUSINESS AND ECONOMICS

Linear Functions : Many economic laws can be represented by the linear function. We will take up a few examples to illustrate the procedure of constructing such functions and finding their solutions (roots).

1. *Demand Function :* In general, the demand function is expressed as :

$$Q_d = a - bp$$

where Q_d is the quantity demanded (or purchased if offered) and p, is the price, a and b are constants.

2. *Supply Function :* In general, the supply function is expressed as :

$$Q_s = c - dp$$

where Q_s is the quantity offered for sale, and p is the price c and d are constants.

3. *Total Cost Function :* In general, the total cost function explicitly can be expressed as:

$$C = C(x)$$

where x is the quantity produced and C is the total cost incurred. However, if total cost of producing x number of units of a particular commodity is analysed in terms of fixed cost F, which is independent of x (with certain limits) and variable cost $V(x)$, which varies with x, then we can write

$$C(x) = F + V(x)$$

The average cost of production or cost per unit is obtained by dividing total cost by the quantity produced. That is $AC(x) = C(x)/x$.

4. *Total Revenue Function :* If $Q(x)$ is the demand for the output of a firm costing Rs. p per unit, then total revenue (R) collected is given by

$$R = p \cdot Q(x)$$

5. *Consumption Function :* In general, the consumption function is expressed as :

$$C = a + cY$$

where C is the total consumption and Y is the national income, a and c are constants.

6. *Investment Function :* The simple investment function is expressed as :

$$I = a + br \; ; \quad a > 0 \quad \text{and} \quad b < 0$$

where I represents investment and r the interest rate.

7. *Production Function :* In general, the production of an item depends upon two input variables, namely, capital (K) and labour (L). Thus, symbolically it is expressed as :

$$P = f(L, K)$$

In economics, the popular Cobb-Douglass production function is defined as:

$$P = a L^{\alpha} K^{\beta} \; ; \qquad \alpha + \beta = 1.$$

Example 6 : A newspaper boy buys papers for Rs. p_1 per paper and sells them at price of Rs. p_2 per paper ($p_2 > p_1$). The unsold papers at the end of the day are sold at Rs. p_3 to a wastepaper dealer ($p_3 < p_1$). Construct the profit function per day for the newspaper boy ?

Solution : The overall profit to the newspaper boy depends on the number of newspapers that he sells in relation to the number he bought at the beginning of the day. Let D be the demand of newspapers and Q be the stock of newspapers.

Case I : If D < Q, then

$$\begin{aligned} \text{Profit} &= p_2 \text{ (number of newspapers sold)} + p_3 \text{ (unsold papers)} - p_1 \text{ (stock)} \\ &= p_2 D + p_3 (Q - D) - p_1 Q = D(p_2 - p_3) + Q(p_3 - p_1) \end{aligned}$$

Case II: If D ≥ Q, then profit = p_2 Q – p_1 Q.

Example 7 : Assume that for a closed economy E = C + I + G, where E is total expenditure, C is expenditure on consumption of goods, I is expenditure on investment on goods and G is Government spending. For equilibrium, we must have E = Y, where Y is the total income received.

For a certain economy, it is given that C = 15 + 0.90 Y, I = 20 + 0.05 Y, and G = 25. Find the equilibrium values of Y, C and I. How will these change if there is no Government spending ?

Solution : Given that E = C + I + G and E = Y. Thus, we have

(*a*) Y = C + I + G = (15 + 0.90 Y) + (20 + 0.05 Y) + 25 = 60 + 0.95 Y

or Y (1 – 0.95) = 60 or Y = 60/0.05 = 1200

For this value of Y, we have

$$C = 15 + 0.90\,Y = 15 + 0.90 \times 1200 = 1095$$

and $$I = 20 + 0.05\,Y = 20 + 0.05 \times 1200 = 80$$

(*b*) If there is no government spending, i.e., G = 0, then closed economy equation becomes

$$Y = C + I = (15 + 0.90\,Y) + (20 + 0.05\,Y) = 35 + 0.95\,Y$$

or $$Y (1 - 0.95) = 35, \text{ i.e., } Y = 35/0.05 = 700$$

For this value of Y, we have

$$C = 15 + 0.90\,Y = 15 + 0.90 \times 700 = 645$$

and $$I = 20 + 0.05\,Y = 20 + 0.05 \times 700 = 55$$

Example 8 : The ABC company is producing an item at unit cost of Rs. 4.

(*a*) If the supply function is $q = 160 + 8p$, where q represents the quantity produced and p represents the market price, then find the total cost function in terms of p.

(*b*) What is the market price at which the company realizes a total profit of Rs. 500.

Solution : (*a*) Total cost = (Cost/unit) (Number of units produced) = $4 \cdot q$

(*b*) Total sales revenue = (Price/unit) (Number of units supplied) = $p \cdot q$

$$\text{Total profit (P)} = \text{Revenue} - \text{Cost} = p \cdot q - 4 \cdot q = (p - 4)\,q = (p - 4)(160 + 8p)$$
$$= 8p^2 + 128\,p - 640$$

Given that P = Rs. 500. Then we have

$$500 = 8p^2 + 128\,p - 640 \quad \text{or} \quad 8p^2 + 128\,p - 1140 = 0$$

$$p = \frac{-128 \pm \sqrt{(128)^2 - 4 \times 8 \times (-1140)}}{2 \times 8} = \frac{-128 \pm 229.92}{16} = 6.37 \text{ or } -22.37$$

The company must charge Rs. 6.37 per unit for its item. The negatvie price $p = -22.37$ is meaningless.

Example 9 : A company is considering a merger proposal. Without the merger, the company will have initial earnings of Rs. 6 per share, and these earings are expected to grow at a rate of 5% per year. If the merger materialises, the earning of the compnay will immediately drop to Rs. 5 per share, but the ratio of growth is expected to increases to 8%. How long will it take to make up the earings if the merger takes place ?

Solution : If y_n be the earings per share at time n without the merger, then $y_n = 6\,(1 + 0.05)^n$. Let x_n be the earning per share with merger at time n, then $x_n = 5\,(1 + 0.08)^n$.

By definition, break-even analysis determines the optimum value of q for which profit P equals zero, i.e.:

$$\text{Total revenue} = \text{Total cost}$$

or $$p \cdot q - (k + v.q) = 0$$

or $$q^* \text{ (optimum)} = \frac{k}{p-(v.q)} = \frac{\text{Fixed cost}}{\text{Selling price} - \text{Variable cost}}$$

Example 11 : A firm produces an item whose production cost function is C = 80 + 4x, where x is the number of items produced. If entire stock is sold at the rate of Rs. 8 then determine the revenue function. Also obtain the 'break-even' point.

Solution : The revenue function is given by R = 8x. Also given that, C = 80 + 4x. Therefore,

$$\text{Profit, P} = R - C = 8x - (80 + 4x) = 4x - 80$$

The break-even point occurs when R – C = 0 or R = C, i.e., $8x = 80 + 4x$ or $x = 20$ (units).

Example 12 : A company producing dry cells introduces production bonus for its employees which increases the cost of production. The daily cost of production C (x) for x number of cells is Rs. (3.5x + 12,000).

(*a*) If each cell is sold for Rs. 6, determine the number of cells that should be produced to ensure no loss.

(*b*) If the selling price is increased by 50 praise, what would be the break-even point ?

(*c*) If at least 6000 cells can be sold daily, what price the company should charge per cell to guarantee no loss ?

Solution : Let R (x) be the revenue due to the sales of x number of cells.

(*a*) Given that, cost of each cell is Rs. 6. Then R (x) = 6x. For no loss, we must have

$$R(x) = C(x) \quad \text{or} \quad 6x = 3.5\,x + 12{,}000 \quad \text{or} \quad x = 12{,}000/2.5 = 4{,}800 \text{ cells.}$$

(*b*) Increased selling price is, Rs. (6 + 0.50) = Rs. 6.5. Thus, R (x) = 6.5 x. Now for break-even point, we must have

$$R(x) = C(x) \quad \text{or} \quad 6.5\,x = 3.5\,x + 12{,}000 \quad \text{or} \quad x = 12{,}000/3 = 4000 \text{ cells.}$$

(*c*) Let p be the unit celling price. Then revenue from the sale of 6000 cells will be, R (p) = 6000 p. Thus, for no loss, we must have

$$R(p) = C(p) \quad \text{or} \quad 6000\,p = 3.5 \times 6000 + 12{,}000 \quad \text{or} \quad p = 33{,}000/6000 = \text{Rs. } 5.5.$$

Example 13 : A hotel charges Rs. 80 a day for each room. However, special concession is available for each room if more than 6 rooms are rented by a group, the rent of a room is decreased by Rs. 3 to a minimum of Rs. 50. Each occupied room requires daily cleaning and other charges of Rs. 10. The hotel also incurs a maintenance charge of Rs. 3 per room if it is not rented. The hotel has 50 rooms.

(*a*) Compute the rent per room that a group has to pay if 15 rooms are rented.

(*b*) Find the profit as a function of the number of rooms rented to a group in case only one group is staying in the hotel at a time.

Solution : (*a*) Since the rent is reduced at the rate of Rs. 3 per room, the rent is reduced by Rs. (13 – 6) × 3 = Rs. 21. Thus, the group has to pay at the rate of Rs. 80 – 21 = Rs. 59 per room.

(*b*) Let P(x), R(x) and C(x) denote respectively the profit, revenue and cost functions. First, we compute R(x).

$$R(x) = 80x, \quad \text{if } 0 \le x \le 6.$$

Since the rent is reduced to a minimum of Rs. 50, we get

$$50 = 80 - 3(x - 6) \Rightarrow x = 16$$

$$\therefore \quad R(x) = [80 - 3(x - 6)]\,x, \text{ if } 6 \le x \le 16$$
$$= x\,(98 - 3x), \quad \text{if } 6 \le x \le 16$$

Also $\quad R(x) = 50x, \quad \text{if } x \geq 16$. Thus,

$$R(x) = \begin{cases} 80x, & 0 \leq x \leq 6 \\ x(98 - 3x), & 6 \leq x \leq 16 \\ 50x, & x \geq 16 \end{cases}$$

Now. let us compute the cost function C(*x*) :

$$C(x) = 10x + 3(50 - x) = 7x + 150$$

[Each rented room requires a maintenance charge of Rs. 10 and the rooms which are not rented required Rs. 3 for maintenance]

Since $\quad P(x) = R(x) - C(x)$, we get

$$= \begin{cases} 73x - 150, & 0 \leq x \leq 6 \\ x(91 - 3x) \times 150, & 6 \leq x \leq 16 \\ 43x - 150, & x \geq 16 \end{cases}$$

Example 14 : A company is considering a merger proposal. Without the merger, the company will have initial earnings of Rs. 6 per share, and these earnings are expected to grow at a rate of 5 % per year. If the merger materialises, the earning of the company will immediately drop to Rs. 5 per share, but the ratio of growth is expected to increase to 8%. How long will it take to make up the earnings if the merger takes place ?

Solution : If y_n is the earning per share at time *n* without the merger, then $y_n = 6(1 + 0.05)^n$. Let x_n be the earning per share with the merger. Then

$$x_n = 5(1 + 0.08)^n$$

We need the value of *n* for which $x_n = y_n \cdot z$, i.e.,

$$5(1.05)^n = 5(1.3)^n \Rightarrow (1.2)\,(1.05)^n = (1.08)^n$$
$$\Rightarrow \log 1.2 + n \log 1.05 = n \log 1.08$$
$$\Rightarrow n(\log 1.08 - \log 1.05) = \log 1.08$$
$$\Rightarrow n(0.0122) = 0.0792 \text{ or } n = 6.49 \text{ years.}$$

Example 15 : A company decides to set up a small production plant for manufacturing electronic clocks. The total cost for initial set-up (fixed cost) is Rs. 9 lakh. The additional cost (i.e., variable cost) for producing each clock is Rs. 300. Each clock is sold at Rs. 750. During the first month, 1500 clocks are produced and sold :

(*a*) Determine the cost funstion C(*x*) for the total cost of produding *x* clocks.

(*b*) Determine the revenue function R(*x*) for the total revenue from the sale of *x* clocks.

(*c*) Determine the profit function P(*x*) for the profit from the sale of *x* clocks.

(*d*) What profit or loss the company incurs during the first month when all the 1500 clocks are sold ?

(*e*) Determine the break-even point.

Solution : (*a*) We know that the cost function includes fixed cost and variable cost. From the given data, fixed cost = Rs. 9,00,000 and variable cost = Rs. 300 per clock. Let C(*x*) denote the cost function to manufacture *x* clocks, then

$$C(x) = \text{Fixed cost} + \text{Variable cost to produce } x \text{ clocks}$$
$$= 9{,}00{,}000 + 300x$$