2 Z-Transforms

2.1 INTRODUCTION TO Z-TRANSFORM

The Z-transform is a convenient and valuable tool for representing, analyzing and designing discrete-time signals and systems. It plays a similar role in discrete-time systems to that which a Laplace transform plays in continuous time systems. In this unit the main objective is to present important concepts of the Z-transform and their application in finding the stability of the discrete time systems.

Z-transform is a powerful tool for determining the transfer function of a system. This is useful to study about stability of a system with respect to pole-zero pattern in z-plane.

The frequency response of a system can be obtained by replacing z by $e^{j\omega}$ within the transfer function of the system.

2.1.1 Definition

The Z-transform of a sequence $x[n]$ is simply defined as

$$Z\left[x[n]\right] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \tag{2.1}$$

and the inverse Z-transform is defined as

$$Z^{-1}\left[X(z)\right] = x[n] = \frac{1}{2\pi j}\int_C X(z)\, z^{n-1}\, dz \tag{2.2}$$

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k} \Leftrightarrow \sum_{k=-\infty}^{\infty} \left|x[k]\right| < \infty \tag{2.3}$$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k} \quad \left(z = re^{j\omega}\right)$$

$$X\left(re^{j\omega}\right) = \sum_{k=-\infty}^{\infty} x[k]\left(re^{j\omega}\right)^{-k} \Leftrightarrow \sum_{k=-\infty}^{\infty} \left|x[k]\right| r^{-k} < \infty \quad \text{(ROC)}$$

2.2 RELATION BETWEEN Z-TRANSFORM AND FOURIER TRANSFORM

As we know

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \tag{2.6}$$

It is obvious that the region of convergence for the Z-transform of $\delta[n]$ is the entire z-plane.

Table 2.1 List of Z-Transform pairs

S. No.	*x[n] for n ≥ 0*	*X(z)*	*ROC*
1.	$\partial[n]$	1	Entire z-plane
2.	$\partial[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
3.	$u[n]$	$\frac{z}{z-1}$	$\|z\| > 1$
4.	$a^n u[n]$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
5.	$-a^n u[-n-1]$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
6.	$e^{-naT} x[n]$	$X(e^{aT} z)$	ROC of $X(z)$
7.	$\sum_{m=0}^{n} x[m]h[n-m]$	$X(z)\ H(z)$	Intersection of ROC of $X(z)$ and ROC of $H(z)$

Table 2.2 Properties of Z-transform

S. No.	*Properties*	*Sequence*	*Z-transform*	*ROC*
1.	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_1 \cap R_2$
2.	Time-shifting	$x[n-n_0]$	$e^{-m} X(z)$	R_x expect for the possible addition or deletion of the origin or infinity
3.	Multiplication by exponential sequence	$a^n x[n]$	$X(a^{-1} z)$	Scaled version of R (i.e., $\|a\|R$ = then set of points $\{\|a\|z\}$ for z in R)
4.	Differentiation	$nx[n]$	$-z\frac{d}{dz}X(z)$	R
5.	Conjugate	$x^*[n]$	$X^*(z)$	R
6.	Time reversal	$x[-n]$	$X(z^{-1})$	$\frac{1}{R}$
7.	Convolution	$x_1[n] \otimes x_2[n]$	$X_1(z)\ X_2(z)$	$R_1 \cap R_2$
8.	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\ X(z)$	At least the intersection of R and $\|z > 0\|$

2.4 ROC AND ITS PROPERTIES

The Z-transform does not converge for all values of z. For any given sequence, the set of values of z for which the Z-transform converges is called region of convergence (ROC), which is governed by condition:

$$\sum_{n=-\infty}^{\infty} x[n]\, r^{-n} < \infty$$

Definition:

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k} \qquad \sum_{k=-\infty}^{\infty} |x[k]|\, r^{-k} < \infty$$

Right-handed: $x[n] = 0\ \forall\ n < N_1$:

$$X(z) = \sum_{k=N_1}^{\infty} x[k]\, z^{-k}$$

If r_0 in ROC, then:

$$\sum_{k=N_1}^{\infty} |x[k]|\, r_0^{-k} < \infty$$

If $r_1 > r_0$, then:

$$\sum_{k=N_1}^{\infty} |x[k]|\, r_1^{-k} < \sum_{k=N_1}^{\infty} |x[k]|\, r_0^{-k} < \infty$$

then r_1 also in ROC, then ROC is *exterior* of circle

Definition:

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k} \Leftrightarrow \sum_{k=-\infty}^{\infty} |x[k]|\, r^{-k} < \infty$$

Left-handed: $x[n] = 0\ \forall\ n > N_2$:

$$X(z) = \sum_{k=-\infty}^{N_2} x[k]\, z^{-k}$$

If r_0 in ROC, then:

$$\sum_{k=-\infty}^{N_2} |x[k]|\, r_0^{-k} < \infty$$

If $r_1 < r_0$, then:

$$\sum_{k=-\infty}^{N_2} |x[k]|\, r_1^{-k} < \sum_{k=-\infty}^{N_2} |x[k]|\, r_0^{-k} < \infty$$

then r_1 also in ROC, then ROC is *interior* of circle

Properties of ROC for Z-transform:

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
2. The ROC does not contain any poles

3. If $x[n]$ is of finite duration, then ROC is the entire z-plane, except possibly $z = 0$ and $z = \infty$

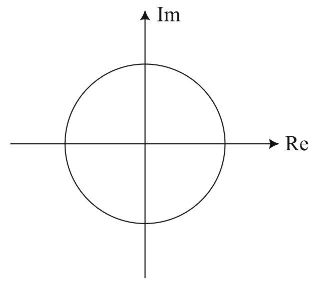


Fig. 2.1 Unit circle in z-plane.

Example 2.1 Obtain the Z-transform of $x(n - m)$

Solution: By definition of Z-transform

$$Z\left[x(n-m)\right] = \sum_{n=m}^{\infty} x(n-m)\, z^{-n}$$

let $n - m = k$

substitution of the above quantity yields

$$\begin{aligned} Z\left[x(n-m)\right] &= \sum_{k=0}^{\infty} x(k)\, z^{-k} z^{-m} \\ &= z^{-m} \sum_{k=0}^{\infty} x(k)\, z^{-k} \\ &= z^{-m} X(z) \end{aligned} \tag{2.7}$$

2.5 TRANSFER FUNCTION

Let the DTLTI system be characterized by the following difference equation

$$y[n] = \sum_{k=0}^{p} a_k x(n-k) - \sum_{k=1}^{q} b_k\, y(n-k) \tag{2.8}$$

Transfer function of the system

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{p} a_k\, z^{-k}}{1 + \sum_{k=1}^{q} b_k\, z^{-k}} \tag{2.9}$$

In any transfer function of Discrete Time Linear Time Invariant (DTLTI) system, if z is replaced by $e^{j\omega}$ the required frequency response of the system can be obtained.

$$H\left(e^{j\omega}\right) = \frac{\sum_{k=0}^{p} a_k \, z^{-jk\omega}}{1 + \sum_{k=1}^{q} b_k \, z^{-jk\omega}} \tag{2.10}$$

ROC: “ring” without poles inside.

Example 2.2 Obtain the transfer function of the following difference equation and obtain its frequency response.

$$y(n) = 0.5y(n-1) + x(n) + x(n-1)$$

Solution: Taking Z-transform on both the sides

$$Y(z) = 0.5z^{-1}\, Y(z) + X(z) + z^{-1} X(z)$$

The transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}}$$

The frequency response can be obtained by replacing z by $e^{j\omega T}$

$$H\left(e^{j\omega T}\right) = \frac{1+e^{-j\omega T}}{1-0.5\, e^{-j\omega T}}$$

$$= \frac{1+\cos\omega T - j\sin\omega T}{1-0.5\cos\omega T + 0.5\, j\sin\omega T}$$

Amplitude response can be represented as

$$\left|H\left(e^{j\omega T}\right)\right| = \sqrt{\frac{(1+\cos\omega T)^2 + \sin^2\omega T}{(1-0.5\cos\omega T)^2 + (0.5\sin\omega T)^2}}$$

The phase response can be represented as

$$H\left(e^{j\omega T}\right) = \tan^{-1}\frac{\sin\omega T}{1+\cos\omega T} - \tan^{-1}\frac{0.5\sin\omega T}{1-0.5\cos\omega T}$$

where T is sampling interval

$$T = \frac{1}{fs}$$

fs is sampling frequency

Example 2.3 Obtain the Z-transform of the following function and find its ROC.

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1$$

Solution:

$$x_1[n] = a^n u[n]$$

$$\Rightarrow \quad X_1(z) = \sum_{k=-\infty}^{\infty} x_1[k]\, z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty}\left(az^{-1}\right) = \Big/\left(\quad az \quad \right) \text{for} \quad \left|az^{-1}\right| < \;\Leftrightarrow |z| > |a|$$

$$X_1(z) = z/(z-a) \Rightarrow \text{ one zero at } z = 0 \text{ and one pole at } z = a$$

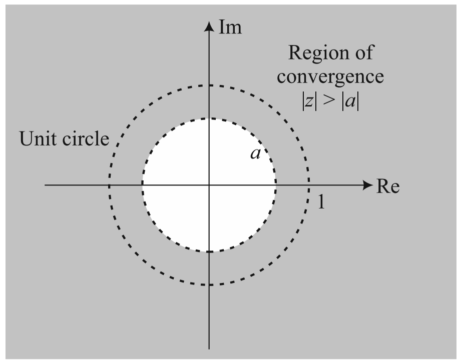


Fig. 2.2 Unit circle in z-plane showing Region of Convergence (ROC).

Example 2.4 Obtain the Z-transform of the following function and find its ROC

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad |a| < 1$$

Solution: $x_2[n] = -a^n u[-n-l] \Rightarrow X_2(z) = \sum_{k=-\infty}^{\infty} x_2[k] z^{-k} = \sum_{k=-\infty}^{-1} -a^k z^{-k} = \sum_{-l-1=-\infty}^{-1} -(az^{-1})^{-l-1}$

$$l = -k-1 \Rightarrow k = -l-1$$

$$X_2(z) = -\sum_{l=\infty}^{0} (a^{-1}z)^{l+1} = -(a^{-1}z) \sum_{k=-\infty}^{\infty} (a^{-1}z)^l = -(a^{-1}z)/1-a^{-1}z \text{ for } |a^{-1}z| < 1$$

$$X_2(z) = 1/(1-az^{-1}) \text{ for } |z| < |a|$$

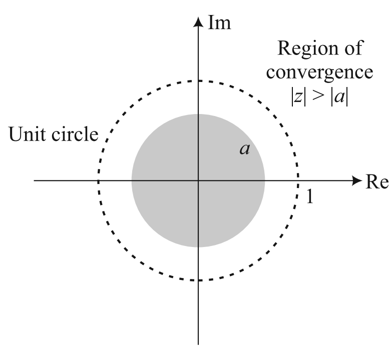


Fig. 2.3 Unit circle in z-plane showing Region of Convergence (ROC).

Example 2.5 Obtain the Z-transform of the following function

$$x[n] = 0.5\, nu[n] + (-0.3)\, nu[n]$$

$$ax[n]+by[n]\leftrightarrow aX(z)+bY(z),\ ROC\quad R_x\cap R_y$$

$$a^n u[n]\leftrightarrow {}^{1}\!/\!_{\left(1-az^{-1}\right)},\ ROC\quad |z|>|a|$$

Solution: $$X(z)=\frac{1}{1-0.5\,z^{-1}}+\frac{1}{1+0.3\,z^{-1}},\ ROC:\quad |z|>|-0.3|\cap|z|>|0.5|$$

$$X(z)=\frac{\left(1+0.3\,z^{-1}\right)+\left(1-0.5\,z^{-1}\right)}{\left(1-0.5\,z^{-1}\right)\left(1+0.3\,z^{-1}\right)},ROC:\quad |z|>0.5$$

$$X(z)=\frac{2-0.2\,z^{-1}}{\left(1-0.5\,z^{-1}\right)\left(1+0.3\,z^{-1}\right)},ROC:\quad |z|>0.5$$

$$X(z)=\frac{2\,z\,(z-0.1)}{(z-0.5)\,(z+0.3)},\ ROC:\quad |z|>0.5$$

Example 2.6 Obtain the Z-transform of the following function

$$x[n]=-\,0.5^n\,u\,[-n-1]+(-\,0.3)^n\,u\,[n]$$

$$ax[n]+by[n]\leftrightarrow aX(z)+bY(z),ROC\quad R_x\cap R_y$$

$$a^n u[n]\leftrightarrow {}^{1}\!/\!_{\left(1-az^{-1}\right)},ROC\quad |z|>|a|$$

$$-a^n u[-n-1]\leftrightarrow {}^{1}\!/\!_{\left(1-az^{-1}\right)},ROC\quad |z|<|a|$$

Solution: $$X(z)=\frac{1}{1-0.5\,z^{-1}}+\frac{1}{1+0.3\,z^{-1}},ROC:\quad |z|>|-0.3|\cap|z|<|0.5|$$

$$X(z)=\frac{\left(1+0.3\,z^{-1}\right)+\left(1-0.5\,z^{-1}\right)}{\left(1-0.5\,z^{-1}\right)\left(1+0.3\,z^{-1}\right)},ROC:\quad 0.3<|z|<0.5$$

$$X(z)=\frac{2-0.2\,z^{-1}}{\left(1-0.5\,z^{-1}\right)\left(1+0.3\,z^{-1}\right)},ROC:\quad 0.3\,|z|<0.5$$

$$X(z)=\frac{2\,z(z-0.1)}{(z-0.5)\,(z+0.3)},ROC:\quad 0.3<|z|<0.5$$

It has 2 poles and 2 zeros

2.6 INVERSE Z-TRANSFORM

2.6.1 Power Series

$$X(z) = \log\left(1 + az^{-1}\right), \quad ROC: |z| > |a|$$

$$\log\left(1 + az^{-1}\right) = \sum_{n-1}^{\infty} \frac{1}{n} \left(-1\right)^{n+1} a^n z^{-n}$$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

$$x[n] = \frac{1}{n} (-1)^{n+1} a^n u[n-1]$$

2.6.2 Inversion Integral

A powerful analytical method determining the inverse Z-transform is the inversion integral method. The function $Y(z)$ can be considered in the complex z-plane. A given coefficient in such a series may be determined by an integral relationship. It can be shown that application of this concept to $y(z)$ yields for the inverse transform.

$$y(n) = \frac{1}{2\pi j} \int_c y(z) z^{n-1} dz \quad (2.11)$$

where c is a contour chosen to include all singularities of the integrand. By Cauchy's residue theorem the integral can be reduced to

$$y(n) = \sum_m \text{Re}s\left[y(z) z^{-1}\right]_{z=p_m} \quad (2.12)$$

where p_m represents a pole of $y(z)z^{n-1}$ and Res [] represents the residue at $z = p_m$

Example 2.7 Find the inverse Z-transform of

$$y(z) = \frac{1}{\left(1 - z^{-1}\right)\left(1 - 0.5 z^{-1}\right)}$$

$$y(z) = \frac{z^2}{(z-1)(z-0.5)}$$

Solution: This can be expressed as

$$y(n) = \sum_m \text{Re}s\left[\frac{z^{n+1}}{(z-1)(z-5)}\right]_{z=p_m}$$

For the poles at $z = 1$ and $z = 0.5$, the residues are calculated as follows

$$\text{Re}s\left[\frac{z^{n+1}}{(z-1)(z-0.5)}\right]_{z=1} = \left[\frac{z^{n+1}}{z-0.5}\right] = 2$$

$$\text{Re}s\left[\frac{z^{z^{n+1}}}{(z-1)(z-0.5)}\right]_{z=0.5} = \left[\frac{z^{n+1}}{z-1}\right]_{z=0.5} = -(0.5)^n$$

$$y(n) = 2 - (0.5)^n$$

Example 2.8 Determine the inverse transform of

$$Y(z) = \frac{1 + 2z^{-1} + z^{-3}}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Solution: Note that the maximum negative power of z in the numerator is larger than for the denominator. Multiplication of the numerator and the denominator by z^3 results in

$$Y(z) = \frac{z^3 + 2z^2 + 1}{z(z-1)(z-0.5)}$$

$$y(n) = \sum_m \operatorname{Re}s\left[\frac{z^{n+1}}{(z-1)(z-5)}\right]_{z = p_m}$$

According to eq. (2.12), we may determine the inverse transform from

$$y(n) = \sum_m \operatorname{Re}s\left[\frac{(z^3 + 2z^2 + 1)z^{n-2}}{z(z-1)(z-0.5)}\right]$$

We must examine z^{n-2} to see if there are any values of n for which there is a pole at the origin. Indeed, for $n = 0$ there is a second-order pole at $z = 0$, and for $n = 1$ there is a simple pole at $z = 0$. However, for $n \geq 2$, the only poles are $z = 1$ and $z = 0.5$. Let us first determine the inverse transform pertinent to this latter range. We have

$$y(n) = \operatorname{Re}s[\;]_{z=1} + \operatorname{Re}s[\;]_{z=0.5}$$

$$= 8 - 13(0.5)^n \quad \text{for} \quad n \geq 2 \tag{2.13}$$

The values of $y(0)$ and $y(1)$ can be determined from the expressions

$$y(0) = \sum_m \operatorname{Re}s\left[\frac{(z^3 + 2z^2 + 1)z^{n-2}}{z(z-1)(z-0.5)}\right]_{z = p_m} \tag{2.14}$$

$$= \operatorname{Re}s[\;]_{z=0} + \operatorname{Re}s[\;]_{z=1} + \operatorname{Re}s[\;]_{z=0.5}$$

$$y(1) = \sum_m \operatorname{Re}s\left[\frac{(z^3 + 2z^2 + 1)z^{n-2}}{z(z-1)(z-0.5)}\right]$$

$$= \operatorname{Re}s[\;]_{z=0} + \operatorname{Re}s[\;]_{z=1} + \operatorname{Re}s[\;]_{z=0.5} \tag{2.15}$$

The reader is invited to demonstrate that the sum of the last two residues in each of equations (2.14) & (2.15) is the same as would be obtained by taking eq. (2.13) and evaluating it for $n = 0$ and $n = 1$ respectively. Thus, instead of performing a complete evaluation of all the residues for $n = 0$ and $n = 1$, it is necessary only to determine the additional residues at $z = 0$ in each case. For eq. (2.13), we have

$$\mathrm{Re}\,s\left[\frac{\left(z^3+2z^2+1\right)}{z^2\left(z-1\right)\left(z-0.5\right)}\right]_{z=0}=6$$

For eq. (2.15) we have

$$\mathrm{Re}\,s\left[\frac{\left(z^3+2z^2+1\right)}{z^2\left(z-1\right)\left(z-0.5\right)}\right]_{z=0}=2$$

This gives $y(0) = 6 + 8 - 13 = 1$

$y(1) = 2 + 8 - 13(0.5) = 3.5$

For $n \geq 2$, the expression of eq. (2.13) is applicable. An alternate way to write $y(n)$ for $n \geq 0$ in one expression is the equation

$$y(n)=6\,\delta(n)+2\,\delta(n-1)+8-13\,(0.5)^n$$

A few values are tabulated in the following Table 2.3.

Table 2.3

n	0	1	2	3	4	5	6	∞
$y(n)$	1	3.5	4.75	6.375	7.1875	7.59375	7.796875	8

Example 2.9 Find the inverse Z-transform of the following

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\quad |z|>\left|\frac{1}{2}\right| \qquad\qquad X(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\quad |z|<\left|\frac{1}{2}\right|$$

Solution: From the equations

$$a^n u[n]\leftrightarrow\frac{1}{1-az^{-1}},\quad |z|>|a| \qquad\qquad -a^n u[-n-1]\leftrightarrow\frac{1}{1-az^{-1}},\quad |z|<|a|$$

Solution using the above equations by visualization

$$x[n]=\left(\frac{1}{2}\right)^n u[n] \qquad\qquad x[n]=-\left(\frac{1}{2}\right)^n u[-n-1]$$

2.6.3 Study of Some Examples Using Partial Fraction Expansion

$$X(z)=\frac{\sum_{k=0}^{q}b_k\,z^{-k}}{\sum_{k=0}^{p}a_k\,z^{-k}}=\frac{b_0\prod_{k=1}^{q}\left(1-c_kz^{-1}\right)}{a_0\prod_{k=1}^{p}\left(1-d_kz^{-1}\right)}=\frac{b_0\,z^{p-q}}{a_0}\,\frac{\prod_{k=1}^{q}\left(z-c_k\right)}{\prod_{k=1}^{p}\left(z-d_k\right)}$$

Partial Fraction Expansion: *q* < *p*, Simple Roots

$$X(z) = \frac{\prod_{k=1}^{q}\left(1 - c_k\ z^{-1}\right)}{\prod_{k=1}^{p}\left(1 - d_k\ z^{-1}\right)} = \sum_{k=1}^{p} \frac{A_k}{\left(1 - d_k\ z^{-1}\right)}$$

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \leftrightarrow x[n] = a^n u[n]$$

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a| \leftrightarrow x[n] = -a^n u[-n-1]$$

$$A_k = \left(\left(1 - d_k\ z^{-1}\right) X(z)\right)\Big|_{z=d_k}$$

Partial Fraction Expansion: *q* < *p*, Simple Roots

$$X(z) = \frac{1 - z^{-1}}{1 + z^{-1} - 6\,z^{-2}}$$

$$z_i^{-1} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm 5}{2} = 2 \vee -3$$

$$X(z) = \frac{1 - z^{-1}}{\left(1 - 2\,z^{-1}\right)\left(1 + 3\,z^{-1}\right)} = \frac{\left(1 - z^{-1}\right) z}{(z-2)(z+3)}$$

$$X(z) = \frac{A_1}{\left(1 - 2\,z^{-1}\right)} + \frac{A_2}{\left(1 + 3\,z^{-1}\right)}$$

$$A_1 = \left(X(z)\left(1 - 2z^{-1}\right)\right)\Big|_{z=2} = \frac{\left(1 - z^{-1}\right)}{\left(1 + 3\,z^{-1}\right)}\Bigg|_{z=2} = \frac{(z-1)}{(z+3)}\Bigg|_{z=2} = \frac{1}{5}$$

$$A_2 = \left(X(z)\left(1 + 3z^{-1}\right)\right)\Big|_{z=-3} = \frac{\left(1 - z^{-1}\right)}{\left(1 - 2z^{-1}\right)}\Bigg|_{z=-3} = \frac{(z-1)}{(z-2)}\Bigg|_{z=-3} = \frac{-4}{-5} = \frac{4}{5}$$

$$X(z) = \frac{1/5}{\left(1 - 2\,z^{-1}\right)} + \frac{4/5}{\left(1 + 3\,z^{-1}\right)}$$

$$X(z) = \frac{1}{1 - az^{-1}}, |z| > |a| \leftrightarrow x[n] = a^n\ u[n]$$

$$X(z) = \frac{1}{1 - az^{-1}}, |z| < |a| \leftrightarrow x[n] = -a^n\ u[-n-1]$$

$$|z| > 3 \Rightarrow x[n] = \frac{1}{5}\,2^n u[n] + \frac{4}{5}(-3)^n\ u[n] = \left(\frac{1}{5}\,2^n + \frac{4}{5}(-3)^n\right) u[n]$$

$$|z| < 2 \Rightarrow x[n] = -\frac{1}{5}\,2^n u[-n-1] - \frac{4}{5}(-3)^n\ u[-n-1] = -\left(\frac{1}{5}\,2^n + \frac{4}{5}(-3)^n\right) u[-n-1]$$

$$A_4 = \left.\frac{1}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right|_{z=4} = \frac{1}{\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)} = \frac{32}{3}$$

Partial Fraction Expansion: $q < p$, Simple Roots

$$X(z) = \frac{-1/6}{1-z^{-1}} + \frac{4}{1-2z^{-1}} + \frac{-27/2}{1-3z^{-1}} + \frac{32/3}{1-4z^{-1}} \qquad \text{ROC: } 2 < |z| < 3$$

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a| \leftrightarrow x[n] = a^n u[n]$$

$$X(z) = \frac{}{az}, \quad |z| < |a| \leftrightarrow x[n] = -a\ u[-n-1]$$

$$x[n] = \left(-\frac{1}{6}\right)u[n] + (4)2^n u[n] + \left(-\frac{27}{2}\right)(-1)3^n\ u[-n-1] + \left(\frac{32}{3}\right)(-1)\ 4^n u[-n-1]$$

$$x[n] = \left(2^{n+2} - \frac{1}{6}\right)u[n] + \left(\frac{3^{n+3}}{2} - \frac{2^{2n+5}}{3}\right)u\,[-n-1]$$

Partial Fraction Expansion: $q >= p$

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, |z > 1|$$

$$X(z) = 2 + \frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, |z > 1|$$

$$X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} = 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

$$\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \overset{2}{\overline{\big)\, z^{-2} + 2z^{-1} + 1}}$$

$$\text{poles} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{4}{2}}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{9-8}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{1}{2}}{2} = 1 \vee \frac{1}{2}$$

$$A_1 = \left[\frac{-1+5z^{-1}}{1-z^{-1}}\right]_{z=1/2} = \frac{9}{-1} = -9$$

$$A_2 = \left[\frac{-1+5z^{-1}}{1-\frac{1}{2}z^{-1}}\right]_{z=1} = \frac{4}{1/2} = 8$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

2.7 Z-DOMAIN STABILITY

2.7.1 Stability

- A system is said to be stable if it produces bounded output for a bounded input (BIBO)
- A system is said to be stable if its impulse response vanishes after sufficiently long time.

$$h[n] \to 0 \text{ as } n \to \infty$$

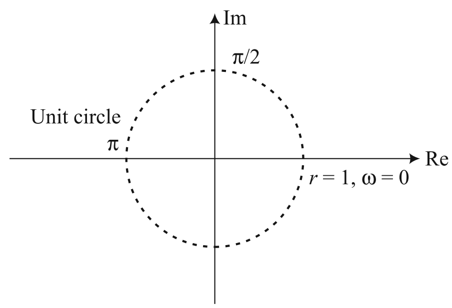


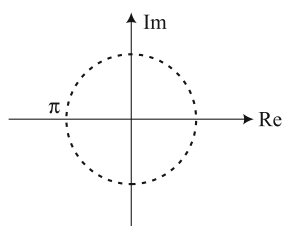
Fig. 2.4 Unit circle in *z*-plane.

System is realizable ⇔ system is stable ∧ system is causal

stable $\Leftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty \Leftrightarrow H(e^{j\varpi})$ *converges* $\Leftrightarrow |z| = 1$ is in ROC

causal $\Leftrightarrow h[n] = 0$ for $n < 0 \Leftrightarrow h[n]$ right-handed $\Leftrightarrow$ ROC is exterior of a circle: $|z| > |a|$

realizable $\Leftrightarrow$ ROC $|z| > |a|$ and includes unit circle $\Leftrightarrow$ all poles inside unit circle



2.7.2 Stability of a DTLTI System

As in the case of a continuous-time system a discrete-time system is said to be stable if every finite input produces a finite output. The stability concept may be readily expressed by conditions relating to the impulse response $h(n)$. These conditions are:

Solution:

(a) Taking the Z-transforms of both sides of the given system difference equation and solving for $H(z)$, we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 + 0.1\, z^{-1} - 0.2\, z^{-2}}$$

The poles and zeros are best obtained by momentarily arranging numerator and denominator polynomials in positive powers of z.

$$H(z) = \frac{z^2 + z}{z^2 + 0.1\, z - 0.2} = \frac{z\,(z+1)}{(z-0.4)\,(z+0.5)}$$

The poles are located at + 0.4 and – 0.5, which are inside the unit circle. Thus, the system is stable.

(b) The impulse response may be obtained by expanding $H(z)$ in a partial fraction expansion according to the procedure of the preceding section.

This yields

$$H(z) = \frac{1.555556\, z}{z - 0.4} - \frac{0.555556\, z}{z + 0.5}$$

Inversion of the above equation yields

$$h(n) = 1.555556\,(0.4)^n - 0.555556\,(-0.5)^n$$

It can be readily seen that the impulse response $h(n)$ vanishes after a sufficiently long time as expected, since this is a stable transfer function.

(c) To obtain the response due to $x(n) = 1$, we multiply $X(z)$ by $H(z)$ and obtain

$$Y(z) = \frac{z^2\,(z+1)}{(z-1)\,(z-0.4)\,(z+0.5)}$$

Partial fraction expansion yields

$$Y(z) = \frac{2.222222\, z}{z-1} - \frac{1.037037\, z}{z-0.4} - \frac{0.185185\, z}{z+0.5}$$

The inverse transform is

$$Y(n) = 2.222222 - 1.037037\,(0.4)^n - 0.185185\,(-0.5)^n$$

2.8 SOME TYPICAL EXAMPLES ON Z-TRANSFORM

Example 2.12 Find $x[n]$ for the following system transfer function.

$$X(z) = \frac{1 + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

Solution:

$$X(z) = \frac{1 + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{\frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$x[n] = Z^{-1}\left[\frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{\frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}\right]$$

$$= \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= \left(\frac{1}{2}\right)^n \left[u[n] + u[n-1]\right]$$

$$= \left(\frac{1}{2}\right)^n \left[u[n] + 2u[n-1] - u[n-1]\right]$$

$$= \left(\frac{1}{2}\right)^n \left[u[n] - u[n-1] + 2u[n-1]\right]$$

$$x[n] = \left(\frac{1}{2}\right)^n \left[\delta[n] + 2u[n-1]\right]$$

Example 2.13 Find *Z*-transform of

$$x[n] = (2)^n u[n-2]$$

Solution:

$$Z\left[u[n]\right] = \frac{1}{1 - z^{-1}}$$

$$Z\left[u[n-2]\right] = \frac{z^{-2}}{1 - z^{-1}}$$

$$Z\left[2^n u[n-2]\right] = \left.\frac{z^{-2}}{1 - z^{-1}}\right| z^{-1} = 2z^{-1}$$

$$= \frac{\left(2z^{-2}\right)^2}{1 - 2z^{-1}} = \frac{4z^{-2}}{1 - 2z^{-1}}$$

Solution:

$$a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

Using the differentiation property of Z-transform (refer Table 2.2, Property 4)

$$na^n\, u[n] \overset{z}{\leftrightarrow} -z\frac{d}{dz}\left(1-\frac{1}{az^{-1}}\right)$$

$$= \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad |z|>|a|$$

Example 2.16 Find $x[n]$ of the following function using convolution property of Z-transform.

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

Solution:

$$X(z) = X_1(z)X_2(z)$$

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$X_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$x_2[n] = \left(-\frac{1}{4}\right)^n u[n]$$

Using convolution property of Z-transform (refer Table 2.2, Property 7)

$$x[n] = x_1[n] * x_2[n]$$

$$= \sum_{k=0}^{n} x_1(n-k)x_2(k)$$

$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n-k}\left(\frac{-1}{4}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{-k}\left(\frac{-1}{4}\right)^k$$

$$= \left(\frac{1}{2}\right)^k\left(-\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{-1}{2}\right)^k$$

$$= \frac{\left(\frac{1}{2}\right)^n \left(1 - \left(-\frac{1}{2}\right)^{n+1}\right)}{1 - \left(-\frac{1}{2}\right)} = \frac{1 - a^{n+1}}{1 - a}$$

$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{2}\right)^n \left(-\frac{1}{2}\right)^{n+1}\right]$$

$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{2}\right)^n \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n\right]$$

$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n\right] u[n]$$

Example 2.17 Find inverse *Z*-transform of the following function under different ROC conditions.

$$X(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{3} z^{-1}\right)}$$

Solution: For $\frac{1}{4} < |z| < \frac{1}{3}$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

For $|z| < \frac{1}{4}$

$$x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

For $|z| > \frac{1}{3}$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

Example 2.18 Find the *Z*-transform of $x(n) = \cos(n\omega)\, u(n)$.

Solution: Given that

$$x(n) = \cos(n\omega)\, u(n)$$

From definition

$$z[x(n)] = x(z) = \sum_{n=-\infty}^{\infty} x(n) \, . \, z^{-n}$$

We know that

$$\cos(n\omega) = \left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right)$$

$\Rightarrow$

$$a^n \cos(n\omega) = a^n \left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right)$$

$$= \frac{a^n\ e^{j\omega n} + a^n\ e^{-j\omega n}}{2}$$

$$= \frac{\left(ae^{j\omega}\right)^n + \left(ae^{-j\omega}\right)^n}{2}$$

Form the definition of Z-transform,

$$z\left[x(n)\right] = x(z) = \sum_{n=0}^{\infty} x(n)\ z^{-n}$$

$\therefore$

$$x(z) = \sum_{n=0}^{\infty} a^n\ .\cos(n\omega).\ z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{\left(a\ .\ e^{j\omega}\right)^n + \left(a\ .\ e^{-j\omega}\right)^n}{2}\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{\left(a\ .\ e^{j\omega}\right)^n z^{-n} + \left(a\ .\ e^{-j\omega}\right)^n z^{-n}}{2}$$

$$= \sum_{n=0}^{\infty} \frac{\left(a\ .\ e^{j\omega} z^{-1}\right)^n + \left(a . e^{-j\omega} z^{-1}\right)^n}{2}$$

$$= \frac{1}{2}\left[\frac{1}{1 - ae^{j\omega}\ .\ z^{-1}} + \frac{1}{1 - ae^{-j\omega}\ .\ z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - ae^{j\omega}} + \frac{z}{z - ae^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z\left(z - ae^{-j\omega}\right) + z\left(z - ae^{j\omega}\right)}{\left(z - ae^{j\omega}\right)\left(z - ae^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - az\ .\ e^{-j\omega} + z^2 - az\ .\ e^{j\omega}}{z^2 - az\ .\ e^{-j\omega} - az\ .\ e^{j\omega} + a^2}\right]$$

$$= \frac{2z^2 - az\left[2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)\right]}{2\left[z^2 - az\left[2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)\right] + a^2\right]}$$

$$1 = A(Z-1)(Z-2)^2 + BZ(Z-2)^2 + C(Z-1)(Z-2)Z + DZ(Z-1)$$

$$1 = A(Z-1)(Z^2-4Z+4) + B(Z^3-4Z^2+4Z) + C(Z^2-Z)(Z-2) + D(Z^2-Z)$$

$$1 = A(Z^3-5Z^2+8Z-4) + B(Z^3-4Z^2+4Z) + C(Z^3-3Z^2+2Z) + D(Z^2-Z)$$

Comparing, Z^3-terms $\Rightarrow A + B + C = 0$

Z^2-terms $\Rightarrow 5A - 4B - 3C + D = 0$

Z-terms $\Rightarrow 8A + 4B + 2C - D = 0$

Constant $\Rightarrow -4A = 1$

$$A = \frac{-1}{4}$$

By solving the above terms, we get

$$B = 1$$

$$C = \frac{-3}{4}$$

$$D = \frac{1}{2}$$

Substituting A, B, C, D from the above equations, then

$$\frac{1}{Z(Z-1)(z-2)^2} = \frac{-1}{4Z} + \frac{1}{(Z-1)} - \frac{3}{4(Z-2)} + \frac{1}{2(Z-2)^2}$$

$$\Rightarrow \quad X(Z) = \frac{-1}{4} + \frac{Z}{(Z-1)} - \frac{3Z}{4(Z-2)} + \frac{Z}{2(Z-2)^2}$$

$$X(Z) = \frac{-1}{4} + \frac{Z}{(Z-1)} - \frac{3Z}{4(Z-2)} + \frac{Z}{2(Z-2)^2}$$

Applying the inverse Z-transform, we get,

$$x(n) = \frac{-1}{4}\delta(n) + (1)^n u(n) - \frac{3}{4}(2)^n u(n) + \frac{1}{2}n(2)^n u(n)$$

Example 2.22 Find the inverse Z-transform of $x(z) = \dfrac{2+z^3+3z^{-4}}{z^2+4z+3}$ $|z| > 0\cdot$

Solution: Given that,

$$x(z) = \frac{2+z^3+3z^{-4}}{z^2+4z+3}|z| > 0$$

$$= \frac{z^{-4}(2z^4+z^7+3)}{(z^2+4z+3)}$$

$$= \frac{\left(2z^4 + z^7 + 3\right)}{z^4\left(z^2 + 4z + 3\right)}$$

$$= \frac{z^7 + 2z^4 + 3}{z^6 + 4z^5 + 3z^4}$$

$$z^6 + 4z^5 + 3z^4 \overline{\big)\, z^7 + 2z^4 + 3} \left(z - 4 + \frac{13}{z} - \frac{38}{z^2} + \frac{118}{z^3} - \frac{338}{z^4} \right.$$

$$\begin{array}{l}
z^7 + 4z^6 + 3z^5 \\
- \quad - \quad - \\
\hline
-4z^6 - 3z^5 + 2z^4 + 3 \\
-4z^6 - 3z^5 + 12z^4 + 3 \\
-4z^6 - 16z^5 - 12z^4 \\
+ \quad + \quad + \\
\hline
13z^5 + 14z^4 + 3 \\
13z^5 + 52z^4 + 39z^3 \\
- \quad - \quad - \\
\hline
-38z^4 - 39z^3 + 3 \\
-38z^4 - 152z^3 - 114z^2 \\
+ \quad + \quad + \\
\hline
113z^3 + 114z^2 + 3 \\
113z^3 + 452z^2 + 339z \\
- \quad - \quad - \\
\hline
-338z^2 - 339z + 3 \\
-338z^2 - 1352z - 1014 \\
+ \quad + \quad + \\
\hline
1013z + 1017 \\
\hline
\end{array}$$

The result is $z - 4 + 13z^{-1} - 38z^{-2} + 113z^{-3} - 338z^{-4}$ (1)

∴ From the definition of Z-transform,

$$z\left[x(n)\right] = x(z) = \sum_{n=-\infty}^{\infty} x(n)\,.\,z^{-n}$$

∴
$$x(z) = \ldots + Z(-1)\,z + x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3} + \ldots \quad (2)$$

By comparing equations (1) and (2), we get,

$$x(-1) = 1$$

$$x(0) = -4$$

$$x(1) = 13$$

$$x(2) = -38$$

Solution:

(i) Given that, $$X(z) = \log\left(\frac{1}{1-az^{-1}}\right),\ |z| > |a|$$

$\therefore$ $$X(z) = -\log\left(1-az^{-1}\right),\ |z| > |a|$$

The power series expansion for log(1 – p) is given as

$$\log(1-p) = -\sum_{n=1}^{\infty} \frac{1}{n}\ p^n\quad |p| < 1$$

The region of convergence is $|z| > |a|$, i.e., $|az^{-1}| < 1$

$\therefore$ Power series expansion of $X(z)$ is given by,

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}\ \left(az^{-1}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(a^n\ z^{-n}\right)$$

$\therefore$ From above equations, $x(n)$ can be defined as

$$x(n) = \left(\frac{a^n}{n}\right),\ \text{ for } n \ge 1$$

$$= 0,\ \text{for } n \quad 0$$

or $$x(n) = \frac{a^n}{n}\, u(n-1)$$

(ii) Given that $$X(Z) = \log\left(\frac{1}{1-a^{-1}\, z}\right),\quad |z| < |a|$$

$\therefore$ $$X(Z) = -\log\left(1-a^{-1}\, z\right),\quad |z| < |a|$$

The region of convergence is,

$$|z| < |a| \text{ i.e., } |a^{-1}\, z| < 1$$

$\therefore$ Power series expansion of $X(z)$ is given by,

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}\ \left(a^{-1}\, z\right)^n$$

$$= \sum_{n=-1}^{-\infty} -\frac{1}{n}\left(a^{-1}\, z\right)^{-n}$$

$\therefore$ From above equations, $x(n)$ can be defined as

$$x(n) = 0,\ \text{for } n \ge 0$$

$$= -\left(\frac{a^n}{n}\right)\ \text{for } n \le -1\ \text{(or)}$$

$$x(n) = -\frac{a^n}{n} u(n-1)$$

12. Determine the inverse Z-transform of the following

$$X(z) = \frac{z(z+0.5)}{(z+0.2)(z+0.4)}$$

13. Obtain the transfer function for the system described by difference equation

$$y(n) + \frac{1}{2} y(n-1) = x(n)$$

14. Obtain the transfer function of the system described by the difference equation given by and hence, impulse response

$$y(n-1) - 4y(n-2) + 3y(n-3) = x(n-1) + x(n-2)$$

15. Find $x(0)$ and $x(\infty)$ for the sequence whose Z-transform is

$$X(z) = \frac{z}{z-3}$$

Multiple-Choice Questions

1. The Z-transform of the unit ramp is given by
   (a) $\frac{z}{(z-1)^2}$ (b) $\frac{z}{z-1}$
   (c) $\frac{z-1}{z}$ (d) $\frac{(z-1)^2}{z}$
2. The Z-transform of $x(0)$ is
   (a) $X(z)$ (b) $X(0)$
   (c) $X(1)$ (d) $\lim_{z \to \infty} X(z)$
3. For causal signals and systems, the Z-transform is defined as
   (a) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ (b) $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$
   (c) $X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n}$ (d) $X(z) = \sum_{n=-\infty}^{1} x(n) z^{-n}$
4. Region of convergence is defined as
   (a) Set of z-values for which the series converges.
   (b) Set of n-values for which the series converges.
   (c) Set of n-values for which the series diverges.
   (d) Set of z-values for which the series diverges.
5. If lower limit of the ROC is greater than the upper limit of ROC the series

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

13. The Z-transform of $\sum_{n=-\infty}^{\infty} x(n)$ is

    (a) $X(1)$ (b) $X(0)$

    (c) $X(\infty)$ (d) $X(z)$

14. The system is said to be stable if and only if $\sum_{n=-\infty}^{\infty} |h(n)|$

    (a) Greater than infinity (b) Equal to infinity

    (c) Less than infinity (d) Equal to zero

15. The Z-transform of $\sum_{n=-\infty}^{l} x(n)$ is

    (a) $X(1)$ (b) $\dfrac{X(z)}{1-z^{-1}}$

    (c) $X(z)$ (d) $X(\infty)$

16. The Z-transform of $a^{-n}\, x(n)$ is

    (a) $X\left(\dfrac{z}{a}\right)$ (b) $X\left(\dfrac{z}{a^2}\right)$

    (c) $X(za)$ (d) $X(z - a)$

17. Unit sample response of a system is

    (a) $h(n) = z[h(n)]$ (b) $h(n) = z^{-1}[H(z)]$

    (c) $h(n) = z^{-1}[X(z)]$ (d) $h(n) = z^{-1}[Y(z)]$

18. The system is said to be causal if and only if

    (a) $h(n) = 0$ for $n < 0$ (b) $x(n) = 0$ for $n < 0$

    (c) $y(n) = 0$ for $n < 0$ (d) $y(n) = 0$ for $n > 0$

19. The system is described by the following difference equation

    $$y(n) - ay\,(n - 1) = x(n)$$

    If the excitation is the unit impulse, the system transfer function is

    (a) $\dfrac{z}{z-a}$ (b) $\dfrac{z}{z+a}$

    (c) $\dfrac{z}{a}$ (d) $\dfrac{a}{z-a}$

20. The inverse Z-transform of $X(z) = \dfrac{z^2}{(z-a)^2}$ if $X(z)$ converges absolutely for some $|z| < |a|$

    (a) $(n + 1)\, a^n$ for $n \geq 0$ (b) $-(n + 1)\, a^n$ for $n \leq -1$

    (c) $-(n + 1)\, a^n$ for $n \geq 0$ (d) $(n + 1)\, a^n$ for $n \leq -1$