2

Numerical Solution of Non-linear Equations

2.1 INTRODUCTION

The most common real-life problems are nonlinear and are not amenable to be handled by analytical methods to obtain solutions of a variety of mathematical problems. Iterative methods are the foremost among the methods developed to obtain approximate solutions.

The method of finding a root of the non-linear equations of the form

$$f(x) = 0 \tag{2.1}$$

where the function $f(x)$ may be algebraic, transcendental or combination of both, plays a major role in the applications of mathematics as problems of such kind occur more frequently in many scientific and engineering mathematical modeling.

The following equations

(i) $x^6 - x - 1 = 0$

(ii) $xe^x - \cos x = 0$

(iii) $\sin(x)\, e^x - 2x - 5 = 0$

can be classified as follows.

(i) The equation $x^6 - x - 1 = 0$ is an algebraic equation of degree 6 having one root nearly at $x = 1.13472413$.

(ii) The equation $xe^x - \cos x = 0$ is a transcendental equation as it contains transcendental functions which has a root nearly about $x = 0.51775736$.

(iii) The equation $\sin(x)\, e^x - 2x - 5 = 0$ is an equation combined of both algebraic and transcendental functions and it has a root nearly about $x = -2.523245230$.

If a number 'α' makes the function of a scalar variable, i.e., $f(x)$ to zero then 'α' is called a zero or a root of the equation $f(x) = 0$. To obtain this root, if one obtains iterates $\{x_1, x_2, ... x_n ...\}$ starting with an initial guess x_0, then this sequence of iterates converge to the root whenever $\lim_{n\to\infty} |x_n - \alpha| = 0$ (or) $\lim_{n\to\infty} x_n = \alpha$.

In this chapter, we shall study the methods of obtaining an approximate solution of equations of the form (2.1) and discuss the importance of each of these methods comparing with one another when required, through some examples.

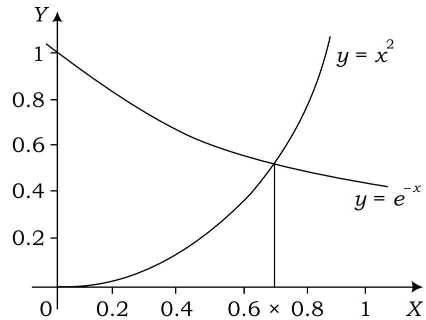

Fig. 2.1

From the above plotting of the graphs, one can see that abscissa of the intersection of two curves is around $x = 0.7$. Therefore, the required approximate root of the equation (2.1.1) is 0.7.

Example 2.2: Obtain graphically the two real roots of the equation

$$x^4 - 3x^3 + 1 = 0$$

Solution: The given equation is $f(x) = x^4 - 3x^3 + 1 = 0$

One can verify that

$$f(0) > 0, f(1) < 0, f(2) < 0, \text{ and } f(3) > 0$$

Therefore, we have

$$f(0) \cdot f(1) < 0, \text{ and } f(2) \cdot f(3) < 0$$

and hence there exists roots in (0, 1) and (2, 3).

We now rewrite the given equation as $x = \sqrt[3]{\dfrac{x^4+1}{3}}$.

Let us plot the graphs for the curves $y = x$ and $y = \left(\dfrac{x^4+1}{3}\right)^{1/3}$ on the same scale in the interval (0, 3) with respect to the same axis forming the following tabular values.

Table 2.2

x	0	0.5	1	1.5	2	2.5	3
$y = x$	0	0.5	1	1.5	2	2.5	3
$y = \sqrt[3]{\dfrac{x^4+1}{3}}$	0.69	0.71	0.87	1.26	1.78	2.37	3.01

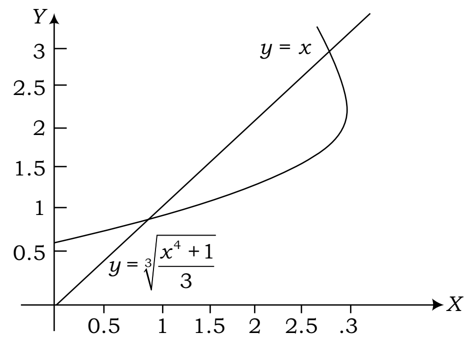

Fig. 2.2

From the above plotting of the curves, we have the abscissae of points of intersection of the curves are near about 0.76 and about 2.96.

Threrefore, the required approximate roots of the given equation in the intervals (0, 1) and (2, 3) are $x = 0.76$ and $x = 2.96$ respectively.

Example 2.3: Find a root of the equation $x - e^{-x} = 0$ correct to 3 decimal places, by using the bisection method.

Solution: Let $f(x) = x - e^{-x}$

Since $f(0) < 0$ and $f(1) > 0$, there is a root in the interval (0, 1).

We now form the following table to obtain the required root on taking $a = 0$ and $b = 1$.

Table 2.3

Sl. No.	a	b	$c = \frac{a+b}{2}$	$f(c)$
1.	0	1	0.5	–0.1065
2.	0.5	1	0.75	0.2776
3.	0.5	0.75	0.625	0.0897
4.	0.5	0.625	0.5625	–0.0073
5.	0.5625	0.625	0.5938	0.0415
6.	0.5625	0.5938	0.5782	0.0172
7.	0.5625	0.5782	0.5704	0.0050
8.	0.5625	0.5704	0.5665	–0.0011
9.	0.5665	0.5704	0.5685	0.0020
10.	0.5665	0.5685	0.5675	0.0006
11.	0.5665	0.5675	0.5670	–0.0002

Since $|f(0.567)| < 0.0005$, the required root of the equation $x - e^{-x} = 0$ correct to three places of decimal is $x = 0.567$.

Note 2.1: The above computed tabulated values are rounded to 4 decimals, though the root is required correct to 3 decimal places, in order to check whether the function value is less than 0.5×10^{-3} to obtain the root up to the desired accuracy.

Remark 2.1: If α is a real root of the equation $f(x) = 0$ correct to N decimal places, then $f(\alpha) < 0.5 \times 10^{-N}$ in magnitude.

Example 2.4: Obtain the smallest positive real root of the equation $e^{-x} - \sin x = 0$ by bisection method, correct to 4 decimal places.

Solution: Let $f(x) = e^{-x} - \sin x$

since $f(0) > 0$, $f(0.5) > 0$ and $f(1) < 0$, the lowest root lies in the interval $\left(\frac{1}{2}, 1\right)$. Now, we form the following table to obtain the root taking $a = 1$ and $b = 0.5$

Table 2.4

Sl. No.	*a*	*b*	$c = \frac{a+b}{2}$	*f(c)*
1.	1	0.5	0.75	−0.20927
2.	0.75	0.5	0.625	−0.04984
3.	0.625	0.5	0.5625	0.03648
4.	0.625	0.5625	0.59375	−0.00722
5.	0.59375	0.5625	0.57813	0.01449
6.	0.59375	0.57813	0.58594	0.0036
7.	0.59375	0.58594	0.58985	−0.00182
8.	0.58985	0.58594	0.5879	0.00088
9.	0.58985	0.5879	0.58888	−0.00047
10.	0.58888	0.5879	0.58839	0.0002
11.	0.58888	0.58839	0.58864	−0.00014
12.	0.58864	0.58839	0.58852	0.00002

Since $f|(0.58852)| = 0.00002 < 0.5 \times 10^{-4}$, the root of the equation $e^{-x} - \sin x = 0$, correct to four decimal places is $x = 0.5885$.

Note 2.2: As the width of the interval is reduced by half at each step in the bisection method, N bisections will be required to have a length of the interval which contains the root, is $\frac{|b-a|}{2^N}$.

And, to have $\frac{|b-a|}{2^N} \leq \varepsilon$ (a small quantity as desired), one can easily deduce that

$$N \geq \ln\frac{|b-a|}{\varepsilon} / \ln 2 \tag{2.2}$$

Condition (2.2) can be verified for the examples (2.3) and (2.4) for $\varepsilon = 0.001$.

2.4 ITERATION METHOD

The iteration method involves transforming the equation (2.1) into the form

$$x = \phi(x) \tag{2.3}$$

and generating a sequence of approximations $x_1, x_2, x_3, ..., x_n, ...$ to a root of the equation (2.1) from the scheme

$$x_{n+1} = \phi(x_n) \tag{2.4}$$

$$(n = 0, 1, 2, ...)$$

Choosing a proper initial approximation x_0.

We now state and prove the sufficient condition for the convergence of the method (2.4).

Theorem 2.1: If I is the interval in which x^* a root of the equation $f(x) = 0$, lies and if $|\phi'(x)| < 1$ for all x in I then the iterative method (2.4) will converge to x^* provided x_0 is properly chosen in I.

Proof: If x^* is a root of $f(x) = 0$, then from (2.3), we have $x^* = \phi(x^*)$ (2.5)

The scheme (2.4) is given by $x_{n+1} = \phi(x_n)$ (2.6)

where x_{n+1} and x_n are the $(n + 1)^{th}$ and n^{th} approximations to x^* respectively.

Now, subtracting (2.6) from (2.5), we get

$$\begin{aligned} x_{n+1} - x^* &= \phi(x_n) - \phi(x^*) \\ &= (x_n - x^*)\, \phi'(\xi) \end{aligned} \tag{2.7}$$

by mean value theorem, where $x_n < \xi < x^*$.

If we let

$$|\phi'(x_n)| \leq K < 1 \quad \text{for all } i = 0, 1, 2, ... \tag{2.8}$$

Then (2.7) can be written as

$$\begin{aligned} |x_{n+1} - x^*| &\leq K \cdot |x_n - x^*| \\ &\leq K \cdot K \cdot |x_{n-1} - x^*| \\ &\leq K \cdot K \cdot K\, |x_{n-2} - x^*| \end{aligned} \tag{2.9}$$

and so on

Proceeding in similar manner, finally we will be left with

$$|x_{n+1} - x^*| \leq K^{n+1} |x_0 - x^*|$$

For a large n, the right-hand side tends to zero. Thus, x_{n+1} converges to x^*. Hence, the proof is complete.

Note 2.3: The iterative method (2.4) has a linear rate of convergence.

Example 2.5: Which of the following forms will converge to a root of the equation $3x - \cos x - 1 = 0$ lies in $\left(0, \frac{\pi}{2}\right)$.

(a) $x = \cos x - 2x + 1$ (b) $x = (1 + \cos x)/3$

Solution: Given $f(x) = 3x - \cos x - 1 = 0$ (2.5.1)

which has a root in $\left(0, \frac{\pi}{2}\right)$.

(a) The equation (2.5.1) is written as

$x = \cos x - 2x + 1$

$= \phi(x)$

So that $\phi'(x) = -(\sin x + 2)$

Since $|\phi'(x)| = |(\sin x + 2)| \not< 1$

for any x in $\left(0, \frac{\pi}{2}\right)$, the iterative method will not converge to a root of (2.5.1) with this form.

(b) In this case, equation (2.5.1) is written as

$x = (1 + \cos x)/3$

$= \phi(x)$

Now, $\phi'(x) = -\sin x/3$

Since $|\phi'(x)| = \left|\frac{\sin x}{3}\right| < 1$ for all x in $\left(0, \frac{\pi}{2}\right)$, the iterative scheme in this case will converge to a root of the equation (2.5.1) by choosing any x_0 in $\left(0, \frac{\pi}{2}\right)$.

Example 2.6: Use iterative method to find a root of the equation $x + e^x = 0$ up to 5 decimals writing it as

(a) $x = -e^x$

(b) $x = x(1 + x + e^x)$

Solution: Given equation is $f(x) = x + e^x = 0$ (2.6.1)

Since $f\left(\frac{-3}{5}\right) < 0$ and $f\left(\frac{-1}{2}\right) > 0$, there exists a root in the interval $\left(\frac{-3}{5}, \frac{-1}{2}\right)$.

(a) Here, equation (2.6.1) is written as

$x = -e^x$

$= \phi(x)$

Now, $\phi'(x) = -e^x$

and $|\phi'(x)| = e^x < 1$ for all x in $(-0.6, -0.5)$

The iterative scheme for the solution of (2.6.1) is

$$x_{n+1} = -e^{x_n} \tag{2.6.2}$$

$(n = 0, 1, 2, \ldots\ldots)$

If we start with $x_0 = -0.6$, then from (2.6.2) we can obtain the following successive approximations.

Table 2.5

n	x_{n+1}	n	x_{n+1}
0	−0.548812	11	−0.567179
1	−0.577636	12	−0.567123
2	−0.561224	13	−0.567155
3	−0.570510	14	−0.567137
4	−0.565237	15	−0.567147
5	−0.568225	16	−0.567141
6	−0.566530	17	−0.567145
7	−0.5674910	18	−0.567142
8	−0.566946	19	−0.567144
9	−0.567255	20	−0.567143
10	−0.567080	21	−0.567143

Hence, the required root of equation (2.6.1) correct to five places of decimal is

$$x = -0.56714$$

(b) We now solve (2.6.1) considering the form

$x = x(1 + x + e^x)$

$= \phi(x)$

Here, $\phi'(x) = 1 + 2x + (x + 1)\, e^x$

It can be verified that $|\phi'(x)| = |1 + 2x + (x + 1)\, e^x| < 1$ for all x in $(-0.6, -0.5)$

Now, taking $x_0 = -0.6$ and applying the iterative scheme

$$x_{n+1} = x_n(1 + x_n + e^{x_n}) \tag{2.6.3}$$

$(n = 0, 1, 2, \ldots)$

the following iterates are obtained.

Table 2.6

n	0	1	2	3	4	5
x_{n+1}	−0.569287	−0.567375	−0.567169	−0.567146	−0.567144	−0.567143

Therefore, the required root correct to 5 decimal places is $x = -0.56714$.

Note 2.4: It may be noted that the scheme (2.6.3) will not converge to the root if we start with $x_0 = -1.5$.

Remark 2.2: Smaller the value of $|\phi'(x)|$, larger the rate of convergence.

Definition 2.1: A sequence of approximations $\{x_0, x_1, x_2, ..., x_n, ...\}$ is said to converge to a root x^* of an equation $f(x) = 0$ with rate of convergence or the order of convergence $p \geq 1$ if $|x_{n+1} - x^*| \leq K \cdot |x_n - x^*|^p$ for some $K > 0$. (2.10)

Definition 2.2: If p is the order of the method and n is the number of functional evaluations per iteration by a method, then the efficiency index of that method is $\sqrt[n]{p}$.

2.5 ACCELERATION OF CONVERGENCE

It is always possible to accelerate the convergence of the method of iteration (2.4) by introducing a parameter α and extrapolating it as

$$\left.\begin{array}{c} x_{n+1} = (1-\alpha)\, x_n + \alpha\phi(x_n) \\ \text{(or)} \\ x_{n+1} = x_n + \alpha(\phi(x_n) - x_n) \end{array}\right\} \quad (2.11)$$

$$(n = 0, 1, 2\ldots)$$

The improvement of convergence majorly depends upon the choice of the relaxation factor α subject to constraints of the given problem. The following are some of the methods which enhance the rate of convergence of the scheme (2.4).

2.6 WEGSTEIN'S METHOD

In this method, the choice of α in the scheme (2.11) is taken as

$$\alpha = \frac{1}{1-\Delta}, \Delta = \frac{\phi(x_n) - x_n}{x_n - x_{n-1}} \quad (2.12)$$

The following steps illustrate the working procedure of this method.

Step 1: Choose an initial approximation x_0 to a root of $f(x) = 0$.

Step 2: Assign $n \leftarrow 1$

Step 3: Calculate $x_n = \phi(x_{n-1})$

Step 4: Find $\Delta = \dfrac{\phi(x_n) - x_n}{x_n - x_{n-1}}$

Step 5: Find $\alpha = \dfrac{1}{1-\Delta}$

Step 6: Calculate $x_{n+1} = x_n + \alpha[\phi(x_n) - x_n]$

Step 7: Stop the process if $|x_{n+1} - x_n|$ is negligible up to the tolerance.

Step 8: Assign $n \leftarrow n + 2$ and go to step 3.

Note 2.5: When $\alpha = 1$ or $\Delta = 0$, the above method reduces to the iterative method (2.4).

2.7 AITKEN'S Δ^2 METHOD

If the method (2.4) converges linearly and if $|\phi'(x^*)| \le K < 1$ where x^* is a root of (2.1), then from (2.9) approximately we have

$$x_{n+1} - x^* = K(x_n - x^*)$$

$$x_n - x^* = K(x_{n-1} - x^*)$$

Solving the above equations for x^* by eliminating K from them, we obtain

$$x^* = x_{n+1} - \frac{(x_{n+1} - x_n)^2}{x_{n+1} - 2x_n + x_{n-1}}$$

Replacing x^* by x_{n+2}, we have

$$x_{n+2} = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}} \tag{2.13}$$

Which is the Aitken's extrapolation method, for a linearly convergent sequences, yields approximations to the root x^* with $x_1 = \phi(x_0)$ and $x_2 = \phi(x_1)$.

Observation 2.1: It is seen that the method (2.11) will accelerate the convergence of the method (2.4) if we choose a proper α in

(i) $(0.5 < \alpha < 1)$ provided $-1 < \phi'(x) < 0$

(ii) $(\alpha > 1)$ provided $0 < \phi'(x) < 1$

Remark 2.3: By Theorem (2.1), one can easily obtain the condition for convergence of the method (2.11) as:

$|1 + \alpha + \alpha\phi'(x)| < 1$ for all x in I (2.14)

The condition (2.14) is same as

$$-1 < 1 - \alpha + \alpha\phi'(x) < 1$$

(or)

$$0 < \alpha < \frac{2}{1 - \phi'(x)} \tag{2.15}$$

for each x.

Solution: Given equation is

$$f(x) = x - \sin x - 6.28 = 0$$

We rewrite this equation as

$$x = 6.28 + \sin x$$
$$= \phi(x)$$

Here $\phi'(x) = \cos x$

Now, the iterative method for the solution of given equation is

$$x_{n+1} = 6.28 + \sin x_n \qquad (2.7.1)$$
$$(n = 0, 1, 2\ldots)$$

The Wegstein's iteration is

$$x_{n+1} = x_n + \alpha(6.28 + \sin x_n - x_n) \qquad (2.7.2)$$
$$(n = 0, 1, 2\ldots)$$

where $\alpha = \dfrac{1}{1-\Delta}$, $\Delta = \dfrac{6.28 + \sin x_n - x_n}{x_n - x_{n-1}}$

The Aitken's Δ^2 process is

$$x_{n+2} = x_{n+1} - \frac{(x_{n+1} - x_n)^2}{x_{n+1} - 2x_n + x_{n-1}} \qquad (2.7.3)$$
$$(n = 0, 1, 2\ldots)$$

where $x_n = 6.28 + \sin x_{n-1}$ and $x_{n+1} = 6.28 + \sin x_n$,
whereas, the extrapolated iterative method with $\alpha = \hat{\alpha}$ is

$$x_{n+1} = x_n - \frac{x_n - 6.28 - \sin x_n}{1 - \cos x_n} \qquad (2.7.4)$$
$$(n = 0, 1, 2\ldots)$$

The first 9 approximations of these methods are tabulated hereunder taking $x_0 = 6$

Table 2.7

n	*Method (2.7.1)* x_n	*Method (2.7.2)* x_n	*Method (2.7.3)* x_{n+1}	*Method (2.7.4)* x_n
1	6.000584502	6.000584502	6.000584502	6.014675016
2	6.001145771	6.001145771	6.001145771	6.015500534
3	6.001684820	6.014762084	6.014705148	6.015503073
4	6.002202612	6.014788597	6.014733649	6.015503073
5	6.002700060	6.015464679	6.014761129	–
6	6.00317805	6.015466049	6.015500802	–

7	6.003637362	6.015501273	6.015500883	–
8	6.004078829	6.015501337	6.015500961	–
9	6.004503186	6.015503135	6.015503073	–

Remark 2.5: The true root of the equation of example (2.7) is 6.01550307297 approximately. Comparing this value with the above tabulated approximations, one can see that the extrapolated iterative method (2.7.4) has a better rate of convergence for obtaining the root of $x = 6.28 + \sin x$.

Example 2.8: Find the largest root of the equation $x^6 - x - 1 = 0$ correct up to 8 places of decimal, starting with $x_0 = 2$ by using:

(i) Iterative method

(ii) Extrapolated iterative method.

Solution: Given

$$f(x) = x^6 - x - 1 = 0 \tag{2.8.1}$$

Which has largest root in the interval $1 < x < 2$.

We rewrite (2.8.1) as

$$x = \sqrt[6]{x+1}$$
$$= \phi(x) \qquad \text{[say]}$$

Now, $|\phi'(x)| = \left|\dfrac{\sqrt[6]{\dfrac{1}{(x+1)^5}}}{6}\right| < 1$

for all x in (1, 2)

The iterative method for solving (2.8.1) is

$$x_{n+1} = \sqrt[6]{x_n + 1}$$
$$(n = 0, 1, 2, 3\ldots..)$$

And the extrapolated iterative method in this case will be of the form

$$x_{n+1} = x_n + \alpha\left[\sqrt[6]{x_n+1} - x_n\right]$$
$$(n = 0, 1, 2, 3\ldots\ldots\ldots\ldots)$$

where

$$\alpha = \frac{1}{\left[1 - \left\{\sqrt[6]{(x_n+1)^{-5}}/6\right\}\right]}$$

The computed results for obtaining the root of equation (i) are tabulated below:

Table 2.8

n	*Iterative Method*	*Extrapolated Iterative Method*	
	x_n	α	x_n
0	2.000000000	–	2.000000000
1	1.200936955	1.071488330	1.143813273
2	1.140515698	1.096827312	1.134725697
3	1.135236648	1.097204065	1.134724138
4	1.134769538	1.094204130	1.134724138
⋮	⋮ ⋮	–	–
⋮	⋮ ⋮	–	–
11	1.134724138	---	–

From the above tabulated values, we have the required largest root of the equations (2.8.1) as x = 1.13472413 which is correct to 8 decimal places.

2.9 METHOD OF FALSE-POSITION (OR) REGULA-FALSI METHOD

In this method, we need to find a sufficiently small interval (a, b) in which the root of an equation $f(x) = 0$ lies as in the case of bisection method, so that the curve $y = f(x)$ crosses the X-axis in between $x = a$ and $x = b$ and also $f(a) \cdot f(b) < 0$.

The false-position method is based on the principle that any small portion of a smooth curve is practically straight for a short distance. Taking this principle into consideration, we fit a straight line passing through the points $P[a, f(a)]$ and $Q[b, f(b)]$ as

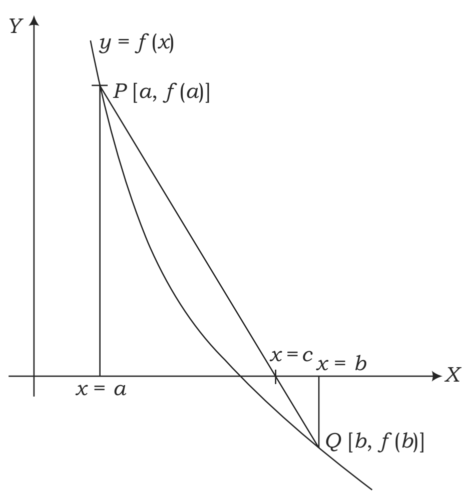

Fig. 2.3

Remark 2.7: Even though the rate of convergence of the secant method is 1.62 approximately, it will fail to converge when $f(x_n) = f(x_{n-1})$ at any stage. But, once secant method converges it will converge more rapidly than the Regula-Falsi method which is always guaranteed to converge.

Example 2.9: Solve the equation given in example (2.8) by (i) Regula-Falsi method and (ii) secant method to obtain the ninth approximate x_9 of the root with $x_0 = 2$ and $x_1 = 1$.

Solution: Let $f(x) = x^6 - x - 1 = 0$

So that $f(2) = 61 > 0$ & $f(1) = -1 < 0$ therefore a root of the equation lies in (1, 2). Taking $x_0 = 2 = a$, $x_1 = 1 = b$, we applied the methods (2.21) and (2.22) and the successive approximations up to 9th approximate are tabulated hereunder.

Table 2.9

n	'C' *of (2.21)*	x_{n+1} *of (2.22)*
1	1.016129032	1.01612902
2	1.030674754	1.190577768
3	1.043716601	1.117655831
4	1.055347031	1.132531550
5	1.065667282	1.134816808
6	1.074783410	1.134723646
7	1.082802820	1.134724138
8	1.089831382	1.134724138

The required root is $x = 1.13472413$ which is correct to 8 decimals. One can see that the more rapid convergence of secant method to the root, over false-position method.

Example 2.10: Use secant method to find a root of the equation $x + e^x = 0$ correct to 8 places of decimal, by tabulating the computed values.

Solution: Given $f(x) = x + e^x = 0$ (2.10.1)

Since $f(-1) < 0$ and $f(0) > 0$, there exists a root in the interval (–1, 0).

The secant method for the solution of (2.10.1) is given by

$$x_{n+1} = x_n - f(x_n) \cdot \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right]$$

$$(n = 1, 2, 3 \ldots)$$

where $f(x_n) = x_n + e^{x_n}$

Let us take $x_0 = -1$ and $x_1 = 0$. Then $f(x_0) = -0.632120559$ and $f(x_1) = 1$. Now, the following table exhibits further approximations.

Table 2.10

n	x_{n+1}	$f(x_{n+1})$
1	−0.612699837	−0.070813948
2	−0.572181412	−0.007888273
3	−0.567102080	0.000064583
4	−0.567143328	−0.000000059
5	−0.567143290	0.000000001
6	−0.567143291	−0.000000001
7	−0.567143290	0.000000001

Since $f(x_8) < 0.5 \times 10^{-8}$, the desired root of the equation correct to eight decimal places is

$$x = -0.56714329.$$

2.11 NEWTON-RAPHSON (N-R) METHOD

To derive the Newton-Raphson method for finding a root x^* of the equation $f(x) = 0$, let x_0 be an initial approximation and let $x_1 = x_0 + h$ where h is a small quantity, be the first approximation to x^*. If x_0 is chosen such that x_0 lies in the neighbourhood of x^*, then x_1 will be nearly equal to x^*.

Thus, $f(x_1) = 0$ approximately.

i.e.,
$$f(x_0 + h) = 0$$

Expanding the above function by means of Taylor's series, one can have

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots\dots\dots\dots = 0 \tag{2.23}$$

Since h is small enough, we can neglect the higher order powers of h starting from h^2 onwards.

Thus, from (2.23), we have

$$f(x_0) + hf'(x_0) = 0$$

which gives $h = -\dfrac{f(x_0)}{f'(x_0)}$

Putting this value of 'h' in $x_1 = x_0 + h$,

we obtain $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

Now, for a better approximation to x^*, if we let $x_2 = x_1 + h$ be the second approximation and proceeding in a similar manner as above, one can easily obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

2.11.2 Condition for Convergence of the Newton-Raphson Method

Comparing the iterative method (2.4) with the N-R method (2.24),

One can have $\phi(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (2.26)

We know that the iterative method converges by Theorem (2.1), if

$|\phi'(x)| < 1$ for all x in I

From (2.26),

$$\phi'(x) = 1 - \frac{f'^2(x) - f(x)\cdot f''(x)}{f'^2(x)} = \frac{f(x)\cdot f''(x)}{f'^2(x)} \quad (2.27)$$

Therefore, the condition for convergence of the N-R method is

$\left|\dfrac{f(x)\, f''(x)}{[f'(x)]^2}\right| < 1$ for all x in I.

Observation 2.2: The convergence of N-R method will be faster if $f'(x)$ is large enough, i.e., the graph of $f'(x)$ is nearly perpendicular to the x-axis.

Observation 2.3: By Theorem (2.1), we can also have the convergence of N-R method provided that the initial approximation x_0 is chosen sufficiently close to the root of $f(x) = 0$.

2.11.3 Newton-Raphson Method has a Quadratic Convergence

Let x^* be a root of $f(x) = 0$ such that $f(x^*) = 0$ and let the small quantities ε_n and ε_{n+1} be the errors at n^{th} and $(n + 1)^{th}$ stages for the approximations x_n and x_{n+1} respectively such that

$$\varepsilon_n = x_n - x^* \text{ and } \varepsilon_{n+1} = x_{n+1} - x^* \quad (2.28)$$

By (2.28), the N-R method (2.24) takes the form

$$\varepsilon_{n+1} + x^* = \varepsilon_n + x^* - \frac{f(\varepsilon_n + x^*)}{f'(\varepsilon_n + x^*)}$$

(or)

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(x^* + \varepsilon_n)}{f'(x^* + \varepsilon_n)}$$

Expanding the right-hand side function by Taylor's theorem, we have

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(x^*) + \varepsilon_n f'(x^*) + \dfrac{\varepsilon_n^2}{2} f''(x^*) + \ldots\ldots}{f'(x^*) + \varepsilon_n f''(x^*) + \ldots\ldots}$$

Taking $f(x^*) = 0$ and neglecting the higher order terms of ε_n starting from ε_n^3 onwards as ε_n is small enough, we obtain

$$\varepsilon_{n+1} = \frac{\varepsilon_n f'(x^*) + \varepsilon_n^2 f''(x^*) + \ldots\ldots - \varepsilon_n f'(x^*) - \frac{\varepsilon_n^2}{2} f''(x^*) + \ldots\ldots}{f'(x^*) + \varepsilon_n f''(x^*)}$$

which give us

$$\varepsilon_{n+1} = \varepsilon_n^2 \left[\frac{f''(x^*)}{2f'(x^*)} \right] \tag{2.29}$$

$$\varepsilon_{n+1} \propto \varepsilon_n^2$$

This relation shows that the subsequent error is proportional to the square of the previous error and hence the N-R method has a quadratic or second order convergence and its efficiency index is $\sqrt[2]{2} \simeq 1.414$.

2.11.4 Newton-Raphson Method Algorithm

Step 1: Choose an initial approximation x_0 to obtain the root of $f(x) = 0$.

Step 2: Calculate $f(x_0)$ and $f'(x_0)$.

Step 3: Evaluate $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Step 4: If $|x_1 - x_0|$ is negligible up to the tolerance stop the process. Otherwise, go to step 5

Step 5: Increase the subscripts of x's by one unit in the above 3 steps and go to step 2.

Example 2.11: Find a root of the equation $\cos x - xe^x = 0$ by Newton's method.

Solution: Let $f(x) = \cos x - xe^x = 0$

Then, $f'(x) = -[\sin x + (x + 1)e^x]$

Since $f(0) > 0$ and $f(1) < 0$, there is a root in the interval (0, 1).

Let us choose $x_0 = 0.5$

Then, the N-R method (2.24) yields

$$x_{n+1} = x_n + \frac{\cos x_n - x_n e^{x_n}}{\sin x_n + (x_n + 1)\, e^{x_n}}$$

$$(n = 0, 1, 2\ldots\ldots\ldots)$$

For $n = 0$, we obtain

$$x_1 = 0.5 + \frac{\cos(0.5) - (0.5)e^{0.5}}{\sin(0.5) + (0.5+1)e^{0.5}}$$

$$= 0.5 + \frac{0.053222}{2.952507}$$

$$= 0.518026$$

Example 2.13: Develop N-R method for finding

Square root of a number M

Reciprocal of a number M

and find $\sqrt{13}$ and $\frac{1}{13}$ correct to 5 decimals.

Solution:

(a) The square root of a number M is a root of the equation $x^2 - M = 0$.

Let $f(x) = x^2 - M$

Then, $f'(x) = 2x$

From (2.24), we have

$$x_{n+1} = x_n - \frac{x_n^2 - M}{2x_n}$$

$$\text{(or) } x_{n+1} = \frac{1}{2}\left(x_n + \frac{M}{x_n}\right) \quad (2.13.1)$$

$(n = 0, 1, 2 ...)$

is the Newton's formula for obtaining square root of a number M.

Since $3 = \sqrt{9} < 13 < \sqrt{16} = 4$, we take $x_0 = 3.5$ to obtain $\sqrt{13}$. From (2.13.1), we have for $n = 0$;

$$x_1 = \frac{\left(x_0 + \frac{13}{x_0}\right)}{2} = 3.60714$$

for $n = 1$;

$$x_2 = \frac{\left(x_1 + \frac{13}{x_1}\right)}{2} = 3.60555$$

for $n = 2$; we obtain $x_3 = 3.60555$

Since $x_2 = x_3$, the square root of 13 correct to 5 decimals is 3.6055.

(b) The reciprocal of a number M is the root of $\frac{1}{x} - M = 0$

Let $f(x) = \frac{1}{x} - M$

Then, $f'(x) = -\frac{1}{x^2}$

From (2.24), we obtain

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - M\right)}{\left(\frac{-1}{x_n^2}\right)}$$

(or) $x_{n+1} = x_n(2 - Mx_n)$ (2.13.2)

$(n = 0, 1, 2, ...)$

is the N-R formula for getting reciprocal of a number M.

Since $0.05 = \frac{1}{20} < \frac{1}{13} < \frac{1}{10} = 0.1,$ we choose $x_0 = 0.075$ to obtain reciprocal of 13.

for $n = 0$, (2.13.2) gives us

$x_1 = x_0(2 - Mx_0) = 0.07688$

for $n = 1$;

$x_2 = 0.07688(2 - 13 \times 0.07688) = 0.07692$

for $n = 2$; $x_3 = 0.07692$

Therefore, the required reciprocal of 13 correct to 5 places of decimal is 0.0769.

2.12 SOME VARIANTS OF NEWTON-RAPHSON METHOD

In this section, we shall discuss some of variants of the N-R method and make note of the importance of these methods over the N-R method in the specific cases when it requires, considering some examples for the comparison.

2.12.1 Modified Newton-Raphson Method (or) Von-Mises Method

This method consists of finding the slope at initial iteration and obtaining these by using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} \tag{2.30}$$

$$(n = 0, 1, 2\ldots)$$

It can be seen from the formula (2.30) that the slopes are not calculated at each and every iteration as in N-R method but are obtained by drawing the parallel lines to $f'(x_0)$ from the points on the curve as shown in the following figure.

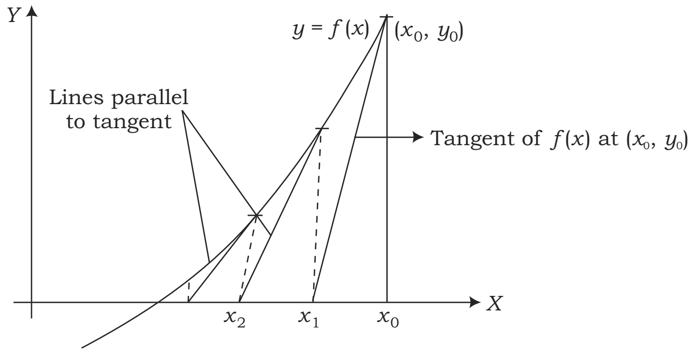

Fig. 2.6

Note 2.8: The Von-Mises method is useful in the cases when the evaluation of $f'(x)$ at every iteration requires a large amount of computational time and in such cases this modified N-R method takes less computational time than the N-R method.

2.12.2 Higher Order Newton's Method (or) Halley's Method

In section (2.11), we have obtained the value of h to get $x_1 = x_0 + h$ by neglecting the higher powers of h starting from the second degree term onwards as

$$h = -\frac{f(x_0)}{f'(x_0)} \tag{2.31}$$

If we neglect the terms in (2.23) from h^3 powers onwards, we get

$$h = -\frac{f(x_0)}{f'(x_0) + \frac{h}{2} f''(x_0)} \tag{2.32}$$

$$= -\frac{f(x_0)}{f'(x_0) - \frac{f(x_0)\, f''(x_0)}{2 f'(x_0)}} \quad \text{by (2.31)}$$

Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0) - \frac{f(x_0)\, f''(x_0)}{2 f'(x_0)}}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f(x_n)\, f''(x_n)}{2 f'(x_n)}} \tag{2.33}$$

$(n = 0, 1, 2\ldots)$

which is the Halley's method having third order convergence.

From (2.30), we have

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \qquad (2.14.2)$$

$$(n = 0, 1, 2, ...)$$

Let $x_0 = 1.5$.

Then, $f'(x_0) = 5.75$

The following approximations are obtained by applying the Von-Mises formula (2.14.2).

Table 2.12

$x_1 = 1.347826087$	$x_9 = 1.324718378$
$x_2 = 1.330316144$	$x_{10} = 1.324718066$
$x_3 = 1.326142416$	$x_{11} = 1.324717985$
$x_4 = 1.325084527$	$x_{12} = 1.324717964$
$x_5 = 1.324812558$	$x_{13} = 1.324717959$
$x_6 = 1.324742389$	$x_{14} = 1.324717958$
$x_7 = 1.324724268$	$x_{15} = 1.324717957$
$x_8 = 1.324719587$	$x_{16} = 1.324717957$

Since $x_{15} = x_{16}$, the required root of equations (2.14.1) is 1.32471795, which is correct to 8 decimal places.

Example 2.15: Solve the problem given in example (2.14) by using Halley's method.

Solution: Given $f(x) = x^3 - x - 1 = 0$ (2.15.1)

Then $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$

The Halley's method (2.33) for the solution of (2.15.1) is

$$x_{n+1} = x_n - \frac{f(x_n)\cdot f'(x_n)}{f'^2(x_n) - \frac{f(x_n)\cdot f''(x_n)}{2}} \qquad (2.15.2)$$

$$(n = 0, 1, 2, ...)$$

Using this formula with $x_0 = 1.5$, we obtained the following approximations

$x_1 = 1.327253219,$ $x_2 = 1.324717968$

$x_3 = 1.324717957,$ $x_4 = 1.324717957$

Since $x_3 = x_4$, the desired root of the equation is $x = 1.324717957$.

Example 2.16: Solve $x^3 - x - 10 = 0$ to obtain a root by using extrapolated N-R method.

Solution: Given $f(x) = x^3 - x - 10$

So $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$

The extrapolated N-R method (2.39) in this case takes the form

$$x_{n+1} = x_n - \frac{(x_n^3 - x_n - 10)}{(3x_n^2 - 1) - \dfrac{(x_n^3 - x_n - 10)(6x_n)}{2(3x_n^2 - 1)}}$$

$$(n = 0, 1, 2, 3 \ldots)$$

Since $f(2) < 0$ and $f(3) > 0$, there is root in the interval (2, 3). Let us choose $x_0 = 2.5$ Then, the above formula gives us the following successive approximations.

$x_1 = 2.3097942563$, $\quad x_2 = 2.3089073198$

$x_3 = 2.3089073197$, $\quad x_4 = 2.3089073197$

Therefore, the root of the equation is $x_3 = 2.3089073197$ which is correct to 10 places of decimal as error reduces cubically in this method.

2.13 NEWTON-RAPHSON METHOD FOR MULTIPLE ROOTS (OR) GENERALIZED NEWTON-RAPHSON (GN-R) METHOD

If x^* is a root of the equation $f(x) = 0$ with multiplicity $m > 1$ then $f(x)$ can be expressed as

$$f(x) = (x - x^*)^m \cdot g(x) \tag{2.40}$$

where $g(x)$ is bounded and $g(x^*) \neq 0$ and

$$f(x^*) = f'(x^*) = f''(x^*) = \ldots = f^{(m-1)}(x^*) = 0 \;\&\; f^{(m)}(x^*) \neq 0$$

The generalized N-R method in the case of (2.34) is defined by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{2.41}$$

which is the N-R method for multiple roots having second order convergence.

2.14 GENERALIZED EXTRAPOLATED NEWTON-RAPHSON (GEN-R) METHOD

If η is a root of (2.1) with multiplicity m, then the generalized Newton-Raphson method is defined as

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{2.42}$$

$$(n = 0, 1, 2 \ldots)$$

We develop generalized extrapolated Newton-Raphson (GEN-R) method by introducing computational parameters α_n of the form

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, x^*_{n+1} \tag{2.43}$$

where x^*_{n+1} is x_{n+1} of the generalized N-R method (2.41).

Now the GEN-R method for multiple roots of equation (2.1) can be defined as

$$x_{n+1} = x_n - m\alpha_n \frac{f(x_n)}{f'(x_n)} \tag{2.44}$$

2.14.1 Convergence Criteria of GEN-R Method

As it is well known that any iterative method of the form $x_{n+1} = \phi(x_n)$ converges if $|\phi'(x_n)| < 1$ for all x in I. Hence, the method (2.44) converges under the condition

$$\mu = |1 - m\alpha_n + m\alpha_n\, \omega_n| < 1 \text{ for all } x \text{ in } I \tag{2.45}$$

where

$$\omega_n = \frac{f(x_n)\, f''(x_n)}{[f'(x_n)^2]} \tag{2.46}$$

The function $f(x)$ in the immediate neighbourhood of $x = \eta$, can be written as

$$f(x) = k\cdot(x - \eta)^m$$

where $k \simeq k(\eta)$ is effectively constant.

Then,
$$f'(x) = k\cdot m(x - \eta)^{m-1}$$
$$f''(x) = k\cdot m(m - 1)\ (x - \eta)^{m-2}$$

We thus have

$$\frac{f(x_n)\, f''(x_n)}{[f'(x_n)^2]} = \frac{k\cdot(x-\eta)^m\ km(m-1)\,(x-\eta)^{m-2}}{[k\cdot m(x-\eta)^{m-1}]^2} = \frac{m-1}{m} \tag{2.47}$$

We need to find a real value of α_n for each iteration, which minimizes μ of (2.45). Since ω_n of (2.46) is positive and real for all x, as noted in the Remark 2.4 in the immediate vicinity of η, we have in general

$$\frac{m-1}{m} \le \omega_n \le \frac{f(x_n)\, f''(x_n)}{[f'(x_n)^2]} \tag{2.48}$$

$$(n = 0, 1, 2\ \ldots)$$

The process of minimizing μ of (2.45) keeping in view of (2.47) with respect to α_n, gives the optimal choice for α_n as

$$m\alpha_n = \frac{2}{2 - \left(\dfrac{m-1}{m} + \dfrac{ff''}{[f']^2}\right)}$$

$$\Rightarrow \qquad \alpha_n = \frac{2}{m + 1 - m\omega_n} \tag{2.49}$$

$$(n = 0, 1, 2\ \ldots)$$

With this optimal choice of α_n, μ of (2.45) takes the form

$$\mu = \left|1 - m\frac{2}{m+1-m\omega_n} + m\frac{2\omega_n}{m+1-m\omega_n}\right|$$

$$= \left|\frac{m+1-m\omega_n - 2m + 2m\omega_n}{m+1-m\omega_n}\right|$$

$$= \left|\frac{1-m(1-\omega_n)}{1+m(1-\omega_n)}\right| \tag{2.50}$$

Which is always less than unity as long as ω_n of (2.46) less than one in magnitude and hence the convergence of the GEN-R method (2.44) is assured.

Note 2.9: It can be seen the GEN-R method (2.44) with the choice of α_n of (2.49) in the case when m = 1 takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \dfrac{f(x_n)f''(x_n)}{2[f'(x_n)^2]}} \tag{2.51}$$

$$(n = 0, 1, 2 \ldots)$$

which is the cubic convergence higher order Newton's method for finding a simple root of the equation (2.1).

2.14.2 Rate of Convergence of Generalized Extrapolated Newton-Raphson Method

Let η be the multiple root of the equation $f(x) = 0$ and ε_{n+1}, ε_n be errors when x_{n+1}, x_n are the $(n + 1)^{\text{th}}$ and n^{th} approximates.

Then, $$x_{n+1} = \eta + \varepsilon_{n+1},\ x_n = \eta + \varepsilon_n \tag{2.52}$$

Substituting these values of x_{n+1} and x_n in (2.44), we have from (2.49)

$$\varepsilon_{n+1} = \varepsilon_n - m\frac{2}{m+1-m\dfrac{ff''}{[f']^2}}\frac{f}{f'}$$

$$= \varepsilon_n - \frac{2ff'}{\left(\dfrac{m+1}{m}\right)[f']^2 - ff''}$$

(ii) When $m = 2$ $\varepsilon_{n+1} \propto \varepsilon_n^3 \left[\frac{1}{24}\left(\frac{f'''}{f''}\right)^2 - \frac{1}{24}\left(\frac{f^{iv}}{f''}\right) \right]$

(iii) When $m = 3$ $\varepsilon_{n+1} \propto \varepsilon_n^3 \left[\frac{1}{2592}\left(\frac{f^{iv}}{f'''}\right)^2 - \frac{1}{2160}\left(\frac{f^{v}}{f'''}\right) \right]$

In general, one can have

$$\varepsilon_{n+1} \propto \varepsilon_n^3 \, k$$

where $k = k_1 \left(\frac{f^{(m+1)}}{f^m} \right)^2 - k_2 \left(\frac{f^{(m+2)}}{f^m} \right)$

where k_1 and k_2 are constants.

Hence, the GEN-R method has a cubic rate of convergence.

Example 2.17: Find the double root of the equation $x^2 - 3x + 2 = 0$ starting with $x_0 = 0$ by GN-R method.

Solution: Let $f(x) = x^3 - 3x + 2$

Then, $f'(x) = 3x^2 - 3$

For $m = 2$, the GN-R method given as

$$x_{n+1} = x_n - 2 \cdot \frac{x_n^3 - 3x_n + 2}{3(x_n^2 - 1)}$$

$$(n = 0, 1, 2, 3, \ldots\ldots\ldots\ldots)$$

Starting with $x_0 = 0$, we obtain the following approximations.

$x_1 = 1.3333333$, $x_2 = 1.0158730$

$x_3 = 1.0000407$, $x_4 = 1.0000407$

Therefore, the required double root of the equation is 1.0000407 which is almost close to the actual double root $x = 1$ of the given equation.

Example 2.18: Solve example 2.17 by using the GEN-R method.

Solution: Let $f(x) = x^3 - 3x + 2$

Then, $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

The GEN-R method (2.44) in this case takes the form

$$x_{n+1} = x_n - \frac{4f(x_n) \cdot f'(x_n)}{3f'^2(x_n) - 2f(x_n) \cdot f''(x_n)} \qquad (2.18.1)$$

For the given problem, (2.18.1) gives us

$$x_{n+1} = x_n - \frac{4(x_n^3 - 3x_n + 2)(3x_n^2 - 3)}{27(x_n^2 - 1)^2 - 12(x_n^3 - 3x_n + 2)\, x_n} \qquad (2.18.2)$$

$$(n = 0, 1, 2, ...)$$

If we choose $x_0 = 0$, then one can have the following successive iterates from (2.18.2)

$x_1 = 0.8888889$, $x_2 = 0.9999372$, $x_3 = 1.0000000$

Therefore, the double root of the equation is $x = 1$.

Example 2.19: Find the triple root of $27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1 = 0$ correct to six places of decimal, using the GEN-R method.

Solution: Given $f(x) = 27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1$

$\Rightarrow f'(x) = 135x^4 + 108x^3 + 108x^2 + 56x + 9$, $f''(x) = 540x^3 + 324x^2 + 216x + 56$

The GEN-R method (2.44) in this case takes the form

$$x_{n+1} = x_n - \frac{6f(x_n)\, f'(x_n)}{4[f'(x_n)]^2 - 3f(x_n)\, f''(x_n)} \qquad (2.19.1)$$

For the given problem, (2.19.1) gives us

$$x_{n+1} = x_n - \frac{6(27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1)(135x^4 + 108x^3 + 108x^2 + 56x + 9)}{4(135x^4 + 108x^3 + 108x^2 + 56x + 9)^2 - 3\begin{bmatrix}(27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1)\\ \times (540x^3 + 324x^2 + 216x + 56)\end{bmatrix}}$$

$$(n = 0, 1, 2, ...) \qquad (2.19.2)$$

Using this formula (2.19.2) with $x_0 = -1$, we obtained the following approximations

$x_1 = -0.3465347$, $x_2 = -0.3333328$, $x_3 = -0.3333324$.

Since the difference between 2nd and 3rd approximations are less than 0.5×10^{-6}, the required solution is $x = -0.333332$

2.15 TWO-STEP ITERATIVE METHODS FOR SOLVING NON-LINEAR EQUATIONS

In this section, we discuss a two-step iterative methods for solving non-linear equations of the form (2.1).

2.15.1 Accelerated Iterative Method

Let α be the exact root of the equation (2.1) in an open interval D in which $f(x)$ is continuous and has well defined first derivative and let x_n be the n^{th} approximate to the root of (2.1), then

Taking α in (2.53) as $(n + 1)^{\text{th}}$ approximate to the root, from (2.53) and (2.60), the two-step accelerated iterative method can be defined as follows.

(i) Calculate

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.62}$$

(ii) Obtain

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}\left[\frac{1}{1+\left(1-\frac{4f(y_n)}{f(x_n)}\right)^{\frac{1}{2}}}\right] \tag{2.63}$$

$(n = 0, 1, 2 ...)$

This method is free from the second derivative and requires two functional evaluations and one of its first derivatives. The efficiency index of this method is $\sqrt[3]{4}$.

2.15.2 Convergence Criteria of Accelerated Iterative Method

If α is the root and x_n is the n^{th} approximate to the root, then expanding $f(x_n)$ about α using Taylor's expansion, we have

$$f(x_n) = f(\alpha) + f'(\alpha)\,e_n + \frac{e_n^2}{2!}f''(\alpha) + \frac{e_n^3}{3!}f'''(\alpha) + \frac{e_n^4}{4!}f^{iv}(\alpha) + \frac{e_n^5}{5!}f^{v}(\alpha) + O(e_n^6)$$

$$= f'(\alpha)\left[e_n + \frac{1}{2!}\frac{f''(\alpha)}{f'(\alpha)}e_n^2 + \frac{1}{3!}\frac{f'''(\alpha)}{f'(\alpha)}e_n^3 + \frac{1}{4!}\frac{f^{iv}(\alpha)}{f'(\alpha)}e_n^4 + \frac{1}{5!}\frac{f^{v}(\alpha)}{f'(\alpha)}e_n^5 + O(e_n^6)\right]$$

$$= f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)\right] \tag{2.64}$$

where $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$, $(j = 2, 3, 4\ldots)$

and, $f'(x_n) = f'(\alpha)\,[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)]$ (2.65)

Now,

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 - (2c_3 - 2c_2^2)\,e_n^3 - (3c_4 - 7c_2 c_3 + 4c_2^3)\,e_n^4 + O(e_n^5) \tag{2.66}$$

From (2.53), (2.62) and (2.66), we have

$$y_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)\,e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\,e_n^4 + O(e_n^5) \tag{2.67}$$

$$f(y_n) = f'(\alpha)\,[c_2 e_n^2 - (2c_2^2 - 2c_3)\,e_n^3 - (7c_2c_3 - 5c_2^3 - 3c_4)\,e_n^4 + O(e_n^5) \tag{2.68}$$

Thus,

$$2\frac{f(x_n)}{f'(x_n)}\left[1+\left(1-4\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{2}}\right]^{-1}$$

$$=\frac{2[e_n-c_2e_n^2-(2c_3-2c_2^2)\,e_n^3-(3c_4-7c_2c_3+4c_2^3)\,e_n^4+O(e_n^5)]}{2[1-c_2e_n+(2c_2^2-2c_3)\,e_n^2+(6c_2c_3-4c_2^3-3c_4)\,e_n^3+O(e_n^4)]}$$

$$=e_n-c_2e_n^2-(2c_3-2c_2^2)e_n^3-(3c_4-7c_2c_3+4c_2^3)e_n^4+O(e_n^5)$$
$$\times[1-\{c_2e_n+(2c_3-2c_2^2)e_n^2+(4c_2^3+3c_4-6c_2c_3)e_n^3+O(e_n^4)]^{-1}$$

$$=[e_n-c_2e_n^2-(2c_3-2c_2^2)\,e_n^3-(3c_4-7c_2c_3+4c_2^3)\,e_n^4+O(e_n^5)]$$
$$\times[1+c_2e_n+(2c_3-2c_2^2)\,e_n^2+(4c_2^3+3c_4-6c_2c_3)\,e_n^3$$
$$+(c_2^2e_n^2+(2c_3-2c_2^2)^2\;e_n^4+2c_2(2c_3-2c_2^2)\,e_n^3+2c_2(4c_2^3+3c_4-6c_2c_3)\,e_n^4$$
$$+(c_2^3e_n^3+2c_2^2(2c_3-2c_2^2)\,e_n^4+c_2^2(2c_3-2c_2^2)\,e_n^4+c_2^4e_n^4]$$

$$=e_n-c_2e_n^2-(2c_3-2c_2^2)\,e_n^3-(3c_4-7c_2c_3+4c_2^3)\,e_n^4+O(e_n^5)$$
$$\times[1+c_2e_n+(2c_3-c_2^2)\,e_n^2+(4c_2^3+3c_4-6c_2c_3+4c_2c_3-4c_2^3+c_2^3)\,e_n^3+O(e_n^4)]$$

$$=[e_n+c_2e_n^2+2c_3e_n^3-c_2^2e_n^3+(c_2^3+3c_4-6c_2c_3)\,e_n^4$$
$$-c_2e_n^2-c_2^2e_n^3-(2c_2c_3+c_2^3)\,e_n^4-(2c_3-2c_2^2)\,e_n^3-c_2(2c_3-2c_2^2)\,e_n^4$$
$$-(3c_4-7c_2c_3+4c_2^3)\,e_n^4+O(e_n^5)]$$

$$=[e_n+(c_2-c_2)\,e_n^2+(2c_3-c_2^2-c_2^2+2c_2^2-2c_3)\,e_n^3+(2c_2^3-c_2^3-4c_2^3+c_2^3+3c_4-3c_4$$
$$-2c_2c_3-2c_2c_3-2c_2c_3+7c_2c_3)\,e_n^4+O(\mathrm{e}_n^5)]$$

$$=e_n+(c_2c_3-2c_2^3)\,e_n^4+O(e_n^5) \tag{2.73}$$

From (2.53), (2.63) and (2.73), we have the rate of convergence of the method (2.63) is four.

2.15.3 Variant of Acccelerated Iterative Method

By expanding $\sqrt{1-4\frac{f(y_n)}{f(x_n)}}$ appearing in the denominator of the method (2.63), we obtain

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left\{\frac{1}{1-\left\{\frac{f(y_n)}{f(x_n)}+\left(\frac{f(y_n)}{f(x_n)}\right)^2+2\left(\frac{f(y_n)}{f(x_n)}\right)^3+\ldots\right\}}\right\} \tag{2.74}$$

And, the above further yields

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1+\frac{f(y_n)}{f(x_n)}+2\left(\frac{f(y_n)}{f(x_n)}\right)^2+5\left(\frac{f(y_n)}{f(x_n)}\right)^3+\ldots\right] \tag{2.75}$$

Considering up to the first and second degree terms of the above expression lying within the brackets of the formula (2.75), the two-step variant of accelerated iterative method is defined by

(i) Calculate

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

(ii) Obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1+\frac{f(y_n)}{f(x_n)}+2\left(\frac{f(y_n)}{f(x_n)}\right)^2\right] \tag{2.76}$$

(n = 0, 1, 2...)

2.15.4 Convergence Criteria of Variant of Accelerated Iterative Method

As done in section (2.15.2), one can easily obtain the error relation as

$$\alpha + e_{n+1} = \alpha + e_n - [e_n + (c_2c_3 - 5c_2^3)\, e_n^4 + O(e_n^5) + \ldots]$$

which gives us

$$e_{n+1} \propto e_n^4 \tag{2.77}$$

Therefore, the method (2.76) has fourth order convergence.

Note 2.10: If we neglect the second degree term lying within the brackets of the variant of accelerated iterative method, one will be left with the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1+\frac{f(y_n)}{f(x_n)}\right] \tag{2.78}$$

which has cubic rate of convergence.

2.15.5 Two-Step Extrapolated Newton-Raphson (ENR) Method

Let α be the exact root of the equation (2.1) in an open interval D in which $f(x)$ is continuous and has well defined first derivative and let x_n be the n^{th} approximate to the root of (2.1)

If we let

$$y_n = x_n + h \tag{2.79}$$

where

$$h = -\frac{f(x_n)}{f'(x_n)}$$

$$= f'(\alpha)\left[e_n + \frac{1}{2!}\frac{f''(\alpha)}{f'(\alpha)}e_n^2 + \frac{1}{3!}\frac{f'''(\alpha)}{f'(\alpha)}e_n^3 + \frac{1}{4!}\frac{f^{iv}(\alpha)}{f'(\alpha)}e_n^4 + \frac{1}{5!}\frac{f^{v}(\alpha)}{f'(\alpha)}e_n^5 + O(e_n^6)\right]$$

$$= f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)\right] \tag{2.85}$$

and $f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)\right]$ (2.86)

$$c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$$

where (j = 2, 3, 4…)

Now, from (2.80) and (2.81), we obtain

$$-\frac{f(x_n)}{f'(x_n)} = -e_n + c_2e_n^2 + (2c_3 - 2c_2^2)\,e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\,e_n^4 + O(e_n^5) \tag{2.87}$$

From (2.83), (2.84) and (2.87), we have

$$y_n = \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)\,e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\,e_n^4 + O(e_n^5)$$

and

$$f(y_n) = f'(\alpha)\,[c_2e_n^2 - (2c_2^2 - 2c_3)\,e_n^3 - (7c_2c_3 - 5c_2^3 - 3c_4)\,e_n^4 + O(e_n^5) \tag{2.88}$$

From (2.84) and (2.88), we have

$$f(x_n) - f(y_n) = f'(\alpha)\,[e_n + (2c_2^2 - c_3)\,e_n^3 + (7c_2c_3 - 5c_2^3 - 2c_4)\,e_n^4 + O(e_n^5)] \tag{2.89}$$

and

$$\frac{f(x_n)}{f(x_n) - f(y_n)} = \left[1 + c_2e_n + c_3e_n^2 + c_4e_n^3 + c_5e_n^4 + O(e_n^5)\right]$$

$$\times [e_n + (2c_2^2 - c_3)\,e_n^3 + (7c_2c_3 - 5c_2^3 - 2c_4)\,e_n^4 + O(e_n^5)]^{-1}$$

$$= 1 + c_2e_n + (2c_3 - 2c_2^2)\,e_n^2 + (3c_4 + 3c_2^3 - 6c_2c_3)\,e_n^3 + O(e_n^4)] \tag{2.90}$$

Finally,

$$\left[\frac{f(x_n)}{f(x_n) - f(y_n)}\right]\frac{f(x_n)}{f'(x_n)} = -e_n - c_2^2e_n^3 - O(e_n^4) \tag{2.91}$$

With (2.91), (2.82) yields

$$e_{n+1} = c_2^2\,e_n^3 + O(e_n^4)$$

Which shows that the method (2.82) has third order convergence.

2.15.7 Two-Step Derivative Free Extrapolated Newton's Method

As it is known that the backward difference approximation for the first derivative for $f'(x)$ at x is

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \tag{2.92}$$

and,

$$\begin{aligned} f(x_n - f(x_n)) = \; & (-c_1^2 + c_1)\, e_n + (c_2 c_1^2 - 3c_2 c_1 + c_2)\, e_n^2 \\ & + (-c_3 c_1^3 + 3c_3 c_1^2 + 2c_1 c_2^2 - 4c_3 c_1 - 2c_2^2 + c_3)\, e_n^3 \\ & + (c_4 + c_2 (c_2^2 + 2c_1 c_3) - 5c_1 c_4 - 5c_2 c_3 \\ & + 6c_1^2 c_4 - 4c_1^3 c_4 + c_1^4 c_4 + 6c_1 c_2 c_3 - 3c_1^2 c_2 c_3)\, e_n^4 + O(e_n^5) \end{aligned} \tag{2.98}$$

Subtracting (2.96) from (2.98), we get

$$\begin{aligned} & f(x_n) - f(x_n - f(x_n)) \\ & = c_1^2 e_n + (-c_2 c_1^2 + 3c_2 c_1)\, e_n^2 + (c_3 c_1^3 - 3c_3 c_1^2 - 2c_1 c_2^2 + 4c_3 c_1 + 2c_2^2)\, e_n^3 \\ & \quad + (-c_2 (c_2^2 + 2c_1 c_3) + 5c_1 c_4 + 5c_2 c_3 - 6c_1^2 c_4 \\ & \quad + 4c_1^3 c_4 - c_1^4 c_4 - 6c_1 c_2 c_3 + 3c_1^2 c_2 c_3)\, e_n^4 + O(e_n^5) \end{aligned} \tag{2.99}$$

Also

$$\begin{aligned} [f(x_n)]^2 & = [c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)]^2 \\ & = c_1^2 e_n^2 + 2c_1 c_2 e_n^3 + (c_2^2 + 2c_1 c_3)\, e_n^4 + (2c_2 c_3 + 2c_1 c_4)\, e_n^5 \end{aligned} \tag{2.100}$$

From (2.99) and (2.100), we get

$$\begin{aligned} \frac{[f(x_n)]^2}{f(x_n) - f(x_n - f(x_n))} = \; & \frac{c_1^2 e_n^2 + 2c_1 c_2 e_n^3 + (c_2^2 + 2c_1 c_3)\, e_n^4 + (2c_2 c_3 + 2c_1 c_4)\, e_n^5 + O(e_n^6)}{\begin{aligned} & \{c_1^2 e_n + (-c_2 c_1^2 + 3c_2 c_1)\, e_n^2 + (c_3 c_1^3 - 3c_3 c_1^2 - 2c_1 c_2^2 + 4c_3 c_1 + 2c_2^2)\, e_n^3 \\ & + (-c_2 (c_2^2 + 2c_1 c_3) + 5c_1 c_4 + 5c_2 c_3 - 6c_1^2 c_4 + 4c_1^3 c_4 \\ & - c_1^4 c_4 - 6c_1 c_2 c_3 + 3c_1^2 c_2 c_3) e_n^4 + O(e_n^5)\} \end{aligned}} \end{aligned}$$

$$= \frac{e_n + 2\dfrac{c_2}{c_1} e_n^2 + \dfrac{(c_2^2 + 2c_1 c_3)}{c_1^2} e_n^3 + \dfrac{(2c_2 c_3 + 2c_1 c_4)}{c_1^2} e_n^4}{\left\{ \begin{aligned} & 1 + \frac{(-c_2 c_1^2 + 3c_2 c_1)}{c_1^2} e_n + \frac{(c_3 c_1^3 - 3c_3 c_1^2 - 2c_1 c_2^2 + 4c_3 c_1 + 2c_2^2)}{c_1^2} e_n^2 \\ & + \left[\frac{\begin{aligned} & (-c_2 (c_2^2 + 2c_1 c_3) + 5c_1 c_4 + 5c_2 c_3 - 6c_1^2 c_4 + 4c_1^3 c_4 \\ & - c_1^4 c_4 - 6c_1 c_2 c_3 + 3c_1^2 c_2 c_3) \end{aligned}}{c_1^2} \right] e_n^3 + O(e_n^4) \end{aligned} \right\}}$$

$$\begin{aligned} = & \left[e_n + 2\frac{c_2}{c_1} e_n^2 + \frac{(c_2^2 + 2c_1 c_3)}{c_1^2} e_n^3 + \frac{(2c_2 c_3 + 2c_1 c_4)}{c_1^2} e_n^4 \right] \\ & \times \left[\begin{aligned} & 1 - \left\{ \frac{(-c_2 c_1^2 + 3c_2 c_1)}{c_1^2} e_n + \frac{(c_3 c_1^3 - 3c_3 c_1^2 - 2c_1 c_2^2 + 4c_3 c_1 + 2c_2^2)}{c_1^2} e_n^2 \right\} \\ & + \frac{(-c_2 c_1^2 + 3c_2 c_1)^2}{c_1^4} e_n^2 + 2\,\frac{(-c_2 c_1^2 + 3c_2 c_1)(c_3 c_1^3 - 3c_3 c_1^2 - 2c_1 c_2^2 + 4c_3 c_1 + 2c_2^2)}{c_1^4} e_n^3 \end{aligned} \right] \end{aligned}$$

$$\times\left[1+\left(\frac{c_2}{c_1}-c_2\right)e_n-\frac{(3c_3c_1^4-c_1^5c_3-c_1^3c_2^2-2c_3c_1^3+2c_1^2c_2^2)}{c_1^4}e_n^2+O(e_n^3)\right] \tag{2.105}$$

with (2.105), (2.95) and (2.96) one can have

$$e_{n+1}+\alpha = e_n+\alpha-\left[e_n+\frac{c_2^2}{c_1}\left(1-\frac{1}{c_1}\right)e_n^3\right] \tag{2.106}$$

which yields

$$e_{n+1} \propto e_n^3$$

which shows the method (2.94) has cubic convergence.

Example 2.20: Find a root of the equation $x^3 + 4x^2 - 10 = 0$ which is near about $x = 1$ by using the two-step accelerated iterative method tabulating all computations.

Solution: Let $f(x) = x^3 + 4x^2 - 10$

then $f'(x) = 3x^2 + 8x$

The two-step accelerated iteration method is given by

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}\left[\frac{1}{1+\left(1-\frac{4f(y_n)}{f(x_n)}\right)^{\frac{1}{2}}}\right]$$

$$(n = 0, 1, 2\ldots\ldots\ldots)$$

where

$$y_n = x_n - \frac{x_n^3+4x_n^2-10}{3x_n^2+8x_n}$$

Taking the initial guess x_0 as unity, the computations for obtaining the solution of the given equation using the above method are tabulated below:

Table 2.13

n	x_n	$f(x_n)$	$f'(x_n)$	y_n	$f(y_n)$	$1-4\frac{f(y_n)}{f(x_n)}$
0	1	−5	11	1.45454545	1.54019534	2.23215627
1	1.36450531	−0.01196310	16.5016667	1.36523027	$0.42541061 \times 10^{-5}$	1.00142241
2	1.36523001	$-0.13500312 \times 10^{-12}$	16.5133991	1.36523001	0	1

From the above tabulated results, we thus have the required root of the equation as $x = 1.36523001$.

Example 2.21: Obtain a root of the equation $x^2 - e^x - 3x + 2 = 0$ tabulating all computations, by using the two-step variant of accelerated iterative method, taking $x_0 = 1$.

Solution: Let $f(x) = x^2 - e^x - 3x + 2$

Then, $f'(x) = 2x - e^x - 3$

The variant of accelerated iterative method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(y_n)}{f(x_n)} + 2\left(\frac{f(y_n)}{f(x_n)}\right)^2\right]$$

$$(n = 0, 1, 2\ldots)$$

where $y_n = x_n - \dfrac{x_n^2 - e^{x_n} - 3x_n + 2}{2x_n - e^{x_n} - 3}$

Taking $x_0 = 1$ and applying the above scheme, the successive approximations are tabulated below:

Table 2.14

n	x_n	$f(x_n)$	$f'(x_n)$	y_n	$f(y_n)$	x_{n+1}
0	1	−2.71828183	−3.71828183	0.26894142	−0.04307326	0.25699012
1	0.25699012	$0.20412214 \times 10^{-2}$	−3.77905211	0.25753026	0.1030958×10^{-6}	0.25753029
2	0.25753029	$-0.88817842 \times 10^{-15}$	−3.77867042	0.25753029	$0.44408921 \times 10^{-15}$	0.25753029

From the above table, we have the required root $x = 0.25753029$.

Example 2.22: Solve $\sin^2(x) - x^2 + 1 = 0$ with $x_0 = 5$, by using

(i) Two-step accelerated iterative method (2.63)

(ii) Variant of two-step accelerated iterative method (2.76).

Tabulate all computations.

Solution:

(i) with $x_0 = 5$, the computations of accelerated iterative method are given below:

Table 2.15

n	x_n	$f(x_n)$	$f'(x_n)$	y_n	$f(y_n)$	$1 - 4\dfrac{f(y_n)}{f(x_n)}$
0	5	−23.0804642	−10.5440211	2.81103774	−6.79658885	−0.17789465

EXERCISES

1. Find graphically the positive root of the equation $x^3 - 6x - 13 = 0$
2. Using graphical method, find an approximate root of the equation $x - 1 = \sin x$.
3. Find the positive real root of the equation $x \log_{10} x = 1.2$ using bisection method.
4. By using bisection method, find an approximate root of the equation $\sin x = \frac{1}{x}$ that lies between $x = 1$ and $x = 1.5$. Carry out computations to obtain the root correct to 3 decimals.
5. Find by the iteration method, the root near 3.8 of the equation $2x - \log_{10} x = 7$ correct to four decimal places.
6. Solve $x = 1 + \tan^{-1} x$ by iteration method.
7. By the Wegstein's method, find the root of the equation $x^3 - 2x - 5 = 0$ that is near about $x = 2$, correct to 3 decimal places.
8. Find a real root of $\cos x = 3x - 1$ by Wegstein's method correct to three decimal places.
9. Find a root of $\cos x - xe^x = 0$ correct to 3 decimal places by Aitken's Δ^2-method, with $x_0 = 0$.
10. Perform Aitken's Δ^2 process to obtain the root of the equation $x_{n+1} = \sin x_n + \frac{1}{2}$ after 3 approximations to the root are available which are obtained by iteration method taking $x = 1$. How many Aitken Δ^2 process iterations are required to obtain the root correct to 6 decimals.
11. Find the root of the equation $\sin x = 1 + x^3$ between -2 and -1 by extrapolated iteration method, correct to 3 decimal places.
12. Find the real root of $x + \log - 2 = 0$ by extrapolated iteration method, correct to 5 decimals places.
13. Solve $x^3 = 2x + 5$ for a positive root by extrapolated iteration method.
14. Find the real root of the equation $3x + \sin x - e^x = 0$ by the method of false-position method correct to four decimal places.
15. Find all real zeros of the function $f(x) = x^3 - x^2 - 24x - 32$ by Regula-Falsi method.
16. A real root of the equation $f(x) = x^3 - 5x + 1 = 0$ lies in the interval (0, 1). Perform four iterations of the secant method.
17. Computer the root of the equation $x^2 e^{\frac{-x}{2}} = 1$ in the interval (0, 2) using the secant method correct to three decimal places.
18. Find a root of $e^x \sin x = 1$ using N-R method correct to 3 decimal places.
19. Using the N-R method, find the root of the equation $e^x = 3x$ that lies between 0 and 1, correct to 6 decimal places.
20. Find all real zeros of $f(x) = x^3 - 9x + 2$ by N-R method.

(iv) Find the positive root of $\sin^2 x - x^2 + 1 = 0$ correct to three decimal places.

(v) Find the real root of the equation $e^x - 4x = 0$ correct to three decimals.

(vi) Find the real root of the equation $\cos x - xe^x = 0$ correct to three decimals.

(vii) Find a real root of the equation $3x + \sin x - e^x = 0$ correct to four decimal places.

(viii) Find a real root of the equation $x^2 - e^x - 3x + 2 = 0$ correct to three decimal places.

(ix) Find the approximate root of $x^4 - x - 10 = 0$ given that the root lies between 1.8 and 2 correct to four decimals.

(x) Find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

(xi) Find root of the equation $xe^{x^2} - \sin^2 x + 3\cos x + 5 = 0$ correct to four decimal places, given that root lies between 0 and 1.

(xii) Find to four places of decimal, the smallest root of the equation $e^{-x} = \sin x$.

(xiii) Find the root of the equation $\sin x = 1 + x^3$ between -2 and -1 correct to 3 places of decimal.

(xiv) Find a real root of $x + \log x - 2 = 0$ correct to 5 decimals.

(xv) Find real root of the equation $\cos x - x = 0$ correct to three decimal places.

(xvi) Compute the root of the equation $x^2 e^{\frac{-x}{2}} = 1$ in the interval (0, 2) correct to three decimal places.

(xvii) Compute the root of the equation $\sin x - \left(\frac{x}{2}\right)^2 = 0$ in the interval [1, 2] and correct to four decimals.

(xviii) Solve the cubic $x_3 - 6x^2 + 8x + 0.8 = 0$ using $x_1 = 2.5$ & $x_2 = 2$ correct up to four decimals.

(xix) Obtain a root of the equation $x^{2.1} - 0.5x - 3 = 0$ using initial values 2 and 2.5.

(xx) The equation $x^4 - 5x^3 - 12x^2 + 76x - 79 = 0$ has two roots close to $x = 2$, find these roots to four decimal places.

(xxi) Find the double root of $x^3 - x^2 - x + 1 = 0$ close to 0.8.

(xxii) Obtain a root, correct to three decimal places for equation $x^3 + x^2 + x + 7 = 0$.

(xxiii) Compute the positive root of $x^3 - 2x - 8 = 0$ correct to two decimal places.

(xxiv) Find a real root of the equation $x^2 \sin^2 x + e^{x^2 \sin x \cos x} - 28 = 0$ correct to three decimal places.

(xxv) Find a real root of $x^4 - 4x - 9 = 0$ correct to 4 decimal places.

(xxvi) Find the root of the equation $xe^x = \cos x$ correct to four decimal places.

(xxvii) Find all real zeros of the function $f(x) = x^3 + 3x^2 - 41$ correct to four decimals

(xxviii) Find the smallest positive root of $x - e^{-x} = 0$ correct to three decimal places.

(xxix) Find the root of the equation $\log x - x + 1.5 = 0$ starting from $x_1 = 2$, $x_2 = 2.50$ compute up to 3 decimals.

(xxx) Find the fourth root of 32 correct to three decimal places.

31. −0.333344167
32. 1.36523001
33. −1.20764783
34. 4
35.

Equation	*Initial guess*	*No. of iterations taken by*		x^*
$f(x) = 0$	x_0	*Method 2.82*	*Method 2.94*	*Root*
(1) $xe^x - \cos x = 0$	0	5	4	0.517757363682458
(2) $e^x \sin x - 1 = 0$	−0.2	5	5	0.588532743981861
(3) $e^x - 1.5 - \tan^{-1} x = 0$	−7	4	4	−14.101269772739964

(i) 0.7391
(ii) 2.02
(iii) 0.61
(iv) 1.404
(v) 3.574
(vi) 0.515
(vii) 0.3604
(viii) 0.257
(ix) 1.8555
(x) 0.6071
(xi) −1.2076
(xii) 0.5885
(xiii) −1.249
(xiv) 1.55714
(xv) 0.739
(xvi) 1.429
(xvii) 1.9337
(xviii) 2.2020
(xix) 1.9265
(xx) 2.2410, 1.7684
(xxi) 1
(xxii) −2.105
(xxiii) 2.33
(xxiv) 3.437 or 4.622
(xxv) 2.6875
(xxvi) 0.5177
(xxvii) −2.8794, −0.6527, 0.5321
(xxviii) 0.567
(xxix) 2.359
(xxx) 2.378
(xxxi) 6.8506
(xxxii) −3.3240, −1.6197, 5.9437
(xxxiii) 0.8526
(xxxiv) 1.28
(xxxv) 1.524
(xxxvi) 2.33
(xxxvii) 2.73
(xxxviii) 2.2395
(xxxix) 1.4
(xl) −1.933
(xli) 3.130
(xlii) 0.619
(xliii) 3.2
(xliv) 1.93