Laws of Conservation

2.1 INTRODUCTION

Laws of conservation are generally the consequences of some underlying symmetry in the universe. In Physics, we come across a number of conservation laws such as conservation of energy, linear momentum, angular momentum, charge, mass, number of protons, neutrons, etc. These conservation laws are very powerful tools for solving the problems in Physics. A revolutionary consequence of conservation laws can be seen in the field of particle physics, by virtue of which we are able to predict the new elementary particles.

2.2 CONSERVATIVE AND NON-CONSERVATIVE FORCES

Conservative Force

If the work done by a force in moving a particle from one point to another depends only on these points and not on the path followed, the force is said to be conservative. The region in which a particle experiences a conservative force is called a conservative force field.

Properties of Conservative Force

(i) The work done by a conservative force around a closed path is always zero.

Proof Let us suppose that a particle moves from the point $A$ to the point $B$ under the influence of a conservative force $\vec{F}$.

Thus, work done,

$$ W = \int_{A}^{B} \vec{F} \cdot d\vec{r} $$
Since the work done by a conservative force depends only on the initial and final points and not on the path followed, the work done along the path $A X B$ and $A Y B$ will be the same, i.e.,

$$\int_{A}^{B} \vec{F} \cdot d\vec{r} = \int_{A}^{B} \vec{F} \cdot d\vec{r} = -\int_{A}^{B} \vec{F} \cdot d\vec{r}$$

or,

$$\int_{A}^{B} \vec{F} \cdot d\vec{r} + \int_{B}^{A} \vec{F} \cdot d\vec{r} = 0$$

or,

$$\oint_{AXBYA} \vec{F} \cdot d\vec{r} = 0$$

Thus, the work done by a conservative force around a closed path is zero. It is an important characteristic of conservative force.

**Note**  A force which is constant and uniform is always conservative.

**Proof**  Work done under such a force in a closed path,

$$W = \oint \vec{F} \cdot d\vec{r} = \oint \vec{F} \cdot d\vec{r} \quad \text{[since force is constant and uniform]}$$

$$= 0 \quad \text{[since } \oint d\vec{r} \text{ is zero for any closed curve]}$$

(ii) **Conservative force is the negative gradient of potential energy.**

**Proof**  We know that the potential energy of a particle at a point $r$ is equal to the work done by an applied force to bring the particle from infinity to that point, i.e., potential energy at a distance $r$,

$$U(r) = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} \text{ where } \vec{F} \text{ is a conservative force.}$$

Putting

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \text{ and } d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k},$$

we get

$$U(r) = -\int (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

or,

$$U(r) = -\int (F_x dx + F_y dy + F_z dz)$$
\[
\begin{align*}
\int_{A}^{B} & [d(x^2y) + d(z^2x)] \\
& = \int_{A}^{B} d(x^2y + z^2x) \\
& = [x^2y + z^2x]_{A(0,1,2)}^{B(5,2,7)} \\
& = (5)^2 \times 2 + (7)^2 \times 5 - 0 - 0 \\
& = 25 \times 2 + 49 \times 5 \\
& = 50 + 245 \\
& = 295 \text{ units}
\end{align*}
\]

### 2.3 SYSTEM OF PARTICLES

A collection of large number of particles (having mass but no size) interacting with one another is called the system of particles. A body may be regarded as a system of particles because a body is formed by the large number of particles which are interacting with one another.

#### Centre of Mass

The centre of mass is a representative point of a system of particles, on which the entire mass of the system is supposed to be concentrated and its motion is same as if the external force acting on the system is supposed to be applied directly on it.

If at any time \( t \), the position vectors of the masses \( m_1, m_2 \ldots m_N \) of \( N \) particles of a system are \( \vec{r}_1, \vec{r}_2 \ldots \vec{r}_N \) respectively with respect to any arbitrary origin \( O \), the position vector of the centre of mass of the system with respect to the origin \( O \) is given by

\[
\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots + m_N \vec{r}_N}{m_1 + m_2 + \ldots + m_N} \quad \ldots(1)
\]

\[
\frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i \quad \ldots(2)
\]

where \( M = \sum_{i=1}^{N} m_i = m_1 + m_2 + \ldots + m_N \) is the total mass of the system and \( \sum_{i=1}^{N} m_i \vec{r}_i \) is known as the first moment of mass for the system.
or,
\[ \vec{V} = \frac{1}{M} (m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_N \vec{v}_N) \]

or,
\[ \vec{V} = \sum_{i=1}^{N} \frac{m_i \vec{v}_i}{M} = \sum_{i=1}^{N} \frac{\vec{p}_i}{M} \]  \hspace{1cm} \text{(5)}

where \( \vec{V} \) is the velocity of the centre of mass; \( \vec{v}_1, \vec{v}_2 \ldots \vec{v}_N \) and \( \vec{p}_1, \vec{p}_2 \ldots \vec{p}_N \) are the velocities and the linear momenta of the particles respectively.

or,
\[ M \vec{V} = \sum_{i=1}^{N} m_i \vec{v}_i = \vec{P} \]  \hspace{1cm} \text{(6)}

where \( \vec{P} \) is the total linear momentum of the system, i.e., total linear momentum of the system is equal to the product of the total mass \( M \) of the system and velocity \( \vec{V} \) of the centre of mass.

If external force \( \vec{F}_{\text{ext}} \) acting on the system is zero, we have
\[ \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt} = 0 \]

or,
\[ \vec{P} = \text{constant} \]

or,
\[ M\vec{V} = \text{constant} \]

or,
\[ \vec{V} = \text{constant} \]

i.e., the velocity of the centre of mass of the system is constant in the absence of external force.

Differentiating Eq. (6) with respect to \( t \), we get
\[ M \frac{d\vec{V}}{dt} = \sum_{i=1}^{N} \frac{d}{dt} (m_i \vec{v}_i) \]
\[ = m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + \ldots + m_N \frac{d\vec{v}_N}{dt} \]
\[ = m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots + m_N \vec{a}_N \]
\[ = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_N \]
\[ M \vec{a} = \sum_{i=1}^{N} \vec{F}_i \]  \hspace{1cm} \text{(7)}

where \( \vec{a} = \frac{d\vec{V}}{dt} \) is the acceleration of the centre of mass; \( \vec{a}_1, \vec{a}_2 \ldots \vec{a}_N \) are the accelerations of the particles and \( \vec{F}_1, \vec{F}_2 \ldots \vec{F}_N \) are the forces acting on the particles having masses \( m_1, m_2 \ldots m_N \) respectively.
Now, let the fourth particle of mass 4 kg be placed at the point \((x_4, y_4, z_4)\) so that the centre of mass of the combination of all the four particles lies at \((1, 1, 1)\). Thus, X-coordinate of the centre of mass of this system will be

\[
X' = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}
\]

or,

\[
1 = \frac{1x_1 + 2x_2 + 3x_3 + 4x_4}{1 + 2 + 3 + 4} = \frac{x_1 + 2x_2 + 3x_3 + 4x_4}{10}
\]

or,  \(x_1 + 2x_2 + 3x_3 + 4x_4 = 10\)  \(\ldots(2)\)

Subtracting Eq. (1) from Eq. (2), we get

\[
4x_4 = 10 - 18 = -8
\]

or,

\[
x_4 = -2
\]

Similarly, we can solve for \(Y\) and \(Z\) coordinates of the centre of mass and will get

\[
y_4 = -2
\]

and

\[
z_4 = -2
\]

Thus, the fourth particle of mass 4 kg must be placed at the position \((-2, -2, -2)\) in order to get the centre of mass of the four particles system at the point \((1, 1, 1)\).

**Example 3** At a given time, two particles of masses 0.1 kg and 0.3 kg have their positions \((2\hat{i} + 5\hat{j} + 13\hat{k})\) metre and \((- 6\hat{i}, + 4\hat{j} - 2\hat{k})\) metre respectively and velocities \((10\hat{i} - 7\hat{j} - 3\hat{k})\) m/sec. and \((7\hat{i} - 9\hat{j} + 6\hat{k})\) m/sec respectively. Find (i) the instantaneous position of the centre of mass and (ii) the velocity of the second particle in a frame of reference attached with the centre of mass.

**Solution**

(i) Position vector of the centre of mass

\[
\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}
\]

\[
= \frac{0.1 (2\hat{i} + 5\hat{j} + 13\hat{k}) + 0.3 (- 6\hat{i} + 4\hat{j} - 2\hat{k})}{0.1 + 0.3}
\]

\[
= \frac{-16\hat{i} + 17\hat{j} + 7\hat{k}}{4} \text{ metre}
\]

(ii) Velocity of the centre of mass

\[
\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}
\]
\[
\begin{align*}
&= 0.1 (10\hat{i} - 7\hat{j} - 3\hat{k}) + 0.3 (7\hat{i} - 9\hat{j} + 6\hat{k}) \\
&= \frac{31\hat{i} - 34\hat{j} + 15\hat{k}}{4} \text{ m/sec.}
\end{align*}
\]

Therefore, velocity of the second particle in the centre of mass frame of reference
\[
\vec{v}_2 - \vec{V} = 7\hat{i} - 9\hat{j} + 6\hat{k} - \frac{31\hat{i} - 34\hat{j} + 15\hat{k}}{4}
\]
\[
= -\frac{3\hat{i} - 2\hat{j} + 9\hat{k}}{4} \text{ m/sec.}
\]

### 2.4 LAW OF CONSERVATION OF LINEAR MOMENTUM

If the resultant external force acting on a system of particles is zero, the total linear momentum of the system remains constant.

Let us consider a system of \( n \)-particles. If \( m_1, m_2 \ldots m_n \) are the masses of the particles and \( \vec{v}_1, \vec{v}_2 \ldots \vec{v}_n \) their respective velocities, the linear momenta of the particles will be \( \vec{p}_1 = m_1\vec{v}_1, \vec{p}_2 = m_2\vec{v}_2 \ldots \vec{p}_n = m_n\vec{v}_n \) respectively.

The total linear momentum of the system,
\[
\vec{P} = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n \tag{1}
\]

Differentiating with respect to time \( t \), we get
\[
\frac{d\vec{P}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \ldots + \frac{d\vec{p}_n}{dt}
\]
or,
\[
\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n \tag{2}
\]
where \( \vec{F} = \frac{d\vec{P}}{dt} \) gives the total force on the system while \( \vec{F}_1 = \frac{d\vec{p}_1}{dt}, \vec{F}_2 = \frac{d\vec{p}_2}{dt} \ldots \) are the forces acting on the particles of masses \( m_1, m_2 \ldots \) respectively.

Each particle may be influenced by two types of forces—internal and external. Internal forces arise due to interaction of particles with one another and are action-reaction forces. From Newton’s third law of motion, action-reaction forces are equal in magnitude but opposite in direction. So, they cancel each other. Therefore, the forces represented in Eq. (2) will obviously be external forces. If the resultant external force is zero, i.e.,
\[
\vec{F} = 0
\]
we have \[ \frac{d\vec{P}}{dt} = 0 \]
or, \[ \vec{P} = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n = \text{constant} \]

Thus, if the resultant external force acting on a system of particles is zero, the total linear momentum of the system remains constant.

**Note** Linear momentum is conserved only when the potential energy is translationally invariant.

**Proof** Let us consider a system of two particles having masses \( m_1 \) and \( m_2 \) and velocities \( \vec{v}_1 \) and \( \vec{v}_2 \) respectively. Let the position coordinates of the particles with respect to origin \( O \) be \( x_1 \) and \( x_2 \).

Thus, the potential energy of the system,
\[ U = U(x_1, x_2) \quad \ldots(1) \]
i.e., \( U \) is a function of position coordinates.

If displacement in the particles \( x' \) (say) does not change the value of \( U \), i.e., \( U \) is translationally invariant
\[ U(x_1 + x', x_2 + x') = U(x_1, x_2) \quad \ldots(2) \]
if
\[ x = x_1 - x_2 \quad \ldots(3) \]
\[ U(x_1 + x', x_2 + x') = U((x_1 + x') - (x_2 + x')) = U(x_1 - x_2) = U(x) \]

Now \[ \frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial x_1} = \frac{\partial U}{\partial x} \quad \text{[since from Eq. (3), } \frac{\partial x}{\partial x_1} = 1] \quad \ldots(4) \]
and \[ \frac{\partial U}{\partial x_2} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial x_2} = - \frac{\partial U}{\partial x} \quad \text{[since from Eq. (3), } \frac{\partial x}{\partial x_2} = -1] \quad \ldots(5) \]

Equating Eqs. (4) and (5), we get,
\[ \frac{\partial U}{\partial x_1} = - \frac{\partial U}{\partial x_2} \]
or, \[ \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0 \quad \ldots(6) \]

But, force on the first particle, \( \vec{F}_1 = - \frac{\partial U}{\partial x_1} \)

and force on the second particle, \( \vec{F}_2 = - \frac{\partial U}{\partial x_2} \)

Substituting in Eq. (6), we get
- \( \sum (\vec{F}_1 + \vec{F}_2) = 0 \)

or,

\[
\frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = 0
\]

or,

\[
\frac{d}{dt} (\vec{P}_1 + \vec{P}_2) = 0
\]

or,

\[
\vec{P}_1 + \vec{P}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant}
\]

i.e., total linear momentum is conserved.

**Example 4** A body at rest explodes, breaking into three pieces. Two pieces, having equal masses, fly off perpendicular to one another with the same speed of 30 m/sec. The third piece has three times the mass of each other piece. What will be the direction and magnitude of its velocity just after the explosion?

**Solution** Let one of the smaller pieces move along the \( x \)-axis and other along the \( y \)-axis.

Momentum of the first small piece = \( m \cdot 30\hat{i} \)

and momentum of the second small piece = \( m \cdot 30\hat{j} \)

where \( m \) is the mass of each small piece.

If the speed of the bigger piece be \( \vec{v} \) m/sec, we have momentum of the bigger piece = \( 3m\vec{v} \)

Since, initially the body was at rest and therefore its linear momentum is zero. As there is no external force acting on the body, according to the law of conservation of linear momentum, the total linear momentum of the three pieces, after explosion, should be zero. Therefore, we have

\[
30m\hat{i} + 30m\hat{j} + 3m\vec{v} = 0
\]

or,

\[
30\hat{i} + 30\hat{j} + 3\vec{v} = 0
\]

or,

\[
3\vec{v} = -30\hat{i} - 30\hat{j}
\]

or,

\[
\vec{v} = -(10\hat{i} + 10\hat{j})
\]

Thus, the magnitude of the velocity of the bigger piece,

\[
v = \sqrt{(10)^2 + (10)^2}
\]

\[
= \sqrt{100 + 100}
\]

\[
= \sqrt{200}
\]

\[
= 10\sqrt{2} \text{ m/sec.}
\]
And the direction with $x$-axis, say,

$$\theta = \tan^{-1} \left( -\frac{10}{10} \right) = \tan^{-1}(-1) = 135^\circ$$

which is same as the angle it makes with $y$-axis.

Thus, just after explosion, the velocity of the bigger piece is $10\sqrt{2}$ m/sec inclined at an angle of $135^\circ$ to the direction of motion of either small piece.

### 2.5 SYSTEM OF VARIABLE MASS: ROCKET

Rocket is one of the best examples of a system of variable mass in which mass of the rocket is continuously decreasing by means of the emergence of hot gases called ‘jet’. The law of conservation of momentum demands that rocket must gain momentum in the direction opposite to that of jet and this gain in momentum is equal and opposite to the momentum lost by the jet.

The fuels used in rocket are of two types (a) solid fuel (e.g., gun powder). There is no need of separate chamber for combustion in this case because solid fuel itself contains the oxidiser. (b) liquid fuel (e.g., liquid hydrogen, liquid paraffin, etc.) The fuel and the oxidiser (e.g., liquid oxygen, hydrogen peroxide or nitric acid) are burnt in the combustion chamber and a jet of hot gases emerges from the tail of the rocket with very high velocity. [Fig. 2.2]

![Fig. 2.2](image)

**Theory:** Let at any instant $t$, mass and velocity of the rocket be $M$ and $V$ respectively in the laboratory frame (an inertial frame). If mass of the jet emerging per second be $\alpha$ and $-v$ be the exhaust velocity of the jet relative to the rocket, time rate of change of mass of rocket $= \frac{dM}{dt} = -\alpha$ (negative sign indicates that mass of the rocket is decreasing with time)
If the velocity of rocket and that of jet be parallel, the velocity of the jet in laboratory frame = \(- \vec{v} + \vec{V}\)

The rate of change of momentum of the jet = \(\alpha (\vec{v} - \vec{V})\)

Thus, force on the rocket, 
\[
\vec{F} = - \alpha (\vec{v} + \vec{V}) = \frac{dM}{dt} (\vec{v} + \vec{V}) \quad \ldots(1)
\]

From Newton’s second law, 
\[
\vec{F} = \frac{dP}{dt} = \frac{d}{dt} (M \vec{V}) \quad \ldots(2)
\]

**CASE I  If the weight of the rocket is neglected**

Using Eqs. (1) and (2), the equation of motion of the rocket can be written as
\[
\frac{d}{dt} (M \vec{V}) = \frac{dM}{dt} (\vec{v} + \vec{V})
\]
or,
\[
M \frac{d\vec{V}}{dt} + \vec{V} \frac{dM}{dt} = - \vec{v} \frac{dM}{dt} + \vec{V} \frac{dM}{dt}
\]
or,
\[
M \frac{d\vec{V}}{dt} = - \vec{v} \frac{dM}{dt} \quad \ldots(3)
\]

Equation (3) gives the instantaneous thrust on the rocket. It can be rewritten as
\[
\frac{d\vec{V}}{dt} = - \frac{\vec{v}}{M} \frac{dM}{dt}
\]

Integrating with respect to \(t\), we get
\[
\vec{V} = - \vec{v} \log_e M + C \quad \ldots(4)
\]
where \(C\) is a constant of integration.

If at \(t = 0\) mass and velocity of the rocket be \(M_0\) and \(V_0\) respectively, we have from Eq. (4),
\[
\vec{V}_0 = - \vec{v} \log_e M_0 + C
\]
or, 
\[
C = \vec{V}_0 + \vec{v} \log_e M_0
\]

Thus, 
\[
\vec{V} = - \vec{v} \log_e M + \vec{V}_0 + \vec{v} \log_e M_0
\]
or, 
\[
\vec{V} = \vec{V}_0 + \vec{v} \log_e \frac{M_0}{M} \quad \ldots(5)
\]

Since mass of the rocket decreases at the rate \(\alpha\), its mass at time \(t\),
\[ M = M_0 - \alpha t \]
or,
\[ M = M_0 \left( 1 - \frac{\alpha}{M_0} t \right) \]
or,
\[ M = M_0 (1 - \beta t) \]

where \( \beta = \frac{\alpha}{M_0} \) is the rate of change of mass in terms of the initial mass \( M_0 \).

Thus, Eq. (5) becomes
\[
\vec{V} = \vec{V}_0 + \vec{v} \log_e \frac{M_0}{M_0 (1 - \beta t)}
\]
or
\[
\vec{V} = \vec{V}_0 + \vec{v} \log_e \frac{1}{(1 - \beta t)}
\]
or,
\[
\vec{V} = \vec{V}_0 - \vec{v} \log_e (1 - \beta t)
\]

**CASE II** If the weight of the rocket is taken into account

In this case, the equation of motion of the rocket moving vertically upwards becomes
\[
\frac{d}{dt} (M\vec{V}) = \frac{dM}{dt} (-\vec{v} + \vec{V}) - Mg
\]
or,
\[
\vec{V} \frac{dM}{dt} + M \frac{d\vec{V}}{dt} = -\vec{v} \frac{dM}{dt} + \vec{V} \frac{dM}{dt} - Mg
\]
or,
\[
M \frac{d\vec{V}}{dt} = -\vec{v} \frac{dM}{dt} - Mg
\]
or,
\[
\frac{d\vec{V}}{dt} = -\frac{\vec{v}}{M} \frac{dM}{dt} - g
\]
or,
\[
\frac{d\vec{V}}{dt} = -\frac{\vec{v}}{M} \frac{dM}{dt} - g dt
\]

Integrating with respect to \( t \), we get
\[
\vec{V} = -\vec{v} \log_e M - gt + C'
\]

where \( C' \) is another constant of integration.

Since at \( t = 0, M = M_0 \) and \( \vec{V} = \vec{V}_0 \), we have
\[
\vec{V}_0 = -\vec{v} \log_e M_0 + C'
\]
or,
\[
C' = \vec{V}_0 + \vec{v} \log_e M_0
\]
Substituting $C'$ in Eq. (7), we get

$$\vec{V} = -\vec{v} \log_e M - gt + \vec{V}_0 + \vec{v} \log_e M_0$$

or,

$$\vec{V} = \vec{V}_0 + \vec{v} \log_e \frac{M_0}{M} - gt$$

...(8)

Since

$$\frac{M_0}{M} = \frac{1}{(1 - \beta t)},$$

we have

$$\vec{V} = \vec{V}_0 - \vec{v} \log_e (1 - \beta t) - gt$$

...(9)

where $t$ is the time of combustion of the fuel.

If $t$ is small, the effect of term $-gt$ is negligible but if $t$ is large, i.e., the fuel burns very slowly, the term $-gt$ may become appreciably large and therefore the final velocity of the rocket may slow down.

In fact, both a slower and a more rapid rate of fuel combustion are equally undesirable for various reasons.

**Note:** In order to achieve high final velocity of the rocket,

(i) The value of exhaust velocity $\vec{v}$ must be large.

(ii) $\frac{M_0}{M}$ should be large, i.e., final mass $M$ of the rocket must be much less than its initial mass $M_0$.

**Limitation of Single Stage Rocket**

The highest possible value of exhaust velocity is the root mean square velocity of the gas molecules at the temperature of combustion chamber. It is approximately equal to 2 km/sec at 3000°C.

If initial velocity of the rocket is zero and ratio $\frac{M_0}{M} = 10$, then from Eq. (5), velocity of rocket $V = 4.6$ km/sec which is much less than the escape velocity (11.2 km/sec) or the orbital velocity near the surface of the earth (8 km/sec).

Thus, a single stage rocket is not suitable for escaping the rocket from earth’s gravitational field or even putting the space satellites in the orbit of the earth.

**Multi-Stage Rocket**

In the multi-stage rocket, two or more rockets are connected in series. The first stage rocket is large and heavy whereas the last stage rocket is the smallest and the lightest.

The first stage rocket is used first. When its fuel is burnt up, it gets detached.
Now the second stage rocket starts on. The velocity of the rocket increases as ratio $M_0/M$ increases. After the detachment of second stage rocket, the velocity further increases and the rocket achieves escape or orbital velocity.

**Example 5** A 5000 kg rocket is set for vertical firing. If the exhaust speed is 500 m/sec, how much gas must be ejected per second to supply the thrust needed (i) to overcome the weight of the rocket (ii) to give the rocket an initial upward acceleration of 19.6 m/sec²?

**Solution**

(i) At any instant, the thrust on the rocket is given by

$$F = M \frac{dV}{dt} = -v \frac{dM}{dt} - Mg$$

Just to overcome the weight of the rocket, the net force $F$ should be zero i.e.,

$$-v \frac{dM}{dt} - Mg = 0$$

or,

$$\frac{dM}{dt} = -\frac{Mg}{v} = -\frac{5000 \times 9.8}{500} = -98 \text{ kg/sec.}$$

It means that the gas must be ejected at the rate of 98 kg/sec just to overcome the weight of the rocket.

(ii) Let the initial upward acceleration of the rocket be $a$. Therefore,

$$F = M \frac{dV}{dt} = -v \frac{dM}{dt} - Mg = Ma$$
or, \[
\frac{dM}{dt} = - \frac{M}{v} (g + a)
\]
\[= - \frac{5000 \times (9.8 + 19.6)}{500} = - 294 \text{ kg/sec}
\]
It means that the gas must be ejected at the rate of 294 kg/sec to give the rocket an initial upward acceleration of 19.6 m/sec².

**Example 6** A rocket starts vertically upward with speed \(v_0\). Show that its speed \(v\) at a height \(h\) is given by

\[v_0^2 - v^2 = \frac{2gh}{1 + \frac{h}{R}}
\]

where \(R\) is the radius of the earth and \(g\) is acceleration due to gravity at earth’s surface. Hence, deduce an expression for maximum height reached by a rocket fired with a speed 90% of the escape velocity.

**Solution** Since the rocket goes to very large heights, acceleration due to gravity \(g\) cannot be assumed to be constant throughout. At the surface of the earth, \(g = \frac{GM}{R^2}\) where \(M\) and \(R\) are the mass and radius of the earth respectively and \(G\) is the universal gravitational constant.

At a height \(h\), acceleration due to gravity

\[g' = \frac{GM}{(R + h)^2} = \frac{GM}{R^2\left(1 + \frac{h}{R}\right)^2} = \frac{g}{\left(1 + \frac{h}{R}\right)^2}
\]

The velocity of the rocket at height \(h\) will be given by

\[v^2 = v_0^2 - 2 \int_0^h g' dh
\]
\[= v_0^2 - 2 \int_0^h \left(\frac{g}{1 + \frac{h}{R}}\right)^2 dh
\]
\[= v_0^2 - 2g \left[- \frac{R}{1 + \frac{h}{R}}\right]^h_0
\]
\[= v_0^2 + 2g \left[\frac{R}{1 + \frac{h}{R}} - R\right]_0
\]
\[ v^2_0 - v^2 = \frac{2gh}{1 + \frac{h}{R}} \]

or,
\[ v^2_0 - v^2 = \frac{2gh}{1 + \frac{h}{R}} \]

The escape velocity is given by \( \sqrt{2gR} \). Therefore, in this case \( v_0 = 0.9\sqrt{2gR} \) and at the heighest point, obviously, \( v = 0 \).

Substituting these values in the relation above, we get

\[ 0.81 \times 2gR - 0 = \frac{2gh_m}{1 + \frac{h_m}{R}} \]

where \( h_m \) is the maximum height attained by the rocket.

or,
\[ 0.81 \times 2gR = \frac{2g h_m R}{R + h_m} \]

or,
\[ 0.81 = \frac{h_m}{R + h_m} \]

Solving this, we get

\[ h_m = \frac{0.81}{0.19} R = 4.26 \approx 4.3R \]

Thus, the maximum height attained by the rocket in this case is 4.3 \( R \).

**Example 7** In a two-stage rocket, the weights of the first and second stage rockets are 100 kg and 10 kg separately and they contain 800 kg and 90 kg of fuels respectively. Find the final velocity that can be achieved with an exhaust velocity of 4.5 km/sec.

(Given \( \log_{10} 10 = 2.3 \) and \( \log_{10} 2 = 0.30 \))

**Solution** At any instant \( t \), the velocity of a rocket is given by

\[ V = V_0 + v \log_e \left( \frac{M_0}{M} \right) \]

Initially, \( V_0 = 0, v = 4.5 \) km/sec

and \( M_0 = 100 + 800 + 10 + 90 = 1000 \) kg

when the first stage rocket ends.

\( M = 100 + 10 + 90 = 200 \) kg

Thus,
\[ V = 0 + 4.5 \log_e \left( \frac{1000}{200} \right) \]
\[ = 4.5 \log_e(10/2) \]
\[ = 4.5 \times 2.3 \times (\log_{10} 10 - \log_{10} 2) \]
\[ = 4.5 \times 2.3 \times (1 - 0.3) \]
\[ = 4.5 \times 2.3 \times 0.7 \]
\[ = 7.245 \text{ km/sec.} \]

Now, this is the initial velocity for the second stage rocket and therefore the final velocity achieved by the rocket at the end of second stage will be
\[ V' = V + v \log_e \left( \frac{M'_{0}}{M'} \right) \]
\[ = 7.245 + 4.5 \log_e \left( \frac{10 + 90}{10} \right) \]
\[ = 7.245 + 4.5 \times 2.3 \log_{10}(10) \]
\[ = 7.245 + 4.5 \times 2.3 \]
\[ = 7.245 + 10.35 \]
\[ = 17.595 \text{ km/sec.} \]

### 2.6 LAW OF CONSERVATION OF ENERGY

**Energy** The energy of a particle is defined as the capacity of doing work and it is measured by the amount of work done by virtue of its motion or position. The energy due to the motion of a particle is called its kinetic energy while the energy stored in the particle due to its position is called potential energy.

**Work-Energy Theorem** Work-energy theorem is stated as “the work done by the resultant force acting on a particle is equal to the change in kinetic energy of the particle.”

**Proof** Let a particle move under the influence of a conservative force \( \vec{F} \) acting along \( x \)-axis. As a result of which the particle is displaced from the position \( x_0 \) to \( x \) in the direction of force.

The work done by the force \( \vec{F} \) in displacing the particle from \( x_0 \) to \( x \),
\[ W = \int_{x_0}^{x} \vec{F} \cdot d\vec{r} = \int_{x_0}^{x} Fdx \quad \ldots(1) \]

But from Newton’s second law,
\[ F = ma = m \frac{dv}{dt} = m \frac{dv}{dx} \frac{dx}{dt} = mv \frac{dv}{dx} \quad \text{[since } v = \frac{dx}{dt}] \]
Substituting for $F$ in Eq. (1), we get

$$W = \int_{x_0}^{x} mv \frac{dv}{dx} \, dx = \int_{v_0}^{v} mv \, dv = m\left[\frac{v^2}{2}\right]_{v_0}^{v}$$

or,

$$W = \frac{1}{2} mv^2 - \frac{1}{2} mv_0^2$$

...(2)

or,

$$W = K - K_0 = \Delta K$$

i.e., the work done on a particle by the conservative force is always equal to the change in kinetic energy of the particle. Thus, work done on a free particle by a conservative force depends only upon the initial and final velocities of the particle and is independent of the nature of force and the path of the particle.

**The Law of Conservation of Energy:** The sum of potential energy and kinetic energy of a particle at any point in the conservative force field remains constant.

**Proof** Let a particle of mass $m$ be displaced from the initial point $i$ to the final point $f$ under the influence of a conservative force $\vec{F}$. From work-energy theorem, the work done $W$ by the force on the particle will be equal to the change in its kinetic energy. i.e.,

$$W = \int_{i}^{f} \vec{F} \cdot d\vec{r} = \frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2$$

...(3)

where $v_i$ and $v_f$ are the initial and final velocities of the particle respectively.

The potential energy difference between these two points of the particle is defined as the work done by the force $\vec{F}$ in displacing the particle from position $i$ to $f$.

i.e.,

$$U_f - U_i = -\int_{i}^{f} \vec{F} \cdot d\vec{r} \text{ (negative sign indicates that the particle is displaced against the conservative force)}$$

or,

$$U_i - U_f = \int_{i}^{f} \vec{F} \cdot d\vec{r}$$

...(4)

where $U_i$ and $U_f$ are the potential energies of the particle at initial and final points respectively.

Equating Eqs. (3) and (4), we get

$$U_i - U_f = \frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2$$

or,

$$U_i + \frac{1}{2} mv_i^2 = U_f + \frac{1}{2} mv_f^2$$

or,

$$U_i + K_i = U_f + K_f = E \text{ (total energy)}$$
i.e., total energy of the particle remains conserved (constant) in a conservative force field.

**Example 8** A pendulum bob has a speed 3 m/sec when it passes through its lowest point. Find its speed when it makes an angle of 60° with the vertical. The length of the pendulum is 0.5 m. (Given \( g = 10 \text{ m/sec}^2 \))

**Solution** The force acting on the bob due to the string is always perpendicular to the velocity of the bob and therefore the work done by the force is zero. Hence, the total mechanical energy will remain constant.

![Fig. 2.4](image)

When the bob having mass \( m \) goes from position \( A \) to position \( B \), the potential energy of the system increases.

Increase in potential energy = \( mgh \)

\[
= mg(l - l \cos \theta)
= mgl(1 - \cos \theta)
\]

Since the total mechanical energy of the system is conserved, this increase in potential energy must be equal to the decrease in the kinetic energy of the system.

Now, decrease in kinetic energy

\[
\frac{1}{2} mv^2 - \frac{1}{2} mv'{}^2
\]

where \( v \) and \( v' \) are the velocities of the bob at positions \( A \) and \( B \) respectively.

Thus,

\[
mgl(1 - \cos \theta) = \frac{1}{2} mv^2 - \frac{1}{2} mv'{}^2
\]

or,

\[
\frac{1}{2} mv'{}^2 = \frac{1}{2} mv^2 - mgl(1 - \cos \theta)
\]

or,

\[
v' = \sqrt{v^2 - 2gl(1 - \cos \theta)}
\]
\[ \sqrt{(3)^2 - 2 \times 10 \times 0.5 \,(1 - \cos 60^\circ)} \]
\[ = \sqrt{9 - 10 \,(1 - \frac{1}{2})} \]
\[ = \sqrt{9 - 5} \]
\[ = 2 \,\text{m/sec.} \]

**Example 9**  A body is allowed to slide on a frictionless track from the rest position under gravity. The track ends in a circular loop of diameter \( D \). What should be the minimum height of the body in terms of \( D \) so that it may complete the loop successfully?

**Solution**

When the body having mass \( m \) slides from point \( A \) to point \( B \), the loss in potential energy = \( mgh \)

This potential energy is converted into the kinetic energy of the body, i.e.,

\[ mgh = \frac{1}{2} \, mv^2 \]

where \( v \) is the velocity of the body at point \( B \).

or,

\[ v^2 = 2gh \]  \ ...(1)

The value of \( v \) should be the least for the least value of \( h \) so that the body just reaches up to the point \( C \). In this position, the necessary centripetal force is provided by the weight of the body.

i.e.,

\[ \frac{mv^2}{(D/2)} = mg \]
or, 

\[ v'^2 = \frac{Dg}{2} \]  

\ldots (2)

Also, applying the law of conservation of energy for points B and C, we get

\[ \frac{1}{2} mv^2 - \frac{1}{2} mv'^2 = mgD \]

or, 

\[ v^2 - v'^2 = 2gD \]  

\ldots (3)

Substituting, the values of \( v^2 \) and \( v'^2 \) from Eqs. (1) and (2), we get

\[ 2gh - \frac{Dg}{2} = 2gD \]

or,  

\[ 2gh = \frac{5}{2} gD \]

or,  

\[ h = \frac{5}{4} D \]

### 2.7 COLLISION

By collision we mean the interaction of particles (or, bodies) with each other. In a collision:

(i) a relatively large force acts on each colliding particle for a relatively short time (i.e., impulse).

(ii) the motion of the colliding particles (or that of at least one of them) changes abruptly.

(iii) there is relatively clean separation of time, like ‘before the collision’ and ‘after the collision’.

(iv) the total momentum of the particles remains conserved.

There are two types of collisions:

(a) elastic collision

(b) inelastic collision

A collision in which both the kinetic energy and the linear momentum of the particles are conserved, is called elastic collision and a collision, in which only linear momentum is conserved while a part of kinetic energy is converted in some other forms, i.e., kinetic energy is not conserved, is called inelastic collision.

### 2.7.1 Elastic One-dimensional Head-on Collision

Let us consider an elastic collision between two non-rotating particles of masses \( m_1 \) and \( m_2 \) moving initially with velocities \( u_1 \) and \( u_2 \) respectively along the line joining their centres. After collision, the particles move with velocities \( v_1 \) and \( v_2 \) respectively along the same straight line.
From the law of conservation of momentum, we have
\[ m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \]
or, \[ m_1(u_1 - v_1) = m_2(v_2 - u_2) \] \( \ldots(1) \)

For the elastic collision, kinetic energy is also conserved, i.e.,
\[ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \]
or, \[ m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \] \( \ldots(2) \)

Dividing Eq. (2) by Eq. (1), we get
\[ u_1 + v_1 = v_2 + u_2 \]
or, \[ u_1 - u_2 = v_2 - v_1 = -(v_1 - v_2) \] \( \ldots(3) \)
i.e., the relative velocity between the particles after an elastic one-dimensional collision is equal and opposite to the relative velocity before the collision.

From Eq. (3), \( v_1 = v_2 - u_1 + u_2 \) and \( v_2 = v_1 + u_1 - u_2 \)

Substituting in Eq. (1), we get
\[ m_1(u_1 - v_1) = m_2(v_1 + u_1 - u_2 - u_2) = m_2(v_1 + u_1 - 2u_2) \]
or, \[ v_1(m_1 + m_2) = u_1(m_1 - m_2) + 2m_2u_2 \]
or, \[ v_1 = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) u_1 + \left( \frac{2m_2}{m_1 + m_2} \right) u_2 \] \( \ldots(4) \)

Also,
\[ m_1(u_1 - v_2 + u_1 - u_2) = m_2(v_2 - u_2) \]
or, \[ 2m_1u_1 + (m_2 - m_1)u_2 = (m_1 + m_2)v_2 \]
or, \[ v_2 = \left( \frac{2m_1}{m_1 + m_2} \right) u_1 + \left( \frac{m_2 - m_1}{m_1 + m_2} \right) u_2 \] \( \ldots(5) \)

**Special Cases:**

**CASE I** If masses of the two particles are same, i.e., \( m_1 = m_2 \), then from Eqs. (4) and (5), we have
\[ v_1 = u_2 \text{ and } v_2 = u_1 \]
i.e., in an elastic one-dimensional collision between two particles of equal mass, the particles simply exchange their velocities.

**CASE II** If one of the particles is at rest. Let the particle of mass \( m_2 \) be at rest, i.e., \( u_2 = 0 \) Then from Eqs. (4) and (5), we have
\[ v_1 = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) u_1 \] and \[ v_2 = \left( \frac{2m_1}{m_1 + m_2} \right) u_1 \] ... (6)

(i) if \( m_1 = m_2 \)

\[ v_1 = 0 \] and \( v_2 = u_1 \)

i.e., if a particle collides with another particle of same mass which is initially at rest, the first particle comes to rest and the second particle acquires the initial velocity of the first particle e.g., hydrogen, whose nucleus (proton) has nearly the same mass as a neutron, is used as a moderator in a fission reactor. Actually the neutrons produced in the reactor from the fission of uranium atoms, move very fast and must be slowed down in order to produce more fission. When a fast moving neutron makes a head-on collision with a hydrogen nucleus at rest, the neutron is brought almost to rest or slowed down.

(ii) if the particle at rest is much more massive than the other, i.e., \( m_1 << m_2 \).

Neglecting the mass \( m_1 \) compared to mass \( m_2 \) in Eq. (6) we get

\[ v_1 \approx -u_1 \] and \( v_2 \approx 0 \)

i.e., when a light particle collides with a massive particle at rest, the velocity of the light particle is approximately reversed and the massive particle remains approximately at rest, e.g., the collision between earth and a ball which is falling vertically downward. If the collision is elastic, the ball rebounds with a reversed velocity and rises to the same height from which it was dropped.

(iii) if the particle at rest is much lighter than the other, i.e., \( m_1 >> m_2 \).

Neglecting the mass \( m_2 \) compared to mass \( m_1 \) in Eq. (6), we get

\[ v_1 \approx u_1 \] and \( v_2 \approx 2u_1 \)

i.e., when a massive particle collides with a lighter particle at rest, the velocity of the massive particle is approximately unchanged but the lighter particle starts moving with approximately twice the velocity of the massive particle.

**Energy Transfer in a Head-on Elastic Collision**

Consider an elastic head on collision between a particle of mass \( m_1 \) moving with velocity \( u_1 \) with another particle of mass \( m_2 \) at rest. After collision, let the particle of mass \( m_2 \) move with velocity \( v_2 \).

The initial kinetic energy of the particle of mass \( m_1 \), \( K = \frac{1}{2} m_1 u_1^2 \) and the kinetic energy gained by the particle of mass \( m_2 \) due to collision,

\[ K' = \frac{1}{2} m_2 v_2^2 \]
Hence, fraction of kinetic energy transferred to the particle of mass \( m_2 \),

\[
\frac{K'}{K} = \frac{\frac{1}{2} m_2 v_2^2}{\frac{1}{2} m_1 u_1^2} = \frac{\frac{1}{2} m_2 \left( \frac{2 m_1}{m_1 + m_2} u_1 \right)^2}{\frac{1}{2} m_1 u_1^2}; \quad [\text{using Eq. (6)}]
\]

\[
= \frac{4 m_1 m_2}{(m_1 + m_2)^2} = \frac{4 m_2/m_1}{(1 + m_2/m_1)^2}
\]

When the mass ratio is unity, i.e., \( m_1 = m_2 \)

or,

\[
\frac{m_2}{m_1} = 1, \quad \frac{K'}{K} = \frac{4}{(1 + 1)^2} = 1
\]

i.e., when \( m_1 = m_2 \), whole of the kinetic energy of the moving particle is transferred to the particle initially at rest.

When \( m_1 > m_2 \) or \( m_1 < m_2 \), the fraction of kinetic energy transferred will be less than 1.

### 2.7.2 Elastic Collision in Two or Three Dimensions

**(i) In the laboratory frame of reference**

Let us consider a particle of mass \( m_1 \) moving with velocity \( \vec{u}_1 \) along \( X \)-axis. It collides with a particle of mass \( m_2 \) initially at rest in the laboratory frame of reference. [Fig. 2.6]

After collision, let the particle of mass \( m_1 \) is scattered by scattering angle \( \theta_1 \) with velocity \( \vec{v}_1 \). Particle of mass \( m_2 \) makes an angle \( \theta_2 \) with the \( X \)-axis and moves with velocity \( \vec{v}_2 \). Since velocities \( \vec{u}_1 \) and \( \vec{v}_1 \) lie in the \( X-Y \) plane, conservation of momentum demands that \( \vec{v}_2 \) will also lie in this plane and has no \( Z \)-component.

From the law of conservation of momentum, we have

\[
m_1 u_1 = m_1 v_1 \cos \theta_1 + m_2 v_2 \cos \theta_2 \quad [\text{along } X\text{-axis}] \quad \cdots (1)
\]
and \( 0 = m_1v_1 \sin \theta_1 - m_2v_2 \sin \theta_2 \) [along Y-axis] \( \ldots (2) \)

Also, from the law of conservation of energy, we have
\[
\frac{1}{2} m_1u_1^2 = \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2 \quad \ldots (3)
\]

On solving these three equations, we can get any unknown variable in which we are interested.

(ii) **In the centre of mass frame of reference**

In the laboratory frame of reference, the velocity \( \vec{V} \) of the centre of mass of the system,

\[
\vec{V} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} = \frac{m_1\vec{u}_1}{m_1 + m_2} \quad [\text{since } \vec{u}_2 = 0] \quad \ldots (4)
\]

The centre of mass frame of reference is attached to the centre of mass of the system and hence the centre of mass is initially at rest in the centre of mass frame. In this frame, the initial and final velocities of the two particles are given by

\[
\vec{u}_1' = \vec{u}_1 - \vec{V}, \quad \vec{u}_2' = - \vec{V} \quad \ldots (5)
\]

and

\[
\vec{v}_1' = \vec{v}_1 - \vec{V}, \quad \vec{v}_2' = \vec{v}_2 - \vec{V} \quad \ldots (6)
\]

Since the velocity of the centre of mass remains zero, total linear momentum of the system must always be zero, i.e., momentum of the two particles must always be equal in magnitude but opposite in direction. Therefore, the velocities \( \vec{v}_1' \) and \( \vec{v}_2' \) after collision must be oppositely directed. [Fig. 2.7]

Now, from Eq. (5), we have
\[
m_1\vec{u}_1' = m_1(\vec{u}_1 - \vec{V})
\]
or, \[
\frac{1}{2} m_1 u_1' = \frac{1}{2} m_1 v_1' = \frac{1}{2} m_1 v_1''
\]

or, \[
v_1' = u_1'
\]

Substituting in Eq. (11), we get

\[
v_2' = u_2'
\]

Thus, Eqs. (12) and (13) show that in the centre of mass frame of reference, the magnitudes of the velocities of the particles remain unchanged in an elastic collision.

**Note** In the centre of mass frame of reference, the scattering angle can have any value but in the laboratory frame of reference the scattering angle cannot have all values.

Since

\[
\tan \theta_1 = \frac{\sin \theta_1}{\cos \theta_1} = \frac{v_1 \sin \theta_1}{v_1 \cos \theta_1}
\]

From Eq. (6), we have

\[
\vec{v}_1 = \vec{v}_1' + \vec{V}
\]

Resolving the velocity along X and Y-axes, we get, [with the help of Figs. 2.6 and 2.7]

\[
\begin{align*}
v_1 \cos \theta_1 &= v_1' \cos \theta + \vec{V} \quad \text{(along X-axis)} \\
v_1 \sin \theta_1 &= v_1' \sin \theta \quad \text{(along Y-axis)}
\end{align*}
\]

From Eqs. (14) and (15), we get

\[
\tan \theta_1 = \frac{v_1' \sin \theta}{v_1' \cos \theta + \vec{V}}
\]

From Eq. (4), we have

\[
V = \frac{m_1 u_1}{m_1 + m_2} = \frac{m_1 (u_1' + V)}{m_1 + m_2} \quad \text{[using Eq. (5)]}
\]

or, \[
(m_1 + m_2) V = m_1 (u_1' + V)
\]

or, \[
m_2 V = m_1 u_1'
\]

or, \[
V = \frac{m_1}{m_2} u_1' = \frac{m_1}{m_2} v_1' \quad \text{[since } v_1' = u_1' \text{]} \quad (17)
\]

Substituting the value of V in Eq. (16), we get
Thus,  
\[ \frac{K_f}{K_i} = \frac{\frac{1}{2}(m_1 + m_2) v^2}{\frac{1}{2} m_1 u_1^2} \]

or,  
\[ \frac{K_f}{K_i} = \frac{(m_1 + m_2) \left( \frac{m_1 u_1}{m_1 + m_2} \right)^2}{m_1 u_1^2} \]  
[using Eq. (2)]

or,  
\[ \frac{K_f}{K_i} = \frac{m_1}{m_1 + m_2} \]  
...(3)

i.e., \( K_f < K_i \) which shows that kinetic energy is lost in an inelastic collision and appears in the form of heat energy in the system.

In the centre of mass frame of reference, motion of particles before and after collision, is described as follows.

In the laboratory frame of reference, velocity of centre of mass,  
\[ \vec{V} = \frac{m_1 \vec{u}_1}{m_1 + m_2} \]

In the centre of mass frame of reference, velocity of first body,  
\[ \vec{u}_1' = \vec{u}_1 - \vec{V} = \vec{u}_1 - \frac{m_1 \vec{u}_1}{m_1 + m_2} \]

or,  
\[ \vec{u}_1' = \frac{m_2}{m_1 + m_2} \vec{u}_1 \]  
...(4)

and velocity of second body,  
\[ \vec{u}_2' = \vec{u}_2 - \vec{V} = 0 - \frac{m}{m_1 + m_2} \vec{u}_1 \]

or  
\[ \vec{u}_2' = - \frac{m_1}{m_1 + m_2} \vec{u}_1 \]  
...(5)

After collision, the particles stick together. The combined mass of the new particle is \((m_1 + m_2)\) and it must be at rest in the centre of mass frame of reference. With respect to the laboratory frame of reference, the new particle has velocity \(v\), which is exactly the same as the velocity of the centre of mass.

**Example 10**  Two particles with position vectors \( \vec{r}_1 = (3\hat{i} + 5\hat{j}) \) m and \( \vec{r}_2 = -(5\hat{i} + 3\hat{j}) \) m move with velocities \( \vec{u}_1 = (4\hat{i} + 3\hat{j}) \) m/sec and \( \vec{u}_2 = (a\hat{i} + 7\hat{j}) \) m/sec. Find (i) value of \( a\), if they collide and (ii) when and where the collision will take place?
Solution

(i) The two particles will collide after a time \( t \), if they have the same position vectors, i.e.,

\[
\vec{r}_1 + \vec{u}_1 t = \vec{r}_2 + \vec{u}_2 t
\]
or, \((\vec{r}_1 - \vec{r}_2) + (\vec{u}_1 - \vec{u}_2) \cdot t = 0\) …(1)

Taking cross product with \((\vec{r}_1 - \vec{r}_2)\), we get

\[
(\vec{r}_1 - \vec{r}_2) \times (\vec{u}_1 - \vec{u}_2) = 0 \quad \text{[since \((\vec{r}_1 - \vec{r}_2) \times (\vec{r}_1 - \vec{r}_2) = 0 \text{ and } t \neq 0\]}

or,

\[
(3\hat{i} + 5\hat{j} + 5\hat{k}) \times (4\hat{i} + 3\hat{j} - a\hat{i} - 7\hat{j}) = 0
\]
or,

\[
8(\hat{i} + \hat{j}) \times [(-4 - a) \hat{i} - 4\hat{j}] = 0
\]
or,

\[
-4\hat{k} - (4 - a)\hat{k} = 0
\]
or,

\[
(-4 - 4 + a) \hat{k} = 0
\]
or,

\[
a - 8 = 0
\]
or,

\[
a = 8
\]

(ii) From Eq. (1), we have

\[
t = - \frac{(\vec{r}_1 - \vec{r}_2)}{(\vec{u}_1 - \vec{u}_2)}
\]

\[
= - \frac{8(\hat{i} + \hat{j})}{-4(\hat{i} + \hat{j})}
\]

\[
= 2 \text{ sec.}
\]

Place of collision
\[
= \vec{r}_1 + \vec{u}_1 t
\]

\[
= (3\hat{i} + 5\hat{j}) + (4\hat{i} + 3\hat{j}) \cdot 2
\]

\[
= 3\hat{i} + 5\hat{j} + 8\hat{i} + 6\hat{j}
\]

\[
= 11\hat{i} + 11\hat{j}
\]

\[
= 11(\hat{i} + \hat{j}) \text{ m}
\]

Example 11 A particle of mass \( m \) moving along \( X \)-axis with a velocity \( v \), strikes another particle of mass 2 \( m \) at rest. After collision, the first particle comes to rest and the other breaks into two equal pieces. One of these pieces starts moving along \( Y \)-axis with a velocity \( v_1 \). Find the velocity of the other piece.
**Solution**

Let the other piece move with velocity $v_2$ at an angle $\theta$ with the $X$-axis as shown in the figure.

Before collision, the total linear momentum of the particles

$$= mv + 2m \times 0 \quad \text{(along X-axis)}$$

$$= mv$$

After collision, the first particle comes to rest. Therefore, its linear momentum will be zero. Now, momentum of the first piece

$$= mv_1 \quad \text{(along Y-axis)}$$

and momentum of the second piece

$$= mv_2 \quad \text{(along its direction of motion)}$$

Applying law of conservation of momentum and taking its components along $X$- and $Y$-axes we have

$$mv_2 \cos \theta = mv \quad \text{(along X-axis)} \ldots (1)$$

and

$$mv_2 \sin \theta = mv_1 \quad \text{(along Y-axis)} \ldots (2)$$

Solving Eqs. (1) and (2), we get

$$v_2 = \sqrt{v^2 + v_1^2}$$

and

$$\tan \theta = \frac{v_1}{v}$$

**Example 12** Two particles of masses $m_1 = 0.2$ kg and $m_2 = 0.5$ kg have velocities $\vec{u}_1 = 10\hat{i}$ m/sec and $\vec{u}_2 = (3\hat{i} + 5\hat{j})$ m/sec just before the collision during which they stick to each other permanently. Calculate:

(i) the velocity of the centre of mass

(ii) the final momentum of the combination in the laboratory frame
\[ K_f = \frac{1}{2} (m_1 + m_2) v^2 \]
\[ = \frac{1}{2} (0.2 + 0.5) \left[ (5)^2 + \left( \frac{25}{7} \right)^2 \right] \]
\[ = \frac{1}{2} (0.7) \left( 25 + \frac{625}{49} \right) \]
\[ = 13.2 \text{ joules} \]

Therefore, fraction of the initial total kinetic energy associated with the motion after collision,
\[ \frac{K_f}{K_i} = \frac{13.2}{18.5} = 0.72 \]

2.8 SCATTERING CROSS-SECTION

When a parallel beam of particles of given energy and momentum is incident upon a target, after interaction they scatter in various directions. The angular distribution of scattered particles is expressed in terms of scattering cross-section.

It gives us information about the nature of force between the incident particles and the target and also their internal structures.

Let us suppose that a uniform parallel beam of \( n \) particles, all of the same mass and energy is incident on a target containing \( N \) number of identical particles or scattering centres. Let us assume that the particles in the beam do not interact with each other.

Fig. 2.9
as solid angle \( d\Omega = \sin \theta d\theta d\phi \) and limit of \( \theta \) and \( \phi \) are 0 to \( \pi \) and 0 to 2\( \pi \) respectively, we have

\[
\sigma = \int_0^\pi \int_0^{2\pi} \left( \frac{d\sigma}{d\Omega} \right) \sin \theta \, d\theta \, d\phi
\]

For the case of central force, \( \frac{d\sigma}{d\Omega} \) is independent of \( \phi \) (as magnitude of central force depends only on \( r \)). Thus,

\[
\sigma = 2\pi \int_0^\pi \left( \frac{d\sigma}{d\Omega} \right) \sin \theta \, d\theta
\]

The unit of total cross-section is \( m^2 \).

**Example 13** A beam of \( \alpha \)-particles with a flux of \( 3 \times 10^8 /m^2 \) sec strikes a thin foil of aluminium, which contains \( 10^{21} \) atoms. A detector of cross-sectional area \( 4 \times 10^{-4} m^2 \) is placed 0.6 m from the target in a direction at right angles to the direction of the incident beam. If the rate of detection of \( \alpha \)-particles is \( 8.1 \times 10^3 /\)sec, compute the differential scattering cross-section.

**Solution** For \( N \) target particles (scattering centres), the number of particles scattered in a solid angle \( d\Omega \) in time \( \Delta t \),

\[
\Delta n = \frac{d\sigma}{d\Omega} \, N \, F \, d\Omega \, \Delta t
\]

where \( F \) is the incident flux and \( \frac{d\sigma}{d\Omega} \) is the differential scattering cross-section.

Thus,

\[
\frac{d\sigma}{d\Omega} = \left( \frac{\Delta n}{\Delta t} \right) \frac{1}{N} \frac{1}{F} \frac{1}{d\Omega}
\]

Here, \( d\Omega \) is the solid angle subtended by the detector at the target for \( \theta = 90^\circ \) and is given by

\[
d\Omega = \frac{dA}{L^2}
\]

where \( dA \) is the area of detector and \( L \) its distance from the target.

Thus,

\[
d\Omega = \frac{4 \times 10^{-4} m^2}{(0.6 m)^2} = 1.1 \times 10^{-3} \text{ Sr.}
\]

Therefore, differential scattering cross-section

\[
\frac{d\sigma}{d\Omega} = \frac{(8.1 \times 10^3/\text{sec})}{10^{21} \times (3 \times 10^8/m^2 \text{ sec}) \times 1.1 \times 10^{-3} \text{ Sr}}
\]

\[
\frac{d\sigma}{d\Omega} = 2.4 \times 10^{-23} \text{ m}^2 \text{ Sr}^{-1}
\]
It means that torque about a point is the time rate of change of angular momentum about that point.

For a system of particles,

$$\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i$$

where \(\vec{r}_i\) and \(\vec{p}_i\) are the position vector and linear momentum of the \(i\)th particle respectively.

and

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i = \frac{d\vec{J}}{dt}$$

where \(\vec{J} = \vec{j}_1 + \vec{j}_2 + \ldots = \sum_i \vec{j}_i\) is the total angular momentum of the system about the fixed point.

In the absence of external torque (i.e., \(\vec{\tau} = 0\)), we have

$$\frac{d\vec{J}}{dt} = 0$$

or,

$$\vec{J} = \text{constant}$$

i.e., total angular momentum of the system remains constant in the absence of external torque. This is the law of conservation of angular momentum.

**Relation between the angular momentum of a system (\(\vec{J}\)) and the angular momentum about its centre of mass (\(\vec{J}_{CM}\))**

Let \(\vec{R}\) and \(\vec{V}\) be the position vector and velocity of centre of mass of the system relative to a fixed point and let \(\vec{r}_{ic}\) and \(\vec{v}_{ic}\) be the position vector and velocity of \(i\)th particle of mass \(m_i\) of the system relative to centre of mass. Thus, position vector and velocity of the \(i\)th particle relative to the fixed point will be \(\vec{r}_i = \vec{R} + \vec{r}_{ic}\) and \(\vec{v}_i = \vec{V} + \vec{v}_{ic}\).

Total angular momentum,

$$\vec{J} = \sum_i m_i (\vec{R} + \vec{r}_{ic}) \times (\vec{V} + \vec{v}_{ic})$$

$$= \sum_i m_i (\vec{R} \times \vec{V}) + \sum_i m_i (\vec{R} \times \vec{v}_{ic}) + \sum_i m_i (\vec{r}_{ic} \times \vec{V}) + \sum_i m_i (\vec{r}_{ic} \times \vec{v}_{ic}) \ldots (1)$$

But \(\vec{r}_{ic} = \vec{r}_i - \vec{R}\)

or,

$$m_i \vec{r}_{ic} = m_i \vec{r}_i - m_i \vec{R}$$

or,

$$\sum_i m_i \vec{r}_{ic} = \sum_i m_i \vec{r}_i - \sum_i m_i \vec{R}$$

$$= \sum_i m_i \vec{r}_i - M\vec{R}$$

where \(M = \sum_i m_i\) is the total mass of the system.

According to the property of centre of mass,
Fig. 2.11

The perpendicular distance between the nucleus and initial direction of proton is called the impact parameter \( b \).

The distance \( NO (= d) \) is called the distance of closest approach of the proton to the nucleus.

Since electrostatic force is a central force, the angular momentum of the proton must remain conserved and thus,

\[
mv_A b = mv_0 d
\]

where \( v_A \) and \( v_0 \) are the velocities of the proton at the points \( A \) and \( O \) respectively.

or,

\[
v_0 = \frac{v_A b}{d} \quad ... (1)
\]

And, from the law of conservation of energy, we have

\[
\frac{1}{2} m v_0^2 + \frac{1}{4\pi \varepsilon_0} \frac{Ze^2}{d} = \frac{1}{2} m v_A^2 + 0 \quad ... (2)
\]

where \( \frac{1}{2} m v_0^2 \) and \( \frac{1}{2} m v_A^2 \) are the kinetic energies of the proton at points \( O \) and \( A \) respectively and \( \frac{1}{4\pi \varepsilon_0} \frac{Ze^2}{d} \) is the potential energy of the proton at the point \( O \). The potential energy of the proton at the point \( A \) is zero because it is far away from the nucleus.

Substituting the value of \( v_0 \) from Eq. (1) in Eq. (2), we have

\[
\frac{1}{2} m \left( v_A \frac{b}{d} \right)^2 + \frac{1}{4\pi \varepsilon_0} \frac{Ze^2}{d} = \frac{1}{2} m v_A^2
\]

or,

\[
\frac{1}{4\pi \varepsilon_0} \frac{Ze^2}{d} = \frac{1}{2} m v_A^2 \left\{ 1 - \left( \frac{b}{d} \right)^2 \right\} \quad ... (3)
\]
Example 15  A lightweight man holds two heavy dumb-bells with outstretched arms while standing on a turntable. A slow push is given to the system, so that he rotates at a rate of 1 revolution per minute. Then he pulls the dumb-bells in towards his chest. If the dumb-bells are originally 60 cm from his axis of rotation and are pulled into 10 cm from the axis of rotation, find the frequency of revolution. Neglect the angular momentum of the man in comparison to that of the dumb-bells.

Solution  The angular momentum of a system rotating with angular velocity $\omega$ is $J = I \omega$ where $I$ is the moment of inertia of the system about the axis of rotation and $\omega = 2\pi n$ where $n$ is the frequency of revolution. If $M$ be the mass of each dumb-bell, the initial angular momentum of the system

$$J_1 = 2Mr_1^2 \omega_1 = 2Mr_1^2 (2\pi n_1)$$

where the angular momentum of the man has been neglected.

Similarly, the final angular momentum of the system will be

$$J_2 = 2Mr_2^2 \omega_2 = 2Mr_2^2 (2\pi n_2)$$

In the absence of external torque, the angular momentum of the system remains conserved and therefore

Initial angular momentum = final angular momentum

or,

$$2Mr_1^2 (2\pi n_1) = 2Mr_2^2 (2\pi n_2)$$

or,

$$n_2 = n_1 \frac{r_1^2}{r_2^2}$$

Given

$n_1 = 1$  revolution per minute

$r_1 = 0.6$ m

and

$r_2 = 0.1$ m

Therefore,

$$n_2 = 1 \times \frac{(0.6)^2}{(0.1)^2}$$

or,

$$n_2 = 36$$

i.e., finally the system will rotate at a rate of 36 revolutions per minute.

Example 16  A neutron of energy 1 MeV passes a proton at such a distance that the angular momentum of the neutron relative to the proton approximately equals $10^{-26}$ erg-sec. What is the distance of closest approach? (Neglect the energy of interaction between the two particles.)

Solution  If $m$ is the mass of the neutron, $\vec{v}$, its velocity and $\vec{r}$, its position vector relative to the proton, its angular momentum $\vec{J} = \vec{r} \times m\vec{v}$
4. Prove that a force which is constant and uniform is always conservative.

5. Show that Lorentz force is conservative. [Bundelkhand 2012]

6. Show that conservative force is the negative gradient of potential energy. [Purvanchal 2012, Meerut 2012, Kanpur 2009]

7. Show that curl of a conservative force is zero. [Purvanchal 2012]


9. Show that the velocity of the centre of mass of the system is constant in the absence of external force. [Bundelkhand 2010]

10. What do you mean by a centre of mass frame of reference? What is its special advantage? [Vidyapeeth 2010]


12. Show that the total linear momentum of a system of particles remains constant if the resultant external force acting on the system is zero.


14. What is the advantage of multi-stage rocket compared to a single stage rocket?

15. Explain the principle of conservation of energy.

16. State and prove work-energy theorem. [Kanpur 2010]


18. What is scattering cross-section? What information do we get from it?

19. Define angular momentum of a particle about a point. [Purvanchal 2010]

20. Show that the time rate of change of angular momentum of a particle is equal to torque acting on it. [Meerut 2012]

21. Show that the total angular momentum of a system remains constant in the absence of external torque. [Kanpur 2006]

22. A man rotates on a turntable with an angular speed \( \omega \). He is holding equal masses at arms length. Without moving his arms, he drops the two masses. What change as occur in his angular speed?

### Long Answer Type Questions

1. What is conservative force? State and prove properties of a conservative force.

2. What is centre of mass? Find out the expression of position vector and velocity of centre of mass of a system of particles. Show that the total linear momentum of the system is equal to the product of the total mass of the system and velocity of the centre of mass.
3. Show that
   (a) the centre of mass of two particles lies on the line joining them;
   (b) the distances of two particles from their centre of mass is the inverse ratio
       of their masses.
4. Write a note on the law of conservation of momentum and its importance in
   Physics.
5. Show that the linear momentum is conserved only when the potential energy
   is translationally invariant.
6. Explain the principle of a rocket. What do you mean by the thrust of a rocket
   and on what factors does it depend?
   Establish the following relation for the velocity of a rocket,
   \[ \bar{V} = \bar{V}_0 + \bar{v} \log_e \frac{M_0}{M} \]
   where the symbols have their usual meanings. How does this relation change
   if the weight of the rocket is taken into account? [Bundelkhand 2012, Kanpur 2011]
7. Show that the speed of a rocket is
   (a) equal to the exhaust speed when the ratio \( \frac{M_0}{M} \) is \( e \).
   (b) equal to twice the exhaust speed when the ratio \( \frac{M_0}{M} \) is \( e^2 \).
8. What is work-energy theorem? Show that the sum of potential energy and
   kinetic energy of a particle at any point in the conservative force field remains
   constant.
9. What do you mean by collision? Show that the relative velocity between the
   particles after an elastic one-dimensional collision is equal and opposite to
   the relative velocity before the collision. [Agra 2012, Meerut 2012]
10. Discuss elastic one-dimensional head-on collision when one of the colliding
    particles is initially at rest for
    (i) \( m_1 = m_2 \)      (ii) \( m_1 << m_2 \) and
    (iii) \( m_1 >> m_2 \)    where \( m_1 \) and \( m_2 \) are the masses of the colliding particles.
11. In an elastic head-on collision between two particles, show that the transference
    of energy is maximum when their mass ratio is unity.
12. In the centre of mass frame of reference, the magnitudes of the velocities of
    the particles remain unchanged in an elastic collision.
13. In a collision of a particle of mass \( m_1 \) and velocity \( u_1 \) with a particle of mass
    \( m_2 \) at rest in the laboratory frame, prove that
\[ \tan \theta_1 = \frac{\sin \theta}{\cos \theta + \frac{m_1}{m_2}} \]

where \( \theta_1 \) and \( \theta \) are the scattering angles in the laboratory and centre of mass frames respectively.

14. In the centre of mass frame of reference, the scattering angle can have any value but in the laboratory frame of reference the scattering angle cannot have all values. Explain it.

15. A bullet having mass \( m \) and speed \( v \) makes a completely inelastic collision with a block of mass \( M (M >> m) \). If the block rises to a height \( h \), find the speed of the bullet.

\[ \text{Ans.} \sqrt{2gh} \frac{(m + M)}{m} \]

16. A bullet having mass \( m \) and speed \( v \) moving horizontally, passes through a pendulum bob of mass \( M \) and emerges with a velocity \( v/2 \). Show that the minimum value of \( v \), which will make the pendulum bob to swing through a complete circle is

\[ \sqrt{5gl} \left( \frac{2M}{m} \right) \]

where \( l \) is the length of the pendulum.

17. Define differential scattering cross-section. On what factors does it depend?

18. Show that the total angular momentum of a system of particles about a fixed point can be expressed as

\[ \vec{J} = \vec{J}_{CM} + \vec{R} \times \vec{P} \]

where \( \vec{J}_{CM} \) is the angular momentum of the system about the centre of mass and \( \vec{R} \times \vec{P} \) is the angular momentum of the centre of mass about the fixed point. [Purvanchal 2010, Kanpur 2009]

19. State the law of conservation of angular momentum. Derive an expression for the distance of closest approach of a proton projected into the coulomb field of a heavy nucleus.
16. Find the minimum horizontal speed that should be given to the bob of a simple pendulum of length \(l\) so that it describes a complete circle.

[Ans. \(\sqrt{5gl}\)]

17. A light rod of length \(l\) has a mass \(m\) attached to its end and is suspended vertically. It is turned through 180° and then released. Find the velocity of the mass and the tension in the rod when the mass reaches its lowest point.

[Ans. 2\(gl\), 5 \(mg\)]

18. A neutron with velocity \(10^5\) m/sec makes an elastic collision with a deuteron at rest. Find the velocities of each particle after collision. (Take mass of neutron = \(1.67 \times 10^{-27}\) kg, and mass of deuteron = \(3.34 \times 10^{-27}\) kg).

[Ans. velocity of neutron = \(3.33 \times 10^4\) m/sec, and velocity of deuteron = \(6.67 \times 10^4\) m/sec, both the particles are oppositely directed]

19. Two particles of masses 0.6 kg and 0.4 kg are moving towards each other along a frictionless horizontal surface with velocities 3 m/sec and 2 m/sec respectively.

(i) Find the final velocities of each particle if the collision is perfectly elastic.

(ii) Find the final velocity of the combined particle if the collision is perfectly inelastic.

(iii) Find the loss of kinetic energy during the inelastic collision.

[Ans. (i) – 1.0 m/sec, 4.0 m/sec (ii) 1 m/sec (iii) 3.0 joule]

20. A bullet of mass \(5 \times 10^{-3}\) kg with velocity \(7 \times 10^2\) m/sec, strikes a ballistic pendulum of mass 1 kg suspended from a weightless cord of length 1 m. After collision, the bullet emerges with a velocity \(2 \times 10^2\) m/sec. Find the height through which the pendulum rises.

[Ans. 0.3 m]

21. A neutron makes a head-on collision with a deuterium nucleus at rest. Find the ratio of the kinetic energies after and before the collision.

[Ans. \(\frac{1}{3}\)]

22. A beam of neutron is passed through paraffin. Its incident flux is \(5 \times 10^{10}\) m\(^{-2}\) s\(^{-1}\). The differential scattering cross-section is measured as \(1.5 \times 10^{-26}\) m\(^2\) Sr\(^{-1}\) at 60°.

Calculate the number of particles scattered per unit time by

(i) a single paraffin molecule, and

(ii) \(10^{22}\) paraffin molecules into a solid angle \(10^{-3}\) Sr.

[Ans. (i) \(7.5 \times 10^{-19}\) s\(^{-1}\) (ii) \(7.5 \times 10^{3}\) s\(^{-1}\)]