2 CHAPTER

State of Stresses

2.0 INTRODUCTION

Stress and strain are most important concepts for a comprehension of the mechanics of solids. The load carrying capacity and sizing of the components depend on the behaviour of material under different kinds of loading. This chapter deals with behaviour of material under uniaxial, biaxial and triaxial states of stress under various combinations of loads. Concept of principal stress and strains is also illustrated for different combinations of loading conditions.

2.1 SIMPLE STRESSES AND STRAINS IN MACHINE ELEMENTS

2.1.1 Stress

When external loads act on a body, internal forces are set up which resist the external forces. This internal force per unit area at any section of the body is known as unit stress or simply stress. It is denoted by the Greek letter sigma (σ). Mathematically,

$$\sigma = \frac{\text{Resisting force}}{\text{Cross-sectional area}} = \frac{F}{A}$$

Thus, stress is defined as the internal resistance developed in the body due to external disturbances over an unit area of its cross section. Stress at a given point does not only depend on the location of the point but also on the plane passing through it. Therefore, it is a second-order tensor.

In SI units, stress is usually expressed in Pascal (Pa) such that 1 Pa = 1 N/m^2. In actual practice, we use bigger units of stress, that is, megapascal (MPa) and gigapascal (GPa), such that

$$1\,\text{MPa} = 1 \times 10^6\ \text{N/m}^2 = 1\,\text{N/mm}^2 \text{ and } 1\ \text{GPa} = 1 \times 10^9\ \text{N/m}^2 = 1\,\text{kN/mm}^2$$

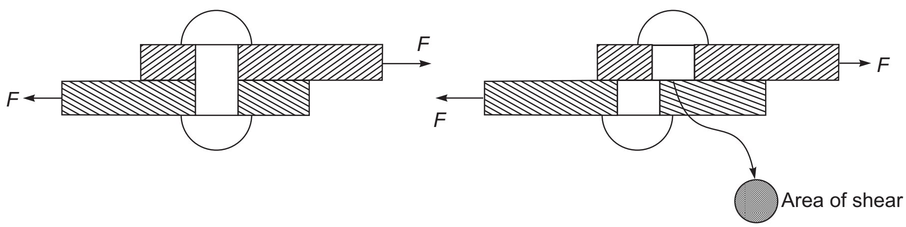
When loads act perpendicular to the axis of the member, they are called normal loads and the corresponding stresses are called normal stresses. Normal stresses are caused by (i) direct loading (may be tensile or compressive) and (ii) bending (both tensile and compressive).

Poisson's Ratio: When a deformable body is subjected to an axial tensile force, elongation as well as lateral contraction occurs. For example, if a metal bar is stretched, both the thickness and width decrease while the length increases. Likewise, the compressive force acting on a body causes it to contract in the direction of force and its sides expand laterally. Poisson's ratio is the ratio of lateral strain to linear strain. In the elastic range, Poisson's ratio lies between 0.25 and 0.33 for most engineering materials.

Typical values of Poisson's ratio for some common materials are given in Table 1.1, p.11 of DDHB.

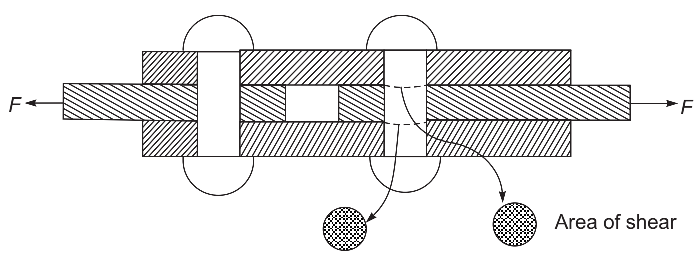
2.1.4 Shear Stress and Strain

Shear stress is induced when a body is subjected to two equal and opposite forces acting tangentially across the resisting section. The rivet shown in Figure 2.3 is subjected to a single shear by tangential tensile load F.



$$\text{Shear stress}, \tau = \frac{\text{Tangential force}}{\text{Area of shear}} = \frac{F}{A_s} \qquad \text{(Eq. 1.1c, DDHB p. 2)}$$

Figure 2.3 Single shear



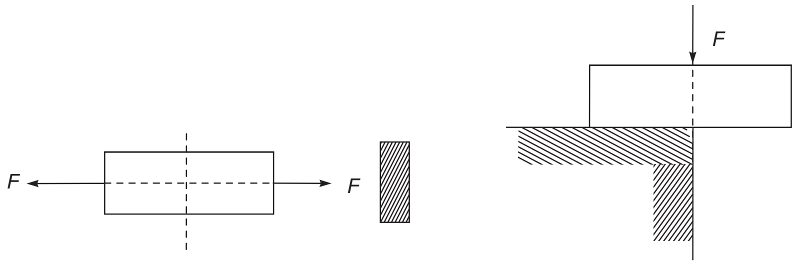
$$\text{Shear stress}, \tau = \frac{\text{Tangential force}}{\text{Area of shear}} = \frac{F}{2A_s}$$

Figure 2.4 Double shear

Figure 2.4 shows double shear, as the force is resisted by two sections of the body. Simple shear stress is accompanied by shear strain γ which is measured in terms of angular deformation.

Shear stresses are caused by (i) direct loading (in the transverse direction), (ii) twisting (torsion) and (iii) bending (only in the case of unequal bending; pure bending will not induce any shear stress in a beam).

Load direction determines the type of stresses induced. Consider Figure 2.5.



Force F is perpendicular to the cross-sectional area. This gives rise to tensile stress.

$$\sigma = \frac{F}{A}$$

Force F is parallel to the cross-sectional area. This gives rise to shear stress.

$$\tau = \frac{F}{A}$$

Figure 2.5

Note that the equations for both kinds of stresses are the same.

2.2 COMPOUND STRESSES

In practice many machine components are subjected to a combination of normal and shear stresses. In such cases, resultant stresses are to be worked out and compared to the design stresses of the components.

In general, the stress system in the case of three-dimensional state stresses can be represented by three normal stresses and three shearing stresses. When direct stresses and shearing stresses act simultaneously on a system, these are called the combined stresses or compound stresses. In a two-dimensional state of stress, a stress field due to external loading may result in normal stress σ_x, σ_y and shear stresses τ_x at a point on the component.

2.2.1 Sign Conventions

There is a system of notations we follow for stresses. Normal stresses are denoted by σ_x, σ_y and σ_z in X, Y and Z directions respectively. Tensile stresses are considered to be positive, while compressive stresses are considered to be negative. Shear stresses are denoted by two subscripts. For example, consider a shear stress denoted by τ_{xy}. The subscript x indicates that the shear stress is on the area which is perpendicular to x axis. The subscript y indicates that the shear stress is acting in the y direction.

(a) **Normal stress:** Tensile stresses are considered to be positive, while compressive stresses are considered to be negative, when the stresses are in the x- or y-direction (Figure 2.6).

$$\sigma_n \times mn \times 1 = (\sigma_x \times pn \times 1) \times \cos\phi$$

$$\sigma_n = \sigma_x \times \frac{pn}{mn} \times \cos\phi = \sigma_x \cos^2\phi \qquad \text{(Eq. 1.6a, DDHB, p. 4)}$$

Now resolving forces parallel to *mn*

$$\tau_n \times mn \times 1 = (\sigma_x \times pn \times 1) \times \sin\phi$$

$$\tau_n = \left(\sigma_x \times \frac{pn}{mn} \times 1\right) \times \sin\phi$$

$$\tau_n = (\sigma_x \times \cos\phi) \times \sin\phi = \frac{2}{2} \times (\sigma_x \times \cos\phi) \times \sin\phi = \frac{\sigma_x}{2}\sin 2\phi \qquad \text{(Eq. 1.6b, DDHB, p. 4)}$$

Maximum and minimum normal in uniaxial state of stress

We know that normal stress $(\sigma_n) = \sigma_x \cos^2\phi$. Maximum normal stress (principal stress) σ_1 acts on cross-section normal to the axis of the bar that is when $\phi = 0$ (Eq. 1.6d, DDHB, p. 4)

Therefore, Maximum normal stress = Maximum principal stress = σ_1 = σ_x (Eq. 1.6c, DDHB, p. 4)

Minimum normal stress acts on the cross section parallel to the axis of the bar, that is, when $\phi = 90$

Therefore, Minimum normal stress = Minimum principal stress = σ_2 = 0 (Eq. 1.6c, DDHB, p. 4)

Shear stresses in uniaxial state of stress

Maximum shear stress occurs when $\sin 2\phi = 1$ or $\phi = 45°$ (Eq. 1.6f, DDHB, p. 4)

Maximum shear stress, $\tau_{max} = \frac{\sigma_x}{2} \times \sin 2\phi = \frac{\sigma_x}{2} = \frac{\sigma_1}{2}$ (Eq.1.6e, DDHB, p. 4)

That is, maximum shear stress induced in a body subjected to uniaxial stress is half of normal stress and is inclined at an angle of $45°$ with respect to reference axis (with vertical).

WORKED OUT PROBLEMS (UNIDIRECTIONAL STATE OF STRESS)

Problem 2.1. A light-duty engine part of diameter 20 mm is made up of grey iron cast iron FG150. Calculate the load (a) when rod is subjected to its full tensile strength, (b) when tested in compression.

Given data: Material: FG150. Diameter = D = 20 mm

Solution:

Both loadings are shown in Figure 2.9

Refer to Table 1.3, DDHB, p. 459.

Tensile strength = $\sigma_y = 150$ MPa, Compressive strength $= \sigma_c = 600$ MPa

Shear strength $\tau = 173$ MPa

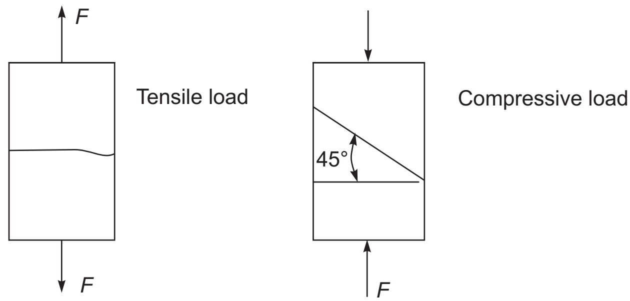


Figure 2.9

(a) When subjected to its full tensile strength

Tensile strength = 150 MPa

Corresponding stress in shear = $\frac{150}{2} = 75 \text{ MPa} < 173 \text{ MPa}$

Hence, the part fails by tension. Failure occurs on the plane of axial stress.

$$F_t = \text{Area} \times \sigma_y = \frac{\pi \times 20^2}{4} \times 150 = 471223 \text{ N} = 47.122 \text{ kN}$$

(b) When subjected to compression

Stress in compression = 600 MPa

Corresponding stress in shear = $\frac{600}{2} = 300 \text{ MPa} > 173 \text{ MPa}$

Hence, the part fails by shear. Failure occurs on the plane 45° to the plane of axial stress.

$$\text{Corresponding compressive stress} = 2 \times \tau = 2 \times 173 = 346 \text{ MPa}$$

$$\text{Corresponding compressive force} = \text{Area} \times \sigma_y$$

$$F_c = \frac{\pi \times 20^2 \times 346}{4} = 108695.9 = 108.69 \text{ kN}$$

Problem 2.2. A circular bar of diameter 25 mm is subjected to an axial force of 20 kN as shown in Figure 2.10. Find the stresses on a plane making an angle 30° to the plane of axial stresses and also on the plane which has maximum shear stress.

Given data: Diameter = D =25 mm, Load F = 20 kN = 20000 N

Solution:

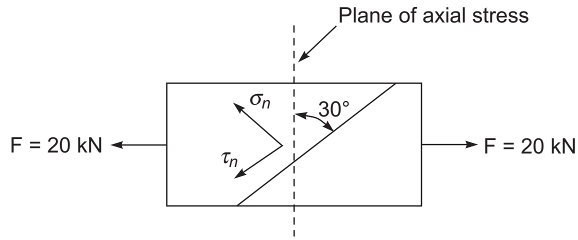


Figure 2.10

Axial stress, $\sigma_x = \frac{F}{A} = \frac{20 \times 1000}{\frac{\pi \times 25^2}{4}} = 40.7435 \text{ MPa}$

The **normal stress** acting on a plane inclined at 30°

$$\sigma_n = \sigma_x \cos^2\phi = 40.7435 \times \cos^2 30 = 30.558 \cong 35.6 \text{ MPa}$$

The **shear stress** acting on a plane inclined at 30°

$$\tau_n = \frac{\sigma_x}{2}\sin 2\phi = \frac{40.7435}{2} \times \sin(2 \times 30) = 17.64 \text{ MPa}$$

Maximum shear stress occurs on a plane where $\phi = 45°$

$$\tau_{max} = \frac{\sigma_x}{2}\sin 2\phi = \frac{40.7435}{2} \times \sin(2 \times 45) = 20.37 \text{ MPa}$$

Problem 2.3. An element of a machine member of rectangular cross section is subjected to a 10 kN unidirectional load using universal tensile testing machine. Determine the cross-sectional area of the member. Take the width of the beam as thrice the thickness. Also find the normal and shear stresses along a plane inclined at 30° (Figure 2.11). The allowable stress for the material is 294 MPa.

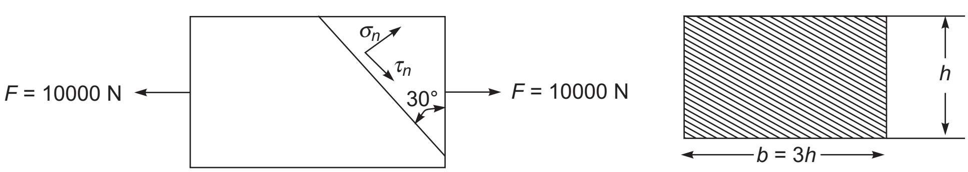


Figure 2.11

Given data:

Force F = 10000 N. $b = 3h$ (rectangular cross section). Allowable stress = 294 MPa. Inclination $\phi = 30°$.

Solution:

Axial stress, $\sigma_x = \dfrac{F}{A}$

$$294\text{ MPa} = \frac{10\times1000}{b\times h} = \frac{10\times1000}{3h\times h}$$

Solving we get, h = 3.367 mm take h = 3.5 and b = 3 × h = 3×3.5 = 11.5 mm

Actual stress induced in beam of cross section 3.5 × 11.5 is

Actual axial stress, $\sigma_x = \dfrac{F}{A} = \dfrac{10\times1000}{b\times h} = \dfrac{10\times1000}{3\times3.5^2} = 272.1\text{ MPa}$

The **normal stress** acting on a plane inclined at 30°

$$\sigma_n = \sigma_x\cos^2\phi = 272.1\times\cos^2 30 = 204.08\text{ MPa}$$

The **shear stress** acting on a plane inclined at 30°

$$\tau_n = \frac{\sigma_x}{2}\sin2\phi = \frac{204.08}{2}\times\sin(2\times30) = 88.369\text{ MPa}$$

2.2.3 Bidirectional State of Stress without Shear

Figure 2.12 shows an element of unit depth subjected to bidirectional stress. ϕ is the angle between horizontal and σ_n.

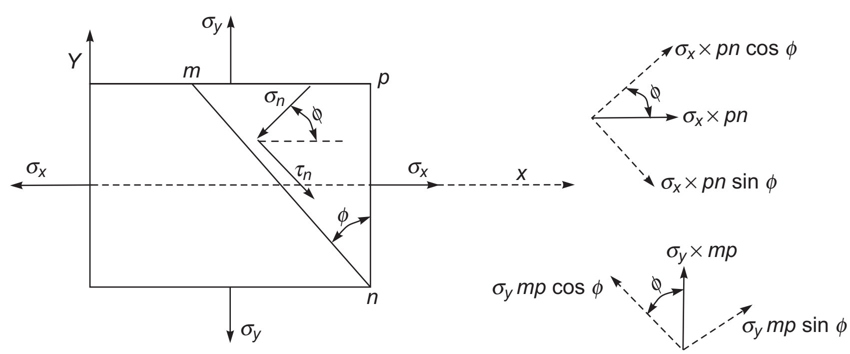


Figure 2.12

Let the block be of unit depth, then considering the equilibrium of forces on the triangular portion *mnp*

Resolving the forces perpendicular to *mn*,

$$\sigma_n \times mn\times1 = (\sigma_x\times pn\times1)\times\cos\phi + (\sigma_y\times mp\times1)\times\sin\phi$$

$$\sigma_n = \left(\sigma_x\times\frac{pn}{mn}\times1\right)\times\cos\phi + \left(\sigma_y\times\frac{mp}{mn}\times1\right)\times\sin\phi$$

2.2.4 Pure Shear

Figure 2.13 shows an element subjected to pure shear. Since the applied and complementary shears are of equal value on x and y planes, they are both given symbol τ_{xy}

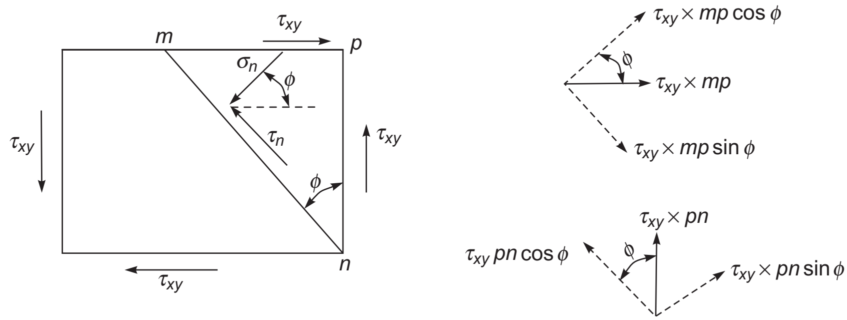


Figure 2.13

ϕ is the angle between the horizontal and σ_n.

Let the block be of unit depth, then considering the equilibrium of forces on the triangular portion mnp,

Resolving the normal to mn

$$\sigma_n \times mn \times 1 = \left(\tau_{xy} \times pn \times 1\right) \times \sin\phi + \left(\tau_{xy} \times mp \times 1\right) \times \cos\phi$$

$$\sigma_n = \left(\tau_{xy} \times \frac{pn}{mn} \times 1\right) \times \sin\phi + \left(\tau_{xy} \times \frac{mp}{mn} \times 1\right) \times \cos\phi$$

$$\sigma_n = \tau_{xy} \cos\phi \sin\phi + \tau_{xy} \sin\phi \cos\phi$$

$$\sigma_n = \tau_{xy} \sin 2\phi$$

Now resolving forces parallel to mn

$$\tau_n \times mn \times 1 = -\left(\tau_{xy} \times pn \times 1\right) \times \cos\phi + \left(\tau_{xy} \times mp \times 1\right) \times \sin\phi$$

$$\tau_n = -\left(\tau_{xy} \times \frac{pn}{mn} \times 1\right) \times \cos\phi + \left(\tau_{xy} \times \frac{mp}{mn} \times 1\right) \times \sin\phi$$

$$\tau_n = -\left(\tau_{xy} \times \cos\phi \times 1\right) \times \cos\phi + \left(\tau_{xy} \times \sin\phi \times 1\right) \times \sin\phi$$

$$\tau_n = -\tau_{xy} \times \cos^2\phi + \tau_{xy} \times \sin^2\phi = -\tau_{xy}\left(\cos^2\phi - \sin^2\phi\right)$$

$$\tau_n = -\tau_{xy} \cos 2\phi$$

The **normal stress** acting on plane $mn = \sigma_n = \tau_{xy} \sin 2\phi$

The **shear stress** acting on plane $mn = \tau_n = -\tau_{xy} \cos 2\phi$

Maximum principal stress $\sigma_1 = \tau_{xy}$

Minimum principal stress $\sigma_2 = -\tau_{xy}$

Angle at which maximum principle stress acts $\phi_p = 45° \text{ or } = 135°$

$$\text{Maximum shear stress, } \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \tau_{xy}$$

Directions of shear stresses are $\phi_s = 0° \text{ or } 90°$

2.2.5 Biaxial State of Stress

Biaxial state of stress: When a component is subjected to different forces in such a way that the stresses produced act on two planes perpendicular to each other and there is no stress on the third plane which is perpendicular to these two planes, the above state of stress is then called biaxial stress.

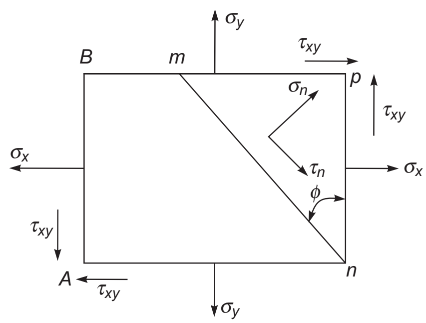
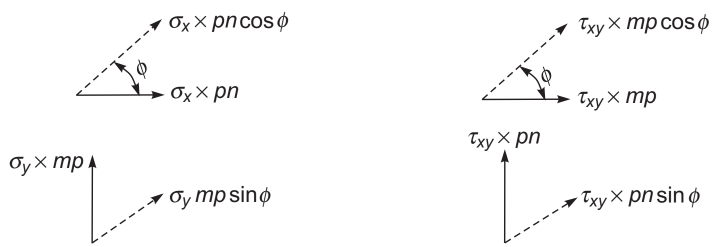


Figure 2.14

Figure 2.14 shows an element subjected to biaxial state of stress along with shear stress. ϕ is the angle between the horizontal and σ_n. Since the applied and complementary shears are of equal value on x and y planes, they are both given symbol τ_{xy}.

The diagram represents a complete stress system for any condition of applied load in two dimensions and represents an addition of stress system previously considered in Figures 2.12 and 2.13



(a) Normal stress components (b) Shear stress components

Figure 2.15

Expression for Maximum and Minimum Principal Stresses

The maximum and minimum normal stresses act on a plane called the major and minor planes respectively. To get an expression for maximum and minimum normal stresses with respect to ϕ, we should use the differential of the normal stress with respect to ϕ and equate it to zero.

$$\sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\phi + \tau_{xy}\sin 2\phi$$

$$\frac{\delta\sigma_n}{\delta\phi} = \frac{\delta}{\delta\phi}\left\{\frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\phi + \tau_{xy}\sin 2\phi\right\} = 0$$

Here, σ_x, σ_y and τ_{xy} are constants, therefore,

$$\left\{(0) + \left(\frac{\sigma_x - \sigma_y}{2}\right) \times (-\sin 2\phi) \times 2 + \tau_{xy}(\cos 2\phi) \times 2\right\} = 0$$

Dividing throughout by − 2 we get,

$$\left(\frac{\sigma_y - \sigma_x}{2}\right) \times (\sin 2\phi) - \tau_{xy}(\cos 2\phi) = 0$$

Rewritten as

$$-\left(\frac{\sigma_y - \sigma_x}{2}\right) \times (\sin 2\phi) + \tau_{xy}(\cos 2\phi) = 0 \qquad \text{(a)}$$

Compare Equation (a) with **Eq. 1.8b, DDHB, p. 5.**

The RHS of Equation (a) is zero. Therefore, we can conclude that shear stress is zero on principal planes.

Rewriting Equation (a) we get,

$$\left(\frac{\sigma_x - \sigma_y}{2}\right) \times (\sin 2\phi) = \tau_{xy}(\cos 2\phi)$$

$$\frac{\sin 2\phi}{\cos 2\phi} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{(Eq. 1.8e, DDHB, p. 5)}$$

$$\phi_{p1,p2} = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

$$\phi_{p2} = \phi_{p1} + 90^\circ$$

That is, maximum and minimum principal planes are orthogonal, or mutually perpendicular to each other.

Expression of the Maximum and Minimum Shear Stresses

The shear stress on an arbitrary plane is given by **Eq. 1.8b, DDHB, p. 5**

$$\tau_n = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\phi + \tau_{xy}\cos 2\phi$$

To get the orientation of maximum and minimum shear stresses, differentiate Equation (b) with respect to ϕ and equate it to zero.

$$\frac{\delta\tau_n}{\delta\phi} = \frac{\delta}{\delta\phi}\left\{-\frac{(\sigma_x - \sigma_y)}{2}\sin 2\phi + \tau_{xy}\cos 2\phi\right\} = 0$$

$$-\frac{(\sigma_x - \sigma_y)}{2}(\cos 2\phi)\times 2 + \tau_{xy}(-\sin 2\phi)\times 2$$

This gives,

$$-(\sigma_x - \sigma_y)(\cos 2\phi) = 2\tau_{xy}(\sin 2\phi) \qquad \text{(e)}$$

$$\frac{\sin 2\phi}{\cos 2\phi} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\phi = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) = \left(\frac{\sigma_x - \sigma_y}{-2\tau_{xy}}\right) \qquad \text{(f)} \qquad \text{Eq. 1.8f, DDHB, p. 5}$$

$$\phi_{s1,s2} = -\frac{1}{2}\tan^{-1}\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) \qquad \text{(g)}$$

$$\phi_{s2} = \phi_{s1} + 90°$$

The orientations of maximum and minimum shear planes are at right angles to each other. Observing Equation (g) for orientation of principal planes and orientation of maximum shear stress, it can be seen that the planes of maximum and minimum shear stress occur at an angle of 45°and 135° with respect to the plane of maximum and minimum normal stress respectively.

From Equation (f), we can draw the right-angle triangle as shown in Figure 2.17.

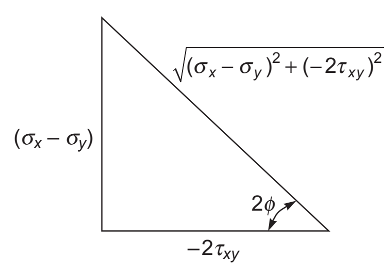


Figure 2.17

We can also use Eq. 1.8f, DDHB, p. 5 to determine the maximum shear stress.

$$\text{Maximum shear stress, } \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{20-0}{2} = 10 \text{ MPa}$$

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{20-0}{2}\right)^2 + 0} = \pm 10 \text{ MPa}$$

Maximum shear stress occurs on a plane where $\phi = 45°$

Problem 2.5. A tension member is formed by connecting with glue, two wooden scantlings, each 50 mm × 100 mm at their ends, which are cut at an angle of 60° as shown in Figure. 2.19. The member is subjected to a pull *F*. Calculate the safe value of *F*, if the permissible normal and shear stresses in the glue used are 3 N/mm^2 and 2N/mm^2 respectively.

Given data: $\sigma_n = 3 \text{ N/mm}^2, \tau = 2 \text{ N/mm}^2,$

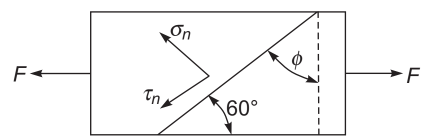


Figure 2.19

Solution:

(a) Normal stress

$$\phi = 30° \text{ (angle with the vertical)}$$

Area, A = 50 × 100 = 5000 mm^2

$$\sigma_x = \text{Normal stress} = \frac{F}{A}$$

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\phi + \tau_{xy} \sin 2\phi \qquad \text{Eq. 1.8a, DDHB, p. 5}$$

Here only σ_x is acting and $\sigma_y = 0$ and $\tau_{xy} = 0$

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\phi + \tau_{xy} \sin 2\phi$$

$$\sigma_n = \left(\frac{\sigma_x + 0}{2}\right) + \left(\frac{\sigma_x - 0}{2}\right)\cos 2\phi + 0 \times \sin 2\phi$$

$$\sigma_n = \left(\frac{\sigma_x}{2}\right) + \left(\frac{\sigma_x}{2}\right)\cos 2\phi = \frac{F}{2A} + \frac{F}{2A}\cos(60) = \frac{F}{2\times 5000} + \frac{F}{2\times 5000}\cos(60) \le 3$$

Solving we get $F \le 20000$ N

Take F = 20000 N.

$$\tau_n = -\frac{0-0}{2}\sin 2\phi + \tau_{xy}\cos 2\phi = +\tau_{xy}\cos 2\phi$$

$$\tau_n = +50\cos 60 = +25 \text{ N/mm}^2 = +25 \text{ MPa}$$

(b) Principal stress

Maximum principal stress (Eq. 1.8c, DDHB, p. 5)

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 0 + \sqrt{(0)^2 + \tau_{xy}^2} = 0 + \sqrt{(0)^2 + (50)^2} = 50 \text{ MPa (tensile)}$$

Minimum principal stress (Eq. 1.8d, DDHB, p. 5)

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 0 - \sqrt{(0)^2 + \tau_{xy}^2} = 0 - \sqrt{(0)^2 + (50)^2} = -50 \text{ MPa}$$

$$\sigma_2 = -\tau_{xy} = -50 \text{ MPa (compressive)}$$

The values of the angle ϕ where either a maximum or minimum normal stress occurs

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\times 50}{0-0} = \infty$$

$$2\phi = 90° \text{ or } 270° \text{ i.e., } \phi = 45° \text{ or } 135°$$

(c) **Shear stress** (Eq. 1.8f, DDHB, p. 5)

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm 50 \text{ MPa}$$

The values of the angle ϕ_s where either a maximum or minimum normal stress occurs

$$\tan 2\phi_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = -\frac{0-0}{2\times(50)} = 0,\ \tan 2\phi_s = 0 \text{ or } \phi_s = 0° \text{ or } \phi_s = 90°$$

Problem 2.7. A point in a certain element is subjected to a horizontal tensile stress of 100 N/mm^2 and vertical shear stress of 60 N/mm^2 as shown in Figure 2.21. Find the magnitude of principal stresses and its location.

tangential, resultant stress and its obliquity on a plane making an angle of 30° with vertical as shown in Figure 2.25. Also find the maximum shear stress.

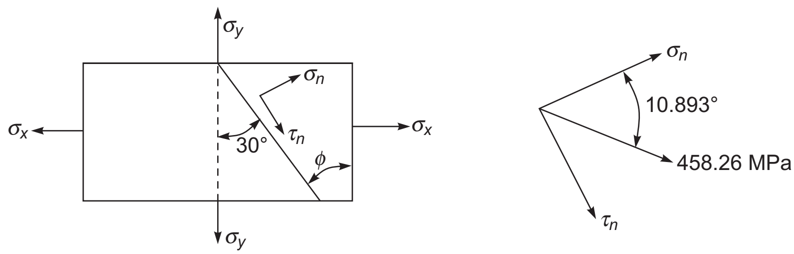


Figure 2.25

Given data: $\sigma_x = 500\,\text{N/mm}^2$, $\sigma_y = 300\ \text{N/mm}^2$, $\phi = 30°$

Solution:

(a) Normal stress

Let us write the generalised biaxial state of stress equation

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\phi + \tau_{xy}\sin 2\phi \qquad \text{Eq. 1.8a, DDHB, p. 5}$$

Here, σ_x and σ_y are present and $\tau_{xy} = 0$

$$\sigma_n = \frac{500+300}{2} + \frac{(500-300)\cos 60}{2} + 0 = 450\ \text{N/mm}^2 = 450\ \text{MPa}$$

(b) Shear stress

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\phi + \tau_{xy}\cos 2\phi \qquad \text{Eq. 1.8b, DDHB, p. 5}$$

$$\tau_n = -\frac{(500-300)\sin 60}{2} = -86.6025\ \text{N/mm}^2 = -86.6025\ \text{MPa}$$

(c) Resultant stress of normal and shear stress on an oblique plane.

$$\sigma_R = \sqrt{\sigma_n^2 + \tau_n^2} = \sqrt{450^2 + (-86.6025)^2} = 458.3\ \text{MPa}$$

$$\tan\phi = \frac{\tau_n}{\sigma_n}$$

$$\phi = \tan^{-1}\left(\frac{86.6025}{450}\right) = 10.893^\circ$$

The angle made by the resultant and the normal stress on an oblique plane is 10.893°

(d) Maximum shear stress (Eq. 1.8f, DDHB, p. 5)

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) = -\left(\frac{500 - 300}{2}\right) = -100 \text{ MPa}$$

$$\tau_{max} = -100 \text{ MPa.}$$

It is negative shear stress and produces clockwise rotation w.r.t the x face.

Problem 2.10. The principal stresses at a point in a bar are 200 N/mm^2 (tensile) and 100 N/mm^2 (compressive). Determine the normal; resultant stress and its direction on a plane inclined at 60° to the horizontal axis as shown in Figure 2.26. Also determine the maximum intensity of shear stress in the material at the point.

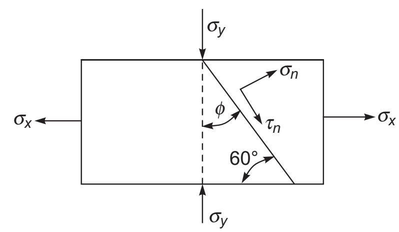


Figure 2.26

Given data: $\sigma_x = 200$ N/mm^2 (tensile), $\sigma_y = 100$ N/mm^2 (compressive)

(a) Normal stress

Let us write the generalised biaxial state of stress equation

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\phi + \tau_{xy}\sin 2\phi \qquad \text{Eq. 1.8a, DDHB, p. 5}$$

Here, $\sigma_x = 200$ MPa and $\sigma_y = -100$ are present and $\tau_{xy} = 0$

Since the plane is inclined 60° to the horizontal, $\phi = 30°$

$$\sigma_n = \left(\frac{200 - 100}{2}\right) + \left(\frac{200 - 100}{2}\right)\cos 60 + 0 = 125 \text{ N/mm}^2 = 125 \text{ MPa}$$

(b) Shear stress

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\phi + \tau_{xy}\cos 2\phi \qquad \text{Eq. 1.8b, DDHB, p. 5}$$

$$\tau_n = -\left(\frac{200 - (-100)}{2}\right)\sin 60 + 0 = -129.9 \frac{\text{N}}{\text{mm}^2} = -129.9 \text{ MPa}$$

Problem 2.13. For the stress element shown in Figure 2.29, find the principal stresses and their directions. **(VTU June/July 2014)**

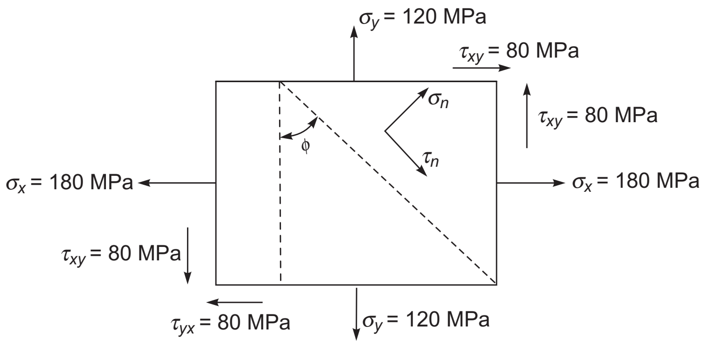


Figure 2.29

Given data:

$$\sigma_x = 180 \text{ MPa}, \ \sigma_y = 120 \text{ MPa}, \ \tau_{xy} = 80 \text{ MPa}$$

Solution:

Principal stress

Maximum principal stress

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{(Eq. 1.8c, DDHB, p. 5)}$$

$$\sigma_1 = \frac{180 + 120}{2} + \sqrt{\left(\frac{180 - 120}{2}\right)^2 + 80^2} = 150 + \sqrt{900 + 6400} = 150 + 85.44 = 235.44 \text{ MPa}$$

Minimum principal stress

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{(Eq. 1.8d, DDHB, p. 5)}$$

$$\sigma_1 = \frac{180 + 120}{2} + \sqrt{\left(\frac{180 - 120}{2}\right)^2 + 80^2} = 150 - \sqrt{900 + 6400} = 150 - 85.44 = 64.56 \text{ MPa}$$

The values of the angle ϕ where either a maximum or minimum normal stress occurs can be found as follows:

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{(Eq. 1.8e, DDHB, p. 5)}$$

$$\tau_{max} = \sqrt{\left(\frac{30+20}{2}\right)^2 + 15^2} = \pm 29.155 \text{ MPa}$$

or $$\tau_{max} = \left(\frac{\sigma_1 - \sigma_2}{2}\right) \quad \text{(Eq. 1.8f, DDHB, p. 5)}$$

$$\tau_{max} = \left(\frac{34.1547 - (-24.155)}{2}\right) = \pm 29.155 \text{ MPa}$$

The values of the angle ϕ_s where either a maximum or minimum normal stress occurs can be found as follows:

$$\tan 2\phi_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad \text{(Eq. 1.8g, DDHB, p. 5)}$$

$$\tan 2\phi_s = -\frac{30-(-20)}{2(15)} = -\frac{50}{30}$$

$$2\phi_s = -59.03°$$

or $$\phi_{s1} = -29.518° \text{ and } \phi_{s2} = -29.518° + 90° = 60.481°$$

Also

$$\phi_{s1} = \phi_{p1} + 45 = 15.481° + 45° = 60.481° \text{ and } \phi_{s2} = 15.481° + 135° = 150.482°$$

2.2.7 Triaxial State of Stress

When a cubical element of a deformable body is under the action of external forces, a stress would act on each of its six faces. If these stresses are resolved into the normal and tangential components to each of the faces, nine stresses act on the element. These are said to be a triaxial stress element. The triaxial stress element shown in Figure 2.31 consists of three normal stresses $\sigma_x, \sigma_y, \sigma_z$ and six shear stresses $\tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{xz}, \tau_{zx}$, all positive. If the element is in equilibrium, then $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{xz} = \tau_{zx}$. The nine components at a point on the element can be represented by a second order tensor.

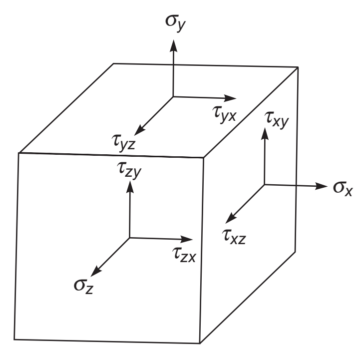


Figure 2.31

$$\sigma_{ij} = \text{state of stress at a point} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

In a triaxial state of stress, there are six faces on which normal and shear forces act and there will be three principal stresses and three principal planes. The principal stresses are obtained by solving the cubic equation.

$$\sigma^3 - I_1\sigma^2 - I_2\sigma + I_3 = 0$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2$$

$$I_3 = \det\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \det\left[\sigma_{\text{ij}}\right]$$

where I_1, I_2, I_3 are called stress invariants.

In the case of triaxial state of stress, resultant unit deformation or strains in X, Y and Z directions are given by Equations 1.12(a), 1.12(b) and 1.12(c), DDHB, p. 7.

Problem 2.15. Find the principal stress for the following state of stress condition that exists in an element.

$$\sigma_{ij} = \begin{Bmatrix} 80 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & -40 \end{Bmatrix} \text{MPa}$$

Solution:

State of stress is given. We can determine principal stresses. The three invariants are

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 80 + 0 - 40 = 40 \text{ MPa}$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 = 80 \times 0 + 0 \times -40 + (-40 \times 80) - 20^2 - 0^2 - 20^2$$

$$= -4000 \text{ MPa}^2$$

$$I_3 = \det\begin{vmatrix} 80 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & -40 \end{vmatrix} = -16000 \text{ MPa}^3$$

$$\sigma^3 - 40\sigma^2 - 4000\sigma + 16000 = 0$$

The roots of the equation are

$$\sigma_1 = 84.896 \text{ MPa}, \ \sigma_2 = 3.865 \text{ MPa}, \ \sigma_3 = -48.761 \text{ MPa}$$

10. Stresses in a two-dimensional stressed body is shown in Figure 2.38. Determine (i) principal stresses and their direction, and (ii) the maximum shear stresses and their planes.

(VTU June/July 2016)

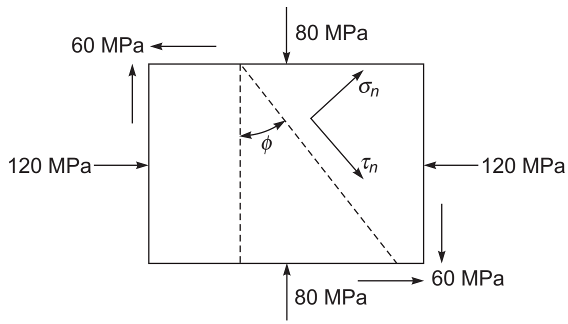


Figure 2.38