CHAPTER 2

Representations of the Lorentz and Poincare Groups and Transformation Properties of Physical Quantities

At the very outset, it is pertinent to mention that finite dimensional irreducible representations (IRRs) of both these groups are not unitary and all unitary representations are infinite dimensional since the groups are non-compact.

2.1 UNITARY REPRESENTATIONS OF THE LORENTZ GROUP (WIGNER, 1939; GEL'FAND, 1963; NAIMARK, 1964; OHNUKI, 1976; TUNG, 1984)

2.1.1 Reduction of Representations and Relabelling of Basis

The Lie algebra for the proper Lorentz group is given by

$$\left[\hat{J}_{\mu\nu}, \hat{J}_{\lambda\sigma}\right] = i\left(\hat{J}_{\sigma\nu}\, g_{\mu\lambda} - \hat{J}_{\lambda\nu}\, g_{\mu\sigma} - \hat{J}_{\mu\lambda}\, g_{\nu\sigma} + \hat{J}_{\mu\sigma}\, g_{\nu\lambda}\right) \tag{1}$$

or equivalently by

$$\left[\hat{J}_{m,} \hat{J}_{n}\right] = i\varepsilon^{mnl}\; \hat{J}_{l} \tag{2}$$

$$\left[\hat{K}_{m}, \hat{J}_{n}\right] = i\varepsilon^{mnl}\; \hat{K}_{l} \tag{3}$$

$$\left[\hat{K}_{m}, \hat{K}_{n}\right] = -\, i\varepsilon^{mnl}\; \hat{J}_{l} \tag{4}$$

related to the eigenvalue of the Casimir operator $\hat{M}^2$ and write eq. (12) as

$$\hat{M}_3|uk\rangle = k|uk\rangle \tag{13}$$

and, further

$$\hat{M}^2|uk\rangle = u(u+1)|uk\rangle \tag{14}$$

Of these, the index u corresponding to the eigenvalue of the Casimir operator $\hat{M}^2$ labels out the IRR, since it is left invariant by group operations and hence, is constant across a given IRR, whereas the other index k, which is the eigenvalue of $\hat{M}_3$ labels out the basis vectors within an IRR. We have

$$\hat{M}_3\hat{M}_\pm|uk\rangle = \hat{M}_\pm\hat{M}_3|uk\rangle + \left[\hat{M}_3, \hat{M}_\pm\right]\left|uk\right\rangle$$

$$= \hat{M}_\pm\hat{M}_3|uk\rangle \pm \hat{M}_\pm|uk\rangle = (\mathrm{k} \pm 1)\ \hat{M}_\pm|uk\rangle \tag{15}$$

Thus, $\hat{M}_\pm|uk\rangle$ is an eigenstate of $\hat{M}_3$ with eigenvalue $k \pm 1$. This means that, in line with our labelling scheme for eigenstates, we must have $\hat{M}_\pm|uk\rangle \infty |u, k \pm 1\rangle$ or $\hat{M}_\pm|uk\rangle = A_m^\pm|u, k \pm 1\rangle$. Similar calculations apply to $SU(2)_N$.

Hence, we can label the IRRs of the direct product algebra $SU(2)_M \otimes SU(2)_N$ that provides the IRRs for the Lorentz group by two numbers u, v corresponding to the two Casimir operators $\hat{M}^2$ and $\hat{N}^2$ of the subalgebras $SU(2)_M$ and $SU(2)_N$ respectively. The basis vectors in the representation space are, then, chosen as $\{|uv; kl\rangle\} \equiv \{|kl\rangle\}$ that consist of the direct products of the basis vectors from the canonical basis $\{|uk\rangle\}$ and $\{|vl\rangle\}$ of the two subalgebras $SU(2)_M$ and $SU(2)_N$ respectively. On this basis, the action of the generators of the Lorentz group is given by the following equations:

$$\hat{J}_3|kl\rangle = \left(\hat{M}_3 + \hat{N}_3\right)|kl\rangle = (k + l)|kl\rangle \tag{16}$$

$$\hat{J}_\pm\left|kl\right\rangle = \left(\hat{M}_\pm + \hat{N}_\pm\right)|kl\rangle = A^\pm(u, k)|k \pm 1, l\rangle + B^\pm(v, l)|k, l \pm 1\rangle \tag{17}$$

$$\hat{K}_3\left|kl\right\rangle = i\left(\hat{N}_3 - \hat{M}_3\right)|kl\rangle = i(l - k|kl\rangle \tag{18}$$

$$\hat{K}_\pm\left|kl\right\rangle = i\left(\hat{N}_\pm - \hat{M}_\pm\right)|kl\rangle = -iA^\pm(u, k)|k \pm 1, l\rangle + iB^\pm(v, l)|k, l \pm 1\rangle \tag{19}$$

It is seen from Eqs. (16) and (17) that the restriction of the representation space to the subgroup of three-dimensional rotations is equivalent to that of a direct product of representations of the $SO(3)$ group labelled by the indices u and v respectively.

We can reduce this direct product of representations to a direct sum of invariant subspaces with respect to rotations. For this purpose, we identify these constituents of the direct product representations in terms of u, v corresponding the eigenvalues of the respective Casimir operators and denote them by $D^u[R]$ and $D^v[R]$ respectively. We denote the direct product of these two representations by $D^{u\times v}[R]$. We designate, as above, the basis vectors in the product representations by $\{|uv; kl\rangle\} \equiv \{|kl\rangle\}$ so that $|kl\rangle = |uk\rangle \otimes |vl\rangle$. The action of $D^{u\times v}[R]$ on the direct product space $W = V_M \otimes V_N$ is obtained by its action on the direct product basis $|kl\rangle = |uk\rangle \otimes |vl\rangle$, e.g.,

$$D^{u\times v}[R]|kl\rangle = D^u[R]^m_k\, D^v[R]^n_l|mn\rangle \tag{20}$$

We show that $D^{u\times v}[R]$ defines a representation of $SO(3)$. This can be easily done by computing the character of the direct product representation. We have $\chi^{u+v} = TrD^{u\times v}[R] = D^{u\times v}[R]^k_k = D^u[R]^i_i\ D^v[R]^j_j = \chi^u\,\chi^v$ establishing our proposition.

We examine the reducibility of $D^{u\times v}[R]$. For this purpose, we write

$$D^{u\times v}[R] \sim \sum_{\lambda\oplus} a_\lambda\, D^\lambda[R] \tag{21}$$

i.e., we attempt to reduce the direct product space W into a direct sum of invariant subspaces W^λ_α, where λ labels out the IRR and $\alpha = 1, 2, \ldots \alpha_\lambda$, distinguishes the spaces that correspond to the same λ. We can, then, choose a basis in W corresponding to any IRR, i.e., any λ, such that the first n^λ_1 basis vectors are in W^λ_1, the next n^λ_1 in W^λ_2, and so on. Therefore, the representation matrices are in block diagonal form in the new basis.

To examine the reducibility of our direct product representation, we consider an infinitesimal rotation about an arbitrary axis $\hat{\mathbf{n}}$. Then, we must have

$$D^u[R_n(d\theta)]\, D^v[R_n(d\theta)] = D^{u\times v}[R_n(d\theta)] \tag{22}$$

whence, to first order in $d\theta$

$$\left(\hat{\mathbf{I}}^u - i\hat{J}^u_n\, d\theta\right).\left(\hat{\mathbf{I}}^v - i\hat{J}^v_n\, d\theta\right) = \hat{\mathbf{I}}^u \otimes \hat{\mathbf{I}}^v - i\left(\hat{J}^u_n \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}^v_n\right) d\theta = \left(\hat{\mathbf{I}}^{u\times v} - i\hat{J}^{u\times v}_n\, d\theta\right) \tag{23}$$

which gives $\hat{J}^{u\times v}_n = \hat{J}^u_n \otimes \hat{\mathbf{I}}^v \times \hat{\mathbf{I}}^u \otimes \hat{J}^v_n$. Thus, we find that the generators of a direct product representation are the sums of the corresponding generators of its constituent representations.

The generators defined by $\hat{J}^{u\times v}_n = \hat{J}^u_n \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}^v_n$ satisfy the same Lie algebra as the original one, i.e., $\left[\hat{J}^{u\times v}_k, \hat{J}^{u\times v}_l\right] = i\varepsilon^m_{kl}\,\hat{J}^{u\times v}_m$. We have

$$\begin{aligned}
\left[\hat{J}^{u\times v}_k, \hat{J}^{u\times v}_l\right] &= \left[\left(\hat{J}^u_k \otimes \hat{\mathrm{I}}^v + \hat{\mathrm{I}}^u \otimes \hat{J}^v_k\right), \left(\hat{J}^u_l \otimes \hat{\mathrm{I}}^v + \hat{\mathrm{I}}^u \otimes \hat{J}^v_l\right)\right] \\
&= \left(\hat{J}^u_{\hat{k}} \otimes \hat{\mathrm{I}}^v + \hat{\mathrm{I}}^u \otimes \hat{J}^v_k\right)\left(\hat{J}^u_l \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}^v_l\right) \\
&\quad - \left(\hat{J}^u_l \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}^v_l\right)\left(\hat{J}^u_k \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}^v_k\right) \\
&= \left(\hat{J}^u_k \otimes \hat{\mathbf{I}}^v\right)\left(\hat{J}^u_l \otimes \hat{\mathbf{I}}^v\right) + \left(\hat{\mathbf{I}}^u \otimes \hat{J}^v_k\right)\left(\hat{\mathbf{I}}^u \otimes \hat{J}^v_l\right) \\
&\quad - \left(\hat{J}^u_l \otimes \hat{\mathbf{I}}^v\right)\left(\hat{J}^u_k \otimes \hat{\mathbf{I}}^v\right) - \left(\hat{\mathbf{I}}^u \otimes \hat{J}^v_l\right)\left(\hat{\mathbf{I}}^u \otimes \hat{J}^v_k\right)
\end{aligned}$$

$$= \left(\hat{J}_k^u \hat{J}_l^u\right) \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \left(\hat{J}_k^v, \hat{J}_l^v\right) - \left(\hat{J}_l^u, \hat{J}_k^u\right) \otimes \hat{\mathbf{I}}^v - \hat{\mathbf{I}}^u \otimes \left(\hat{J}_l^v, \hat{J}_k^v\right)$$

$$= \left[\hat{J}_k^u, \hat{J}_l^u\right] \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \left[\hat{J}_k^v, \hat{J}_l^v\right] = i\varepsilon_{kl}^m \left(\hat{J}_m^u \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}_m^v\right) = i\varepsilon_{kl}^m J_m^{u\times v}$$

Our next step is to regroup the basis vectors of the product representation $\{|kl\rangle\}$ to form invariant subspaces thereby leading to the reduction of the direct product representation. We also have

$$\hat{J}_3^{u\times v}\Big|kl\Big\rangle = \left(\hat{J}_3^u \times \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \times \hat{J}_3^v\right)(|uk\rangle \times |vl\rangle) = (k + l)\,(|uk\rangle \times |vl\rangle) = (k + l)|kl\rangle \tag{24}$$

so that $|kl\rangle$ is an eigenvector of $\hat{J}_3^{u\times v}$ with eigenvalue $(k + l)$.

Further, $(\hat{J}^{u\times v})^2 = \left(\hat{J}_3^{u\times v}\right)^2 + \hat{J}_3^{u\times v} + \hat{J}_-^{u\times v}\hat{J}_+^{u\times v}$ whence

$$\left(\hat{J}^{u\times v}\right)^2\Big|uk\Big\rangle \otimes|vl\rangle = \left(\hat{J}_3^u \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}_3^v\right)\left(\hat{J}_3^u \otimes \hat{\mathrm{I}}^v + \hat{\mathrm{I}}^u \otimes \hat{J}_3^v\right)\Big|uk\Big\rangle \otimes|vl\rangle$$

$$+ \left(\hat{J}_3^u \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}_3^v\right)\Big|uk\Big\rangle \otimes|vl\rangle + \left(\hat{J}_-^u \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}_-^v\right)\left(\hat{J}_+^u \otimes \hat{\mathrm{I}}^v + \hat{\mathrm{I}}^u \otimes \hat{J}_+^v\right)|uk\rangle \otimes|vl\rangle$$

$$= (k + l)^2|uk\rangle \otimes|vl\rangle + (k + l)|uk\rangle \otimes |vl\rangle + \left(\hat{J}_-^u \otimes \hat{\mathbf{I}}^v + \hat{\mathbf{I}}^u \otimes \hat{J}_-^v\right)$$

$$[A^+(u, k)|u, k + 1\rangle \otimes|v, l\rangle + B^+(v, l)|uk \otimes|v, l + 1\rangle]$$

$$= (k + l)^2|uk\rangle \otimes|vl\rangle + (k + l)|uk\rangle \otimes |vl\rangle + A^-(u, k + 1)$$

$$A^+ (u, k)|u, k\rangle \otimes|v, l\rangle + B^-(v, l + 1)B^+ (v, l)|u, k\rangle \otimes|v, l\rangle$$

so that $|uk\rangle \otimes |vl\rangle$ is also an eigenvector of $\left(\hat{J}^{u\times v}\right)^2$.

We also know from the elementary theory of angular momentum in three dimensions, of which the group $SO(3)$ provides a representation, that the range of the eigenvalues of the generator $\hat{M} \equiv \hat{J}_3^u$ extends from $-u \le k \le +u$, where u relates to the eigenvalue of the Casimir operator $\hat{M}^2$. Similarly, the range of l extends from $-v \le l \le +v$. Therefore, the highest eigenvalue of $\hat{J}_3^{u\times v}$ is $u + v$ and there is only one eigenvector corresponding to this eigenvalue given by $|uv\rangle$. There will be two eigenvectors corresponding to the eigenvalue $u + v - 1$ of $\hat{J}_3^{u\times v}$ given by $|u - 1, v\rangle$ and $|u, v - 1\rangle$ and so on. The lowest eigenvalue of $\hat{J}_3^{u\times v}$ is $-u - v$ and there is only one eigenvector corresponding to this eigenvalue given by $|-u,-v\rangle$. Since the eigenvalues of $\hat{J}_3^{u\times v}$ change in integral steps, the total number of eigenvalues of $\hat{J}_3^{u\times v}$ will be $2(u + v) + 1$ and the corresponding eigenvectors can be obtained by applying the "lowering operator $\hat{J}_-^{u\times v}$" to the highest state $|uv\rangle$.

Since there is only one state with the eigenvalue $u + v$ and there exists no state with a higher eigenvalue of $\hat{J}_3^{u\times v}$, this state i.e. $|uv\rangle$ is the highest member of the basis of the IRR labelled by $u + v$. The remaining basis vectors can be generated by successive application of the lowering operator $\hat{J}_-^{u\times v}$, as mentioned above. The total number of basis vectors in this representation will be $2(u + v) + 1$ (because the range of eigenvalues of $\hat{J}_3^{u\times v}$ is from $u + v$ through 0 to $-(u + v)$) and they constitute the basis of an invariant subspace corresponding to the IRR labelled by $u + v$ relating it to the eigenvalue of $(\hat{J}^{u\times v})^2$.

As mentioned above, there are two linearly independent states corresponding to the eigenvalue $u + v - 1$ of $\hat{J}_3^{u\times v}$ given by $|u - 1, v\rangle$ and $|u, v - 1\rangle$. One of these states is included as the second element of the basis in the IRR labelled by $u + v$. The second state with eigenvalue $u + v - 1$ (orthogonal to the first with the same eigenvalue), will constitute the highest basis vector in another IRR that is labelled by the index $u + v - 1$. This basis will consist of $2(u + v - 1) + 1 = 2(u + v) - 1$ vectors which can be obtained by using the "lowering operator $\hat{J}_-^{u\times v}$."

For the eigenvalue $u + v - 2$ of $\hat{J}_3^{u\times v}$, there will be three linearly independent vectors viz. $|u, v - 2\rangle$, $|u - 1, v - 1\rangle$, $|u - 2, v\rangle$, one of which is included as the third element in the basis of the invariant subspace labelled by $u + v$, the second being included as the second element in the basis of the invariant subspace labelled by $u + v - 1$ and the third will be the highest element of the basis of the invariant subspace labelled by $u + v - 2$.

Now, if $u > v$, the above series will terminate when the eigenvalue of $\hat{J}_3^{u\times v}$ is $u - v$ (because the lowest allowed value of l is $-v$) with the last set of linearly independent vectors being $|u, -v\rangle$, $|u - 1, -v + 1\rangle, \ldots, |u - 2v, v\rangle$. Further, the space labelled by $u - v$ will have a basis consisting of $2(u - v) + 1$ linearly independent vectors (because the range of eigenvalues of $\hat{J}_3^{u\times v}$ in this case will be from $u - v$ through 0 to $-(u - v)$). A similar analysis holds if $v > u$ whence the series will terminate when the eigenvalue of $\hat{J}_3^{u\times v}$ is $v - u$. Hence, we conclude that this iterative process will yield an invariant subspace in each step with the index of the IRR diminishing by one until the index reaches $|u - v|$. The number of vectors that span the invariant subspace in this case will be $2|u - v| + 1$.

The aggregate number of independent basis states of all the IRRs taken together are:

$$[2(u + v) + 1] + [2(u + v - 1) + 1] + \ldots + [2|u - v| + 1] = (2u + 1)(2v + 1) \tag{25}$$

as it should be.

If we represent the new basis vectors by $\{|jm\rangle\}$, where $m = -j, \ldots, +j$ and $j = |u - v|, \ldots, u + v$, then the transformation matrix between the two sets is given by

$$|jm\rangle = \langle kl(uv)\ jm\rangle |kl\rangle \tag{26}$$

$$|kl\rangle = \langle jm(uv)\ kl\rangle |jm\rangle \tag{27}$$

where the coefficients of the basis vectors on the right hand side are called the Clebsch Gordon coefficients. The above notation for the Clebsch Gordon coefficients involves summing over the repeated indices, viz. kl and jm in eqs. (26) and (27) respectively. The indices in parenthesis (uv) are not to be summed over. Further $\langle jm(uv)kl\rangle = \langle kl(uv)jm\rangle^*$ so that the unitarity condition becomes $\langle jm(uv)kl\langle = \langle kl(uv)jm\rangle$.

The above discussion enables us to formulate another scheme for the labelling of IRRs. We can use the labels (j_0, j_1) to identify the IRR, where $j_0 = |u - v|$ and $j_1 = u + v$ are the lowest and highest values of the angular momentum in the representation.

2.1.2 Effect of the Group Generators on the Basis

In view of the foregoing, the representation space carries the label (j_0, j_1) and the vectors within a representation carry the labels (j, m). We also know, from the theory of angular momentum in three dimensions that j can be an integer or a half integer and $-j \le m \le +j$. The action of the generators of rotations $\hat{J}_3^{u\times v}$ on a basis vector is given by

$$\hat{J}_3^{u\times v}\,|jm\rangle = m|jm\rangle \tag{28}$$

$$\hat{J}_\pm^{u\times v}|jm\rangle = A_m^\pm\,(j, m)|jm \pm 1\rangle \tag{29}$$

The value of j, being related to the Casimir operator $(\hat{J}^{u\times v})^2$ does not change by the group operations i.e., rotations.

The effect of the generators of the Lorentz boosts $\hat{K}_i$ on the basis vectors is slightly more complicated. In view of eq. (3), the three components of $\hat{K}_i$ constitute the Cartesian components of a 3-vector under rotations. Applying the Wigner Eckart theorem to obtain the matrix elements between irreducible physical states represented in an appropriate basis, we obtain

$$\langle j'm'|\hat{K}_3|jm\rangle = A_j^{j'}\,\langle j'm'(1, j)0m\rangle \tag{30}$$

and

$$\langle j'm'|\hat{K}_\pm|jm\rangle = \mp\,\sqrt{2A_j^{j'}}\,\langle j'm'(1, j) \pm 1m\rangle \tag{31}$$

The coefficients $A_j^{j'}$ represent the reduced matrix elements given by $\langle j'|\hat{K}_3|j\rangle$. $\langle j'm'(1, j)0m\rangle$ and $\langle j'm'(1, j) \pm 1m\rangle$ are the Clebsch Gordon coefficients. The factor of $\mp\sqrt{2}$ arises in Eq. (31) because of the fact that $\hat{K}_3 \mp (2)^{-1/2}\,\hat{K}_\pm$ constitute a normalized set of irreducible spherical tensor operators. Taking the values of the Clebsch Gordon coefficients from published tables, noting that the coefficients $\langle j'm'(1, j)nm\rangle$ vanish unless $j' = j - 1, j, j + 1$ and $m' = m + n,$ we define

$$A_j^+ = A_j^{j+1}\,[(j + 1)\,(2j + 1)]^{-1/2} \tag{32}$$

$$A_j = A_j^j\,[j(j + 1)]^{-1/2} \tag{33}$$

$$A_j^- = A_j^{j-1}\,[j(2j + 1)]^{-1/2} \tag{34}$$

and write eqs. (30) and (31) as

$$\hat{K}_3|jm\rangle = |j - 1m\rangle\,[(j + m)\,(j - m)]^{1/2}\,A_j^- + |jm\rangle\,mA_j + |j + 1m\rangle\,[(j + m + 1)\,(j - m + 1)]^{1/2}\,A_j^+ \tag{35}$$

and

$$\hat{K}_{\pm}|jm\rangle = \mp|j-1 m\pm 1\rangle\, [j \mp m)(j \mp m - 1)]^{1/2} A_j^- + |jm \pm 1\rangle\, [(j \mp m)(j \pm m + 1)]^{1/2} A_j \pm |j+1 m \pm 1\rangle\, [(j \pm m + 1)(j \pm m + 2)]^{1/2} A_j^+ \tag{36}$$

2.1.3 Computing the Coefficients A_j^+, A_j and A_j^-

To obtain constraints on the coefficients A_j^+, A_j and A_j^- and hence, compute their admissible values, we mandate that the group Lie algebra must be satisfied. We start by requiring that $\left[\hat{K}_+, \hat{K}_-\right] = -2\hat{J}_3$ or equivalently

$$\left[\hat{K}_+, \hat{K}_-\right]|jm\rangle = -2\hat{J}_3|jm\rangle \tag{37}$$

Using eq. (36), the left-hand side of eq. (37) becomes

$$\begin{aligned}\left[\hat{K}_+, \hat{K}_-\right]|jm\rangle &= |j-1, m\rangle\, 2\sqrt{(j+m)(j-m)}\, A_j^-[(j-1)A_{j-1} - (j+1)A_j] \\ &- |jm\rangle\,(2m)\,[2j+3)A_{j+1}^- A_j^+ + A_j^2 - (2j-1)A_j^- A_{j-1}^+] \\ &+ |j+1, m\rangle\, 2\sqrt{(j+m+1)(j-m+1)}\, A_j^+[jA_j + (j+2)A_{j+1}]\end{aligned} \tag{38}$$

and the right hand side gives

$$-2\hat{J}_3|jm\rangle = -|jm\rangle\,(2m) \tag{39}$$

Equating eqs. (38) and (39), we get

$$[(j-1)A_{j-1} - (j+1)A_j]A_j^- = 0 \tag{40}$$

$$A_j^+[(j+2)A_{j+1} - jA_j] = 0 \tag{41}$$

$$(2j-1)A_j^- A_{j-1}^+ - A_j^2 - (2j+3)A_{j+1}^- A_j^+ = 1 \tag{42}$$

Equations (40) and (41) lead to the same constraint. Since the allowed values for j are $0, \frac{1}{2}, 1, \ldots$, they are bounded from below. Therefore, it must have a least value, say j_0. From eq. (35) and (36), it then follows that $A_{j0}^- = 0$. However, when $A_j^{\pm} \neq 0$, the expressions within the square brackets of eqs. (40) and (41) must vanish. For $A_j^- \neq 0$, eq. (41) gives

$$A_{j+1} = \frac{j}{j+2} A_j = A_{j0} \prod_{k=j_0}^{j} \frac{k}{k+2} = A_{j_0} \frac{j_0(j_0+1)}{(j+1)(j+2)} \tag{43}$$

The second step is simply the expansion of the recursion relation and the final step involves cancelling of common factors in the numerator and denominator. Writing $t = -i(j_0 + 1)A_{j_0}$, we have, from Eq. (43)

$$A_j = \text{it}\, \frac{j_0}{j(j+1)} \tag{44}$$

Using Eq. (44) and writing $B_j^2 = -A_j^- A_{j-1}^+$, eq. (42) becomes

$$-\sum_{k'=j_0+1}^{j}\frac{2k'-1}{k'^2(k'-1)^2}=\sum_{k'=j_0+1}^{j}\frac{k'^2-2k'+1-k'^2}{k'^2(k'-1)^2}$$

$$=\sum_{k'=j_0+1}^{j}\left[\frac{1}{k'^2}-\frac{1}{(k'-1)^2}\right]=\frac{1}{j^2}-\frac{1}{j_0^2} \tag{48}$$

Combining these results, we get

$$B_j^2=\frac{(j^2-j_0^2)(j^2-t^2)}{j^2(4j^2-1)} \tag{49}$$

whence, using $B_j^2 = -A_j^- A_{j-1}^+$, we can write

$$A_j^- = B_j \eta_j \text{ and } A_{j-1}^+ = -B_j \eta_j^{-1} \tag{50}$$

with η_j being arbitrary.

2.1.4 Computing *t* and η_j

The admissible values of t and η_j are subject to the constraints of unitarity of the representation and the orthonormality of the canonical basis. The requirement of unitarity mandates that $\hat{K}_3^\dagger = \hat{K}_3$ and $\hat{K}_\pm^\dagger = \hat{K}_\mp$ whence, from eqs. (35) and (36), we get

$$A_j = A_j^* \text{ and } A_j^- = -(A_{j-1}^+)^* \tag{51}$$

Substituting eqs. (44), (49) and (50) in eq. (51), we get

$$j_0(t+t^*)=0 \text{ and } |B_j|\,(|\eta_j|^2 - e^{-2i\beta_j}) = 0 \tag{52}$$

where we have obtained the second equation by writing $B_j = |B_j|e^{i\beta_j}$ so that $B_j^* = |B_j|e^{-i\beta_j}$ which gives $A_j^- = B_j\eta_j = -(A^+_{j-1})^* = -(-B_j^*\eta_j^{-1*}) = B_j^*\eta_j^{-1*}$ whence $|B_j|e^{i\beta_j}|\eta_j|^2 - |B_j|e^{-i\beta_j} = 0 = |B_j|\,(|\eta_j|^2 - e^{-2i\beta_j})$. If $|B_j| \neq 0$, then $|\eta_j|^2 - e^{-2i\beta_j} = 0$ whence $\beta_j = 0$ and $|\eta_j|^2 = 1$. Thus, η_j is an arbitrary phase factor associated with the basis vectors. Conventionally, we choose $\eta_j = 1$ for all values of j.

The first part of eq. (52) mandates that either (i) $t^* = -t$, i.e., t is imaginary or (ii) $j_0 = 0$. IRRs corresponding to: (i) are known as the Principal Series while those corresponding to (ii) are classified as Complementary Series. It is pertinent to mention here that the parameter t gets identified with j_1 in the usual (j_0, j_1) representation.

2.1.5 Principal Series and Complementary Series

(i) Writing $t = -i\omega$ with $\omega \in \mathbf{R}$, we have $A_j = \dfrac{\omega j_0}{j(j+1)}$ and $B_j^2 = \dfrac{(j^2-j_0^2)(j^2+\omega^2)}{j^2(4j^2-1)} > 0$ is automatically satisfied so that $\omega \in \mathbf{R}$ is unconstrained. This set of conditions yields the principal series of unitary representations of the Lorentz group.

(ii) $j_0 = 0$ implies that $A_j = 0$ and $B_j^2 = \frac{(j^2 - t^2)}{(4j^2 - 1)}$. Since we require B_j^2 to be real and positive for $j =$ 1, 2, 3,... we must have $1 - t^2 \geq 0$ or $-1 \leq t \leq +1$. These constitute the complementary series.

2.2 FINITE DIMENSIONAL REPRESENTATIONS OF THE LORENTZ GROUP (GEL' FAND, 1963, TUNG, 1984)

The results of the preceding section carry over in entirety to determine the finite dimensional representations of the Lorentz group upto eq. (50) since we have nowhere used the prescription of unitarity upto that stage (eq. (50)). To obtain the finite dimensional representations, we can, therefore, assume that results upto eq. (50) are already established and shall proceed therefrom. The local isomorphism of the proper Lorentz group to the direct product group $SU(2)_M \otimes SU(2)_N$ restricts the permissible values of j of the canonical basis $\{|jm|\}$ to $|u - v| \leq j \leq u + v$. We now determine the values of j_0 and $t \equiv j_1$ for the case of finite dimensional representations. As in the foregoing paragraphs j_0 is defined by $j_0 = |u - v|$ and it is the lowest value for j (which is bounded from below). Let us further assume that $j_{\max} = j_m = u + v$.

Now, from eqs. (35) and (36), we see that for this maximum value of $j \equiv j_m$, A_j^+ must necessarily vanish, i.e., $A_{jm}^+ = 0$. Using $B_j^2 = -A_j^- A_{j-1}^+$, this implies that $B_{j_m+1}^2 = -A_{j_m+1}^- A_{jm}^+ = 0$ whence from eq. (49), $(j_m + 1)^2 = t^2$ or $t = \pm (j_m + 1)$. Using $\hat{M}_m = \frac{1}{2}(\hat{J}_m + i\hat{K}_m)$ and $\hat{N}_m = \frac{1}{2}(\hat{J}_m + i\hat{K}_m)$ with $m, n = 1, 2, 3$, we may write

$$\hat{M}^2 = \frac{1}{4}[\hat{J}_3^2 - \hat{K}_3^2 + 2i\hat{J}_3\hat{K}_3 + 2(\hat{J}_3 + i\hat{K}_3) + \hat{J}_-\hat{J}_+ - \hat{K}_-\hat{K}_+ + i(\hat{K}_-\hat{J}_+ + \hat{J}_-\hat{K}_+)] \tag{53}$$

$$\hat{N}^2 = \frac{1}{4}[\hat{J}_3^2 - \hat{K}_3^2 - 2i\hat{J}_3\hat{K}_3 + 2(\hat{J}_3 - i\hat{K}_3) + \hat{J}_-\hat{J}_+ - \hat{K}_-\hat{K}_+ - i(\hat{K}_-\hat{J}_+ + \hat{J}_-\hat{K}_+)] \tag{54}$$

Since these operators are Casimir operators, they commute with all the group generators and, therefore, by Schur's Lemma, get mapped into a constant multiple of the identity operator. We proceed to determine their eigenvalues from the above expansion by operating on our canonical basis vectors. It is emphasized here that the choice of the vector is irrelevant since the eigenvalues of Casimir operators remain unchanged for the basis vectors in a given IRR. Operating $\hat{M}^2$ and $\hat{N}^2$ on $|j_m\, j_m\rangle$ and using eqs. (53) and (54); (35) and (36), we have

$$\hat{M}^2|j_m j_m\rangle = \frac{1}{4}[\hat{J}_m^2 - j_m^2 A_{jm}^2 + 2ij_m^2 A_{jm} + 2j_m(1 + iA_{jm})]|j_m j_m\rangle \tag{55}$$

$$\hat{N}^2|j_m j_m\rangle = \frac{1}{4}[\hat{J}_m^2 - j_m^2 A_{jm}^2 - 2ij_m^2 A_{jm} + 2j_m(1 - iA_{jm})]|j_m j_m\rangle \tag{56}$$

$\xi = \tanh^{-1}(v/c)$, $-\infty \leq \xi \leq +\infty$. In fact, Lorentz boosts can be construed as spatiotemporal rotations that mix the spatial and time components of the four-dimensional spacetime by hyperbolic (unbounded) angles.

The non-compact nature of the proper Lorentz group may also be attributed, equivalently, to the non-positive definite nature of the Minkowski metric, which shows up again as the negative sign in the RHS of eq. (4).

As mentioned earlier, the Lie algebra of the Lorentz group consisting of eqs. (5–7) is identical to that of $SU(2)_M \otimes SU(2)_N$ and there exists a local isomorphism between the two so that every representation of the latter provides a finite dimensional representation of the former. The important point, however, is that whereas $SU(2)$ and hence, the direct product $SU(2)_M \otimes SU(2)_N$ is compact, the proper Lorentz group is not. The reason is that while $SU(2)_M \otimes SU(2)_N$ has $\hat{M}$, $\hat{N}$ as the generators and, therefore, group elements can be obtained by exponentiation of $i\hat{M}$, $i\hat{N}$ with a real parameter, the proper Lorentz group has $\hat{J}$, $\hat{K}$ as the generators so that group elements are obtained by exponentiation of $i\hat{J}$, $i\hat{K}$. Now, because of the interelationship $\hat{M}_m = \frac{1}{2}(\hat{J}_m + i\hat{K}_m)$ and $\hat{N}_m = \frac{1}{2}(\hat{J}_m + i\hat{K}_m)$, the two sets of generators $\hat{M}$, $\hat{N}$ and $\hat{J}$, $\hat{K}$ cannot be Hermitian simultaneously. Hence, the representations generated by both these sets cannot simultaneously be unitary.

An important aspect of representations of the Lorentz group is the physical relevance of different classes of representations. Although the non-unitary representations cannot be realized as physical particle states (because all symmetry operations must be represented by unitary operators on the space of physical states), physical variables like position, momentum, angular momentum, energy momentum tensor, particle wavefunctions and fields all do transform as finite dimensional representations. However, physical particle states themselves transform as the unitary (infinite dimensional) representations.

2.3 UNITARY REPRESENTATIONS OF THE POINCARE GROUP (WIGNER, 1939; GEL'FAND, 1963; NAIMARK, 1964; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994)

Since the Poincare group is non-compact, all its unitary representations are infinite dimensional. They may conveniently be derived by the method of induced representation. Before applying this method to the actual construction of the IRRs of the Poincare group, we introduce the steps involved in the procedure and illustrate them with the example of the two-and three-dimensional Euclidean groups.

(a) We define the two-dimensional Euclidean group E_2 as consisting of all continuous linear transformations on the two-dimensional Euclidean space $\mathbf{R}_2$ that leave the length of the 2-vectors invariant. Representing an arbitrary vector $\mathbf{x}$ in $\mathbf{R}_2$ in terms of its components (x^1, x^2), we can write a general linear transformation of this vector as $x'^i = R^i_j x^j + a^i$, $i, j = 1, 2$. The condition of "length preservation", then, mandates that the matrix R should be an orthogonal matrix for, writing the transformation equation in matrix form as $\mathbf{x}' = R.\mathbf{x} + A$, where $A \equiv (a^1\ a^2)^T$ whence $|\mathbf{x}|^2 = \mathbf{x}^T\mathbf{x} = |\mathbf{x}' - A|^2 = (R.\mathbf{x})^T.(R.\mathbf{x}) = \mathbf{x}^T R^T R.\mathbf{x}$ whence $R^T = R^{-1}$. Thus, the homogeneous part of the transformation corresponds to "rotations" in two-dimensional space while the inhomogeneous part corresponds to uniform translations of all points. Hence, to study the representations of E_2, we need to examine the nuances of the translational group and the two-dimensional rotation group.

(b) We first consider the representations of the one-dimensional translational group designated T_1. An arbitrary element of this group corresponds to translation by a distance a and is denoted by $\hat{T}(a)$. The action of $\hat{T}(a)$ on a state $|x\rangle$ localized at the point x is given by $\hat{T}(a)|x\rangle = |a + x\rangle$. The operator $\hat{T}(a)$, then possesses the properties

$$\text{(i) } \hat{T}(a)\,\hat{T}(b) = \hat{T}\,(a+b) \tag{65}$$

$$\text{(ii) } \hat{T}(0) = \hat{\mathbf{I}} \tag{66}$$

$$\text{(iii) } \left[\hat{T}(a)\right]^{-1} = \hat{T}(-a) \tag{67}$$

so that the elements $\hat{T}(a)$ form a one parameter group designated as above by T_1. If the infinitesimal generator of this one parameter group is designated by $\hat{P}$, then $\hat{T}(da) = \hat{\mathbf{I}} - i\hat{P}\,da$, whence $\hat{T}(a + da) = \hat{T}(a)\,\hat{T}(da) = \hat{T}(a)\,[\hat{\mathbf{I}} - i\hat{P}\,da]$. Expanding $\hat{T}(a + da)$ as a Taylor series around $\hat{T}(a)$ to first order, we obtain $\hat{T}(a+da) = \hat{T}(a) + \dfrac{d\hat{T}(a)}{da}da$, whence $\dfrac{d\hat{T}(a)}{da} = -i\hat{P}\,\hat{T}(a)$ which has the solution $\hat{T}(a) = e^{-i\hat{P}a}$ in view of the boundary condition $\hat{T}(0) = \hat{\mathbf{I}}$. All IRRs of T_1 are one dimensional. For unitary representations, the generator $\hat{P}$ must be a Hermitian operator whose action on a state vector $|p\rangle$ is defined by $\hat{P}|p\rangle = p|p\rangle$ where p is the eigenvalue of the operator $\hat{P}$ corresponding to the eigenstate $|p\rangle$. If $\hat{T}(a) \to \hat{T}^p(a) \equiv e^{-i\hat{P}a}$ then $\hat{T}^p(a)|p\rangle \equiv e^{-i\hat{P}a}|p\rangle = e^{-ipa}\,|p\rangle$ with p being unrestricted.

(c) We now discuss the irreducible representations of the two-dimensional rotation group. For this, we consider a system that is symmetric under rotations in a plane around a fixed point. Let e_1, e_2 be a Cartesian orthonormal basis in the two-dimensional space so that an arbitrary vector can be represented as $x^i e_i \equiv x^1 e_1 + x^2 e_2$. Let this vector be rotated through an angle θ in the counter-clockwise direction. Then, if the transformed vector is given by $x'^i e_i \equiv x'^1 e_1 + x'^2 e_2$, by elementary geometry, we have $x'^1 = x^1 \cos\theta - x^2 \sin\theta$, $x'^2 = x^1 \sin\theta + x^2 \cos\theta$ or, in matrix notation $\mathbf{x}' = R(\theta)\mathbf{x}$ wherein $\mathbf{x} = (x^1\ x^2)^T$, $\mathbf{x}' = (x'^1\ x'^2)^T$ and $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The transformation rule for components, therefore, is $x'^j = R^j_i\, x^i$. Since rotation leaves the length of the vector invariant, the rotation matrix must be orthogonal, e.g., $|\mathbf{x}|^2 = \mathbf{x}^T\mathbf{x} = |\mathbf{x}'|^2 = (R.\mathbf{x})^T.(R.\mathbf{x}) = \mathbf{x}^T R^T R.\mathbf{x}$, whence $R^T = R^{-1}$. This can also be explicitly verified for the above general form of the two-dimensional rotation matrix. Furthermore, $R^T = R^{-1}$ implies that $(\det R)^2 = \det R \det R^T = \det R \det R^{-1} = \det RR^{-1} = \det \mathbf{I} = 1$, whence $\det R = \pm 1$. Orthogonal matrices satisfying the condition $\det R = +1$ represent rotations in two-dimensional space. They constitute a group by themselves, called the special orthogonal group designated $SO(2)$.

Orthogonal matrices with $\det R = -1$ correspond to rotations coupled with spatial inversions. This set of matrices is not connected to the identity transformation by a continuous change of parameters.

integer. Denoting these integral values of j by m we label the corresponding representation by this integer m and write.

$$\hat{J}\,|m\rangle = m|m\rangle \tag{74}$$

$$\hat{U}^{m}(\phi)|m\rangle = e^{-im\phi}|m\rangle \tag{75}$$

The defining equation for $R(\theta)$, viz. $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a two-dimensional representation of the group $SO(2)$. It must, therefore, be reducible. It is, actually, equivalent to a direct sum of the $m = \pm 1$ representations. This can be established by diagonalizing the matrix corresponding to the operator $\hat{J}$, i.e., $\hat{J} = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}$ which can be easily done. The eigenvalues of $\hat{J}$ are found to be ± 1 with the corresponding normalized eigenvectors in the orthonormal Cartesian basis $\{e_1, e_2\}$ being $e_\pm = \frac{1}{\sqrt{2}}\,(\mp\, e_1 - ie_2)$. With respect to the new eigenbasis we, have $\hat{J}\,e_\pm = \pm\, e_\pm$ and $R(\phi)\, e_\pm = e^{\mp i\phi}\, e_\pm$.

(d) We, now, revert our attention to the two-dimensional Euclidean group E_2 that consists of translations and rotations in two-dimensional space. A general transformation in this space can, therefore, be written in terms of the components of the state vectors as $x'^1 = x^1\cos\theta - x^2\sin\theta + b^1$, $x'^2 = x^1\sin\theta + x^2\cos\theta + b^2$. We denote this transformation by $g(b, \theta)$. Using the above explicit representation of an arbitrary transformation in E_2, we can write the group composition law as $g(b_2, \theta_2)\, g(b_1, \theta_1) = g(b_3, \theta_3)$, where $\theta_3 = \theta_1 + \theta_2$ and $b_3 = R(\theta_2)b_1 + b_2$. This also enables determination of the group inverse. e.g., $g(b', \theta')\, g(b, \theta) = g(0, 0)$, whence $\theta' = -\theta$ and $b' = -R(-\theta)b$ giving $g^{-1}(b, \theta) = g(-R(-\theta)\, b, -\theta)$. The subset of E_2 consisting of elements $\{g(0, \theta) = R(\theta)\}$ constitutes the subgroup of rotations in two-dimensional space $SO(2)$ with the representation $\hat{R}(\theta) = e^{-i\hat{J}\theta}$. On the other hand, the subgroup $\{g(b,0) = T(b)\}$ constitutes the subgroup of translations T_2 in two dimensions with two independent one parameter subgroups generated by $\hat{P}_1$and $\hat{P}_2$ respectively. Since the translations in x^1 and x^2 directions are independent of each other, the corresponding generators $\hat{P}_1$ and $\hat{P}_2$ commute and we can write a general translation as $\hat{T}(b) = e^{-i\hat{\mathbf{p}}.\mathbf{b}} = e^{-i\hat{P}_1 b^1}\, e^{-i\hat{P}_2 b^2}$. We can write the general group element of E_2, $g(b, \theta)$ as $g(b, \theta) = T(b)\, R(\theta)$ for, the right-hand side is $T(b)\, R(\theta) = g(b, 0)\, g(0, \theta) = g(b, \theta)$.

The general group element of E_2, $g(b, \theta)$ can be conveniently represented by 3×3 matrices as $g(b,\ \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & b^1 \\ \sin\theta & \cos\theta & b^2 \\ 0 & 0 & 1 \end{pmatrix}$ with the generators of rotations being represented by $\hat{J} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and the two translation generators in the x^1 and x^2 directions by $\hat{P}_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\hat{P}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$ respectively.

Using the explicit representations of $\hat{J}$, $\hat{P}_1$ $\hat{P}_2$ given above, we can easily verify that $[\hat{P}_1\ \hat{P}_2] = 0$, $[\hat{J}\ \hat{P}_k] = i\varepsilon^{km}\ \hat{P}_m$, where e^{km} is the two dimensional unit antisymmetric tensor with $e^{kk} = 0$, $e^{km} = 1$, k, m cyclic. Using the Campbell Baker Hausdorff formula, viz. $e^A B e^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}[A, B]_n$, where $[A, B]_n = [A, [A, B]_{n-1}]$ and $[A, B]_0 = 1$, we have $e^{-i\hat{J}\theta}\ \hat{P}_k\, e^{i\hat{J}\theta} =$

$\sum_{n=0}^{\infty}\frac{1}{n!}[-i\hat{J}\theta,\ \hat{P}_k]_n = \sum_{n=0}^{\infty}\frac{(-i\theta)^n}{n!}\ [\hat{J},\ \hat{P}_k]_n$. Noting that $[\hat{J}\ \hat{P}_1] = i\hat{P}_2$, $[\hat{J}\ \hat{P}_2] = -\,i\hat{P}_1$ so that $[\hat{J}, [\hat{J}\ \hat{P}_1]] = i[\hat{J},\ \hat{P}_2] = -\,i^2\ \hat{P}_1$ etc., we have

$$e^{-i\hat{J}\theta}\ \hat{P}_1 e^{i\hat{J}\theta} = \left[\mathbf{I} - \frac{i^2}{2!}(-i\theta)^2 + \frac{i^4}{4!}(-i\theta)^4 - \ldots\right]\hat{P}_1 + \left[\frac{i}{1!}(-i\theta) - \frac{i^3}{3!}(-i\theta)^3 + \frac{i^5}{5!}(-i\theta)^5 - \ldots\right]\hat{P}_2$$

$$= \left[\mathbf{I} - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \ldots\right]\hat{P}_1 + \left[\frac{1}{1!}\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \ldots\right]\hat{P}_2$$

$$= \cos\theta\ \hat{P}_1 + \sin\theta\ \hat{P}_2 \tag{76}$$

with a similar expression for $\hat{P}_2$ so that we may write as a general expression $e^{-i\hat{J}\theta}\ \hat{P}_k\, e^{i\hat{J}\theta} = \hat{P}_m R(\theta)^m_k$. Proceeding similarly, we also obtain

$$e^{-i\hat{J}\theta}\ \hat{\mathbf{P}}.\mathbf{b}\ e^{i\hat{J}\theta} = e^{-i\hat{J}\theta}\ \hat{P}_1 b^1\ e^{i\hat{J}\theta} + e^{-i\hat{J}\theta}\ \hat{P}_2 b^2\ e^{i\hat{J}\theta}$$

$$= \hat{P}_1 R(\theta)^1_1\ b^1 + \hat{P}_2 R(\theta)^2_1\ b^1 + \hat{P}_1 R(\theta)^1_2\ b^2 + \hat{P}_2 R(\theta)^2_2\ b^2$$

$$= \hat{P}_1\,[R(\theta)^1_1\ b^1 + R(\theta)^1_2\ b^2] + \hat{P}_2\,[R(\theta)^2_1\ b^1 + R(\theta)^2_2\ b^2]$$

$$= \hat{P}_1\, b'^1 + \hat{P}_2\, b'^2 = \hat{\mathbf{P}}.\mathbf{b}' = \hat{P}_m R(\theta)^m_n\ b^n \tag{77}$$

where $b'^m = R(\theta)^m{}_1\, b^1 + R(\theta)^m{}_2\, b^2 = R(\theta)^m_k\ b^k$ so that $\mathbf{b}' = R(\theta).\mathbf{b}$ which leads us to

$$e^{-i\hat{J}\theta}\ \hat{T}(\mathbf{b})\ e^{i\hat{J}\theta} = e^{-i\hat{J}\theta} e^{-i\hat{\mathbf{P}}.\mathbf{b}}\ e^{i\hat{J}\theta} = e^{-i\hat{J}\theta}\left(\sum_{n=0}^{\infty}\frac{1}{n!}(-i\,\hat{\mathbf{P}}.\mathbf{b})^n\right)e^{+i\hat{J}\theta}$$

$$= e^{-i\hat{\mathbf{P}}\,R(\theta)\mathbf{b}} = \hat{T}\,[R(\theta)\mathbf{b}] \tag{78}$$

This leads us to another approach to the composition rule for this group as

$$g(a, \theta)\ g(b, \phi) = T(a)\ R(\theta)\ T(b)\ R(\phi) = T(a)\ [R(\theta)T(b)\ R(-\theta)]\ R(\theta)\ R(\phi)$$

$$= T(a)\ T[R(\theta)\,b]\ R(\theta + \phi) = T[R(\theta)b + a]\ R(\theta + \phi)$$

$$= g[R(\theta)b + a,\ \theta + \phi] \tag{79}$$

$$= R(\theta)\ \hat{P}_1 R(\theta)^k_1|\mathbf{p}_0\rangle + R(\theta)\hat{P}_2\, R(\theta)^k_2|\mathbf{p}_0\rangle$$
$$= pR(\theta)^k_{\ 1}\ R(\theta)|\mathbf{p}_0\rangle = p'^k\ R(\theta)|\mathbf{p}_0\rangle \tag{80}$$

where, in the second step, we have used $e^{-i\hat{J}\theta}\ \hat{P}_k\ e^{i\hat{J}\theta} = \hat{P}_m\ R(\theta)^m_k$, in the third step, the orthogonality of the rotation matrix $R(-\theta)^i_j = R(\theta)^j_i$ and, in the final step $\hat{P}_l|p\rangle,= p|\mathbf{p}_0\rangle\ \hat{P}_2|\mathbf{p}_0\rangle = 0$. The above equation implies that corresponding to the eigenvector $|\mathbf{p}_0\rangle$ with eigenvalue p, the operator $\hat{P}_k$, on rotation, generates the eigenvector $R(\theta)|\mathbf{p}_0\rangle$ with eigenvalue $p'^k = pR(\theta)^k_1$. Further

$$\hat{\mathbf{P}}.\mathbf{b}\ R(\theta)|\mathbf{p}_0\rangle = R(\theta)\ [R(\theta)^{-1}\ \hat{\mathbf{P}}.\mathbf{b}\ R(\theta)]|\mathbf{p}_0\rangle$$
$$= R(\theta)\ [R(\theta)^{-1}\ (\hat{P}_1\, b^1 + \hat{P}_2\, b^2)\ R(\theta)]|\mathbf{p}_0\rangle$$
$$= R(\theta)\ \hat{P}_l\, R(-\theta)^l_k\ b^k|\mathbf{p}_0\rangle = pR(\theta)\ R(-\theta)^l_k\ b^k|\mathbf{p}_0\rangle$$
$$= p''\ R(\theta)|\mathbf{p}_0\rangle \tag{81}$$

where $p'' = [pR(-\theta)^1_1\, b^1 + pR(-\theta)^1_2\ b^2]$ showing that $R(\theta)|\mathbf{p}_0\rangle$ is an eigenvector of $\hat{\mathbf{P}}$.

Let $|\mathbf{p}\rangle = R(\theta)|\mathbf{p}_0\rangle$. Since $R(\theta) = e^{-i\hat{J}\theta}$ is unitary, the normalization of both $|\mathbf{p}\rangle$ and $|\mathbf{p}_0\rangle$ will be the same.

The set of vectors $\{|\mathbf{p}\rangle\}$ constitutes a basis of an irreducible vector space that is invariant under E_2 for

(i) $\{|\mathbf{p}\rangle\}$ is closed under all group operations in E_2, viz.

$$\hat{T}(b)|\mathbf{p}\rangle = e^{-i\hat{\mathbf{P}}b}|\mathbf{p}\rangle = e^{-i\mathbf{p}b}|\mathbf{p}\rangle = e^{-ip''}|\mathbf{p}\rangle \tag{82}$$

since $|\mathbf{p}\rangle$ is an eigenstate of $\hat{\mathbf{P}}b$ with eigenvalue p''.

$$R(\phi)|\mathbf{p}\rangle = R(\phi)\ R(\theta)|\mathbf{p}_0\rangle = R(\phi + \theta)|\mathbf{p}_0\rangle = |\mathbf{p}'\rangle \tag{83}$$

(ii) If $\mathbf{p} \neq \mathbf{p}'$, then the corresponding vectors $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ are orthogonal being eigenvectors of a Hermitian operator $\hat{\mathbf{P}}$ corresponding to different eigenvalues so that $\langle\mathbf{p}'|\mathbf{p}\rangle = 0$.

We finally examine the normalization of the basis vectors. For $\mathbf{p} \neq \mathbf{p}'$ we know that $\langle\mathbf{p}'|\mathbf{p}\rangle = 0$. For $\mathbf{p} = \mathbf{p}'$, we proceed by noting that $\hat{\mathbf{P}}^2$ commutes with all the three generators of the group for

$$[\hat{J},\ \hat{P}^2_k] = [\hat{J},\ \hat{P}_k]\ \hat{P}_k + \hat{P}_k\ [\hat{J},\ \hat{P}_k] = i\varepsilon^{kl}\ \hat{P}_l\ \hat{P}_k + i\varepsilon^{kl}\hat{P}_k\ \hat{P}_l \tag{84}$$

which gives

$$[\hat{J},\ \hat{\mathbf{P}}^2] = \sum_{k=1}^{2}\left[\hat{J}, \hat{P}^2_k\right] = i\sum_{k=1}^{2}\left(\varepsilon^{kl}\ \hat{P}_l\ \hat{P}_k + \varepsilon^{kl}\ \hat{P}_k\ \hat{P}_l\right) = i(\hat{P}_2\hat{P}_1 - \hat{P}_2\ \hat{P}_1) = 0 \tag{85}$$

Thus, $\hat{\mathbf{P}}^2$ is a Casimir operator and, therefore, has a unique eigenvalue for each irreducible representation. Its eigenvalues p^2 are positive definite. Since these eigenvalues are invariant under all group operations, the labelling of irreducible representations can be done in terms of these eigenvalues. The eigenvectors (basis vectors) in a particular irreducible representation (that is identified by the eigenvalue of the Casimir operator $\hat{\mathbf{P}}^2$) are then labelled by the continuous label θ and we write $|\mathbf{p}\rangle \equiv |p, \theta\rangle$. Further, from $|\mathbf{p}\rangle = R(\theta)|\mathbf{p}_0\rangle$, we find that there exists a one-one correspondence between these basis vectors and the subgroup of rotations $\{R(\theta)\} \equiv SO(2)$. We, therefore, use the invariant measure $\frac{d\theta}{2\pi}$ of the subgroup of rotations for measuring these basis vectors and adopt the normalization scheme $\langle \mathbf{p}'|\mathbf{p}\rangle = \langle p, \theta'|p, \theta\rangle = 2\pi\delta(\theta' - \theta)$.

(e) As another illustration of the induced representation method, we derive the induced representations of the three dimensional Euclidean group E_3. The group operations include the subgroup consisting of translations $\{T_3: T(b)\}$ in the three spatial directions with infinitesimal generators $\hat{P}_1, \hat{P}_2, \hat{P}_3$ and rotations in three-dimensional space $\{SO(3): R(\alpha, \beta, \gamma)\}$ with corresponding generators $\hat{J}_1, \hat{J}_2, \hat{J}_3$. The group elements are generated, as usual, by exponentiation of the infinitesimal generators so that $\hat{T}(\mathbf{b}) = e^{-i\hat{\mathbf{P}}.\mathbf{b}} = e^{-i\hat{P}_i b^i}$ and $R(\alpha, \beta, \gamma) = e^{-i(\hat{J}_1\alpha + \hat{J}_2\beta + \hat{J}_3\gamma)}$. The Lie algebra consisting of the six generators $\hat{P}_i, \hat{P}_i$ satisfies the following:

$$[\hat{P}_i, \hat{P}_j] = 0 \tag{86}$$

$$[\hat{P}_k, \hat{J}_l] = i\varepsilon_{klm}\,\hat{P}^m \tag{87}$$

$$[\hat{J}_k, \hat{J}_l] = i\varepsilon_{klm}\,\hat{J}^m \tag{88}$$

where ε_{klm} is the three-dimensional totally antisymmetric Levi Civita tensor.

The two operators $\hat{\mathbf{P}}^2$ and $\hat{\mathbf{J}}.\hat{\mathbf{P}}$ commute with all the group elements and therefore, constitute the Casimir operators of the group E_3. This is shown as follows:

$$[\hat{\mathbf{P}}^2, \hat{P}_k] = \left[\sum_l \hat{P}_l^2, \hat{P}_k\right] = \sum_l \left[\hat{P}_l^2, \hat{P}_k\right] = \sum_l \{\hat{P}_l[\hat{P}_l, \hat{P}_k] + [\hat{P}_l, \hat{P}_k]\hat{P}_l\} = 0$$

$$[\hat{\mathbf{P}}^2, \hat{J}_l] = \left[\sum_k \hat{P}_k^2, \hat{J}_l\right] = \sum_k \left[\hat{P}_k^2, \hat{J}_l\right] = \sum_k \{\hat{P}_k[\hat{P}_k, \hat{J}_l] + [\hat{P}_k, \hat{J}_l]\hat{P}_k\}$$

$$= \mathrm{i}\sum_k (e_{klm}\,\hat{P}_k\,\hat{P}^m + \varepsilon_{klm}\,\hat{P}^m\,\hat{P}_k) = 0 \tag{89}$$

because e_{klm} is antisymmetric in the indices k, m. Similarly,

$$[\hat{\mathbf{J}}.\hat{\mathbf{P}}, \hat{P}_l] = \left[\sum_k \hat{J}_k\,\hat{P}_{k,}\,\hat{P}_l\right] = \sum_l \left[\hat{J}_k\,\hat{P}_k, \hat{P}_l\right] = \sum_k \{\hat{J}_k[\hat{P}_k, \hat{P}_l] + [\hat{J}_k, \hat{P}_l]\hat{P}_k\}$$

$$= \sum_k \{[\hat{J}_k, \hat{P}_l]\hat{P}_k\} = -i\sum_k \varepsilon_{lkm}\,\hat{P}^m\,\hat{P}_k = 0 \tag{90}$$

$$\hat{\mathbf{P}}|p, \lambda; \hat{\mathbf{P}}| = \mathbf{p}|p, \lambda; \hat{\mathbf{P}}\rangle \tag{96}$$

Since the factor group $\frac{E_3}{T_3} \cong SO(3)$, a trivial representation of E_3 can be directly induced from the representations of $SO(3)$ by the mapping $g = TR \to D^J(R)$ in which all pure translations are mapped into the identity element. In this representation all generators of translations are mapped into the null operator and both the Casimir operators $\hat{\mathbf{P}}^2$ and $\hat{\mathbf{J}}.\hat{\mathbf{P}}$ have zero eigenvalues.

Let us now consider the subspace of the given vector space that is characterized by a standard vector $\hat{\mathbf{P}} = \hat{\mathbf{P}}_0 = (0, 0, 1) = e_3$. Now, all the group elements in the factor group $\frac{E_3}{T_3} \cong SO(3)$, i.e., the group of rotations that leave the subspace corresponding to the standard vector $\hat{\mathbf{P}}_0$ invariant constitute a subgroup of the factor group $\frac{E_3}{T_3} \cong SO(3)$. This is proved as follows.

Let us assume that V_0 is the subspace corresponding to the standard vector $\hat{\mathbf{P}}_0$. Further, let G_0 be the set of the elements of the factor group $\frac{E_3}{T_3}$ that leave the space V_0 invariant. (We define V_{inv} to be an A invariant subspace of a given linear transformation $A: V \to V$ in a vector space V, if and only if $Av \in V_{inv}\ \forall\ v \in V_{inv}$, i.e., $AV_{inv} \subset V_{inv}$). Then, if $g \in G_0$ and $\hat{p} \in V_0$, we must have $g\,\hat{p} \in V_0$. This implies that $g_2\, g_1\, \hat{p} \in V_0$ for $g_1, g_2 \in G_0$ so that $g_2\, g_1 \in G_0$, whence G_0 is closed under the group composition and, therefore, a subgroup of $\frac{E_3}{T_3}$. This subgroup is called the little group of $\hat{\mathbf{P}}_0$.

Since $\hat{\mathbf{P}}_0$ is aligned along the x^3 axis, its value remains unchanged by rotations that are carried out about this (x^3) axis. These rotations are generated by $\hat{J}_3$ by exponentiation, i.e., $R_3(\phi) = e^{-i\hat{J}_3\phi}$. The little group of $\hat{\mathbf{P}}_0$, therefore, consists of rotations in the $x^1 - x^2$ plane and is isomorphic to the group $SO(2)$.

We know from the preceding analysis of the $SO(2)$ group that all its irreducible representations are one dimensional and can be labelled by a single index λ that is the eigenvalue of the generator $\hat{J}_3$. Furthermore, these eigenvalues, being eigenvalues of $\hat{J}_3$ can take all integral values as is well known from the standard theory of angular momentum. Since we have defined our standard vector $\hat{\mathbf{P}} = \hat{\mathbf{P}}_0 = (0, 0, 1) = e_3$, it follows that $\hat{P}_1|\mathbf{p}_0\rangle = \hat{P}_2|\mathbf{p}_0\rangle = 0$ and $\hat{P}_3|\mathbf{p}_0\rangle = \mathbf{p}_0|\mathbf{p}_0\rangle$ so that $\hat{\mathbf{P}}|\mathbf{p}_0\rangle = \mathbf{p}_0|\mathbf{p}_0\rangle$. Also, $\hat{\mathbf{J}}.\hat{\mathbf{P}}|\mathbf{p}_0\rangle = \mathbf{p}_0\hat{\mathbf{J}}|\mathbf{p}_0\rangle = (0, 0, p).\ (\hat{J}_1, \hat{J}_2, \hat{J}_3)|\mathbf{p}_0\rangle = p\hat{J}_3|\mathbf{p}_0\rangle = p\lambda|\mathbf{p}_0\rangle$ where $p = |\,\mathbf{p}_0\,|$.

The basis vectors of the subspace corresponding to the standard vector $\hat{\mathbf{p}}_0$, under transformations of its little group (rotations in two-dimensional space, viz. $SO(2)$) will satisfy

$$R_3(\phi)|p, \lambda; \hat{\mathbf{P}}_0\rangle = e^{-i\hat{J}_3\phi}|p, \lambda; \hat{\mathbf{P}}_0\rangle = e^{-i\lambda\phi}|p, \lambda; \hat{\mathbf{P}}_0\rangle \tag{97}$$

because $\hat{J}_3|\mathbf{P}_0\rangle = \lambda|\mathbf{p}_0\rangle$. Similarly, under translations, we have

$$\hat{T}(\mathbf{b})|p, \lambda; \mathbf{p}_0\rangle = e^{-i\hat{\mathbf{P}}.\mathbf{b}}|p, \lambda; \mathbf{p}_0\rangle = e^{-i\mathbf{p}_0\mathbf{b}}|p, \lambda; \mathbf{p}_0\rangle \tag{98}$$

which follows from $\hat{\mathbf{P}}|\mathbf{p}_0\rangle = \mathbf{p}_0|\mathbf{p}_0\rangle$.

Since the group has two Casimir operators $\hat{\mathbf{P}}^2$ and $\hat{\mathbf{J}}.\hat{\mathbf{P}}$, the labelling of the IRRs of E_3 has to be done using two indices. We use p, λ for this purpose. Since p, λ correspond, respectively, to the Casimir operators $\hat{\mathbf{P}}^2$ and $\hat{\mathbf{J}}.\hat{\mathbf{P}}$ of E_3 their values are invariant for a particular IRR of E_3 which enables labelling of the IRRs using these two indices. Within an IRR, the vectors are identified by the unit vector index $\hat{\mathbf{P}}$.

To construct the full vector space of a particular IRR of E_3 that is labelled by p, λ, we generate the new basis vectors from $|p, \lambda; \mathbf{p}_0\rangle$ by operating thereon by rotations that do not belong to the little group of $\hat{\mathbf{p}}_0$. We have

$$|p, \lambda; \hat{\mathbf{p}}\rangle = R(\omega, \theta, 0)|\, p, \lambda; \hat{\mathbf{p}}_0\rangle \tag{99}$$

where $\hat{\mathbf{p}} = R(\omega, \theta, 0)\hat{\mathbf{p}}_0$. The remaining labels p, λ will remain unchanged by these rotations as such rotations do not result in a new IRR but generate the remaining basis vectors of the same IRR.

It remains to establish that the basis vectors generated by $|p, \lambda; \hat{\mathbf{p}}\rangle = R(\omega, \theta, 0)|p, \lambda; \hat{\mathbf{p}}_0\rangle$ (where in $\hat{\mathbf{p}} = (\omega, \theta)$ in polar coordinates) have the necessary group composition properties of E_3. For this, we have $\hat{T}(\mathbf{b})|p, \lambda; \hat{\mathbf{p}}\rangle = e^{-i\hat{\mathbf{p}}.\mathbf{b}}|p, \lambda; \hat{\mathbf{p}}\rangle = e^{-ipb}|p, \lambda; \hat{\mathbf{p}}\rangle$. Further,

$$\begin{aligned} R(\alpha, \beta, \gamma)|p, \lambda; \hat{\mathbf{p}}\rangle &= R(\alpha, \beta, \gamma)\, R(\omega, \theta, 0)|p, \lambda; \hat{\mathbf{p}}_0\rangle \\ &= R(\omega', \theta', 0)\, [R^{-1}(\omega', \theta', 0)\, R(\alpha, \beta, \gamma)\, R(\omega, \theta, 0)]|p, \lambda; \hat{\mathbf{p}}_0\rangle \\ &= R(\omega', \theta', 0)\, R(0, 0, \psi)|p, \lambda; \hat{\mathbf{p}}_0\rangle \\ &= R(\omega', \theta', 0)\, e^{-i\lambda\psi}|p, \lambda; \hat{\mathbf{p}}_0\rangle = e^{-i\lambda\psi}|p, \lambda; \hat{\mathbf{p}}'\rangle \end{aligned} \tag{100}$$

where $\hat{\mathbf{p}}' = (\omega', \theta')$ polar coordinates.

The fact that a ψ exists satisfying $R^{-1}(\omega', \theta', 0)\, R(\alpha, \beta, \gamma)\, R(\omega, \theta, 0) = R(0, 0, \psi)$ follows from the fact that the three rotations $R(\omega', \theta', 0)^{-1}\, R(\alpha, \beta, \gamma)\, R(\omega, \theta, 0)$ when acting on the state $|p, \lambda; \hat{\mathbf{p}}_0\rangle$ produce

$$\begin{aligned} R(\omega', \theta, 0)^{-1}\, R(\alpha, \beta, \gamma)\, R(\omega, \theta, 0)|p, \lambda; \hat{\mathbf{p}}_0\rangle & \\ &= R(\omega', \theta', 0)^{-1}\, R(\alpha, \beta, \gamma)|p, \lambda; \hat{\mathbf{p}}\rangle \quad \hat{\mathbf{p}} = (\omega, \theta) \\ &= R(\omega', \theta', 0)^{-1}\, R_3(\alpha)\, R_2(\beta)\, R_3\,(\gamma)|p, \lambda; \hat{\mathbf{p}}\rangle \\ &= R(\omega', \theta'\, 0)^{-1}|p, \lambda; \hat{\mathbf{p}}'\rangle\, e^{-i\lambda\psi} = |p, \lambda; \hat{\mathbf{p}}_0\rangle\, e^{-i\lambda\psi} \end{aligned} \tag{101}$$

(since $R(\alpha, \beta, \gamma)|p, \lambda; \hat{\mathbf{p}}'\rangle = |p, \lambda; \hat{\mathbf{p}}'\rangle\, e^{-i\lambda\psi}$ with $\hat{\mathbf{p}}' = (\omega', \theta')$ so that $R(\omega', \theta', 0)^{-1}|p, \lambda; \hat{\mathbf{p}}'(= \omega', \theta')\rangle = R(\omega', \theta', 0)^{-1}\, [R(\omega', \theta', 0)|p, \lambda; \hat{\mathbf{p}}_0\rangle] = |p, \lambda; \hat{\mathbf{p}}_0\rangle$ thereby leaving the standard vector $\hat{\mathbf{p}}_0$ invariant. Hence, the product of the three rotations $R(\omega', \theta', 0)^{-1}\, R(\alpha, \beta, \gamma)\, R(\omega, \theta, 0)$ constitutes a rotation about the x^3 axis and can, therefore, be represented as $R(0, 0, \psi)$ and, is also a member of the little group of $\hat{\mathbf{p}}_0$.

construct scalars as products of the vector ($\hat{P}_\mu$) and tensor ($\hat{J}_{\mu\nu}$) quantities. In the context, we define vectors and scalars as:

(i) An operator $\hat{A}_\mu$, $\mu = 0, 1, 2, 3$ is classified as a vector if it transforms as $[\hat{J}_{\mu\nu}, \hat{A}_\lambda] = i(g_{\nu\lambda}\hat{A}_\mu - g_{\mu\lambda}\hat{A}_\nu)$;

(ii) An operator $\hat{A}_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ is classified as a tensor of second rank if it transforms as a product of vector components $\hat{A}_\mu \hat{A}_\nu$ under the action of the generators $\hat{J}_{\mu\nu}$. Similarly, higher rank tensors can be defined.

It is well known that we can obtain scalars from vectors and tensors by the operation of contraction (index convolution). In our case, we have to generate scalars from the two sets of generators $\{\hat{P}_\mu\}$and $\{\hat{J}_{\mu\nu}\}$. Let us define the vectors $\hat{W}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\hat{J}^{\nu\rho}\hat{P}^\sigma$ and $\hat{\Gamma}_\lambda = \hat{J}_{\lambda\mu}\hat{P}^\mu$. Then, we have $\hat{W}_\mu \hat{P}^\mu = 0$, $\hat{\Gamma}_\mu \hat{P}^\mu = 0$, $[\hat{P}_\mu, \hat{W}_\nu] = 0$, $[\hat{W}_\mu, \hat{W}_\nu] = i\varepsilon_{\mu\nu\rho\sigma}\hat{P}^\rho \hat{W}^\sigma$, $[\hat{W}_\mu, \hat{\Gamma}_\sigma] = -i\hat{P}_\mu \hat{W}_\sigma$ $[\hat{\Gamma}_\mu, \hat{P}_\nu] = i(\delta_{\mu\nu}\hat{P}_\lambda \hat{P}^\lambda - \hat{P}_\mu \hat{P}_\nu)$, $[\hat{\Gamma}_\mu, \hat{\Gamma}_\nu] = -i\hat{J}_{\mu\nu}\hat{P}_\lambda \hat{P}^\lambda$, $\hat{P}^\lambda \hat{J}_{\lambda\mu}\hat{J}^\mu_\sigma = \hat{P}_\sigma\left(6 - \frac{1}{2}\hat{j}^2\right) - \frac{1}{2}\varepsilon_{\sigma\mu\nu\rho}$ $\hat{J}^{\mu\nu}\hat{W}^\rho + 2i\hat{\Gamma}_\sigma$, $= \hat{J}^2 = \hat{J}_{\lambda\rho}\hat{J}^{\lambda\rho}$,

$$\hat{J}_{\mu\lambda}\hat{J}^{\lambda\sigma}\hat{J}_{\sigma\nu} = \frac{1}{2}(ig_{\mu\nu} - \hat{J}_{\mu\nu})\hat{J}^2 + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\hat{J}^{\rho\sigma}\left(\frac{1}{8}\varepsilon_{\alpha\beta\lambda\gamma}\hat{J}^{\alpha\beta}\hat{J}^{\lambda\gamma}\right) + 3i\hat{J}_{\mu\lambda}\hat{J}^\lambda_\nu$$

It is seen from the above that all scalars that can be constituted by the convolution of $\{\hat{P}_\mu\}$ and $\{\hat{J}_{\mu\nu}\}$ can be expressed as the linear combinations of the entities $\hat{J}^2$, $\hat{P}_\mu \hat{P}^\mu$ $\hat{W}_\mu \hat{W}^\mu$, $\hat{\Gamma}_\lambda \hat{\Gamma}^\lambda$, $\hat{W}_\lambda \hat{\Gamma}^\lambda$ and $\varepsilon_{\alpha\beta\lambda\gamma}\hat{J}^{\alpha\beta}\hat{J}^{\lambda\gamma}$. These six scalars exhaust the set of all independent scalars that can be constructed from $\{\hat{P}_\mu\}$ and $\{\hat{J}_{\mu\nu}\}$. Furthermore, out of these six scalars only $\hat{P}_\mu \hat{P}^\mu$, $\hat{W}_\mu \hat{W}^\mu$ commute with all the generators of the Poincare algebra and hence, constitute the two main Casimir operators of the group. This is established below:

$$[\hat{P}^\mu \hat{P}_\mu, \hat{P}_\lambda] = \hat{P}^\mu [\hat{P}_\mu \hat{P}_\lambda] + [\hat{P}^\mu \hat{P}_\lambda]\hat{P}^\mu = 0 \tag{108}$$

$$\begin{aligned}
[\hat{P}_0^2 - \hat{\mathbf{P}}^2, \hat{J}_n] &= [\hat{P}_0^2, \hat{J}_n] - [\hat{\mathbf{P}}^2, \hat{J}_n] \\
&= \hat{P}_0[\hat{P}_0, \hat{J}_n] + [\hat{P}_0, \hat{J}_n]\hat{P}_0 - [\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2, \hat{J}_n] \\
&= -(\hat{P}_1[\hat{P}_1, \hat{J}_n] + [\hat{P}_1, \hat{J}_n]\hat{P}_1) - (\hat{P}_2[\hat{P}_2, \hat{J}_n] + [\hat{P}_2, \hat{J}_n]\hat{P}_2) \\
&\quad - (\hat{P}_3[\hat{P}_3, \hat{J}_n] + [\hat{P}_3, \hat{J}_n]\hat{P}_3) = 0
\end{aligned} \tag{109}$$

$$\begin{aligned}
[\hat{P}_0^2 - \hat{\mathbf{P}}^2, \hat{K}_n] &= [\hat{P}_0^2, \hat{K}_n] - [\hat{\mathbf{P}}^2, \hat{K}_n] \\
&= \hat{P}_0[\hat{P}_0, \hat{K}_n] + [\hat{P}_0, \hat{K}_n]\hat{P}_0 - [\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2, \hat{K}_n] \\
&= -i\hat{P}_0\hat{P}_n - i\hat{P}_n\hat{P}_0 - (\hat{P}_1[\hat{P}_1, \hat{K}_n] + [\hat{P}_1, \hat{K}_n]\hat{P}_1)
\end{aligned}$$

$$- (\hat{P}_2[\hat{P}_2, \hat{K}_n] + [\hat{P}_2, \hat{K}_n]\hat{P}_2) - (\hat{P}_3[\hat{P}_3, \hat{K}_n] + [\hat{P}_3, \hat{K}_n]\hat{P}_3)$$

$$= -i\hat{P}_0\,\hat{P}_n - i\hat{P}_n\,\hat{P}_0 + i\delta_{1n}\,(\hat{P}_1\hat{P}_0 + \hat{P}_0\hat{P}_1)$$

$$+ i\delta_{2n}\,(\hat{P}_2\hat{P}_0 + \hat{P}_0\hat{P}_2) + i\delta_{3n}\,(\hat{P}_3\hat{P}_0 + \hat{P}_0\hat{P}_3) = 0 \tag{110}$$

The commutativity of $\hat{C}_1 = \hat{P}_\mu\,\hat{P}^\mu$ with the generators of the Poincare group may also be directly inferred from the fact that:

(i) Being the scalar product of a 4-vector with itself, it is invariant under homogeneous Lorentz transformations and hence, commutes with the generators of such Lorentz transformations.

(ii) It is also invariant under translations because of the commutativity of the generators of translations, i.e., the abelian nature of the translations subgroup.

We shall now establish that $\hat{C}_2 = \hat{W}_\lambda\,\hat{W}^\lambda$ (called the Pauli Lubanski vector) constitutes the second Casimir operator of the Poincare group. To establish this, we obtain the following properties of this vector:

(i) $\hat{W}^\lambda\,\hat{P}_\lambda = 0$ (111)

(ii) $[\hat{W}^\lambda, \hat{P}^\mu] = 0$ (112)

(iii) $[\hat{W}^\lambda, \hat{J}^{\mu\nu}] = i(\hat{W}^\nu g^{\mu\lambda} - \hat{W}^\mu g^{\lambda\nu})$ (113)

(iv) $[\hat{W}^\lambda, \hat{W}^\sigma] = i\varepsilon^{\lambda\sigma\mu\nu}\,\hat{W}_\mu, \hat{P}_\nu$ (114)

Property (i) follows from the fact that in $\hat{W}^\lambda\,\hat{P}_\lambda = \frac{1}{2}\,\varepsilon^{\lambda\mu\nu\sigma}\,\hat{J}_{\mu\nu}\,\hat{P}_\sigma\,\hat{P}_\lambda$, the four dimensional Levi Civita tensor is antisymmetric in the summation indices λ, σ whereas the product of 4-momenta $\hat{P}_\sigma\,\hat{P}_\lambda$ is symmetric in these indices. This property implies the orthogonality of the vectors $\hat{W}$ and $\hat{P}$.

To prove (ii), we note that

$$[\hat{W}^\lambda, \hat{P}^\mu] = \frac{1}{2}[\varepsilon^{\lambda\alpha\beta\sigma}\,\hat{J}_{\alpha\beta}\,\hat{P}_\sigma\,\hat{P}^\mu] = \frac{1}{2}\,\varepsilon^{\lambda\alpha\beta\sigma}\,\{\hat{J}_{\alpha\beta}\,[\hat{P}_\sigma, \hat{P}^\mu] + [\hat{J}_{\alpha\beta}, \hat{P}^\mu]\hat{P}_\sigma\}$$

$$= \frac{1}{2}\varepsilon^{\lambda\alpha\beta\sigma}\,[\hat{J}_{\alpha\beta}, \hat{P}^\mu]\hat{P}_\sigma = \frac{i}{2}\varepsilon^{\lambda\alpha\beta\sigma}\,(\delta^\mu_\alpha\,\hat{P}_\beta - \delta^\mu_\beta\,\hat{P}_\alpha)\hat{P}_\sigma$$

$$= \frac{i}{2}\varepsilon^{\lambda\mu\beta\sigma}\,\hat{P}_\beta\,\hat{P}_\sigma - \frac{i}{2}\varepsilon^{\lambda\alpha\mu\sigma}\,\hat{P}_\alpha\,\hat{P}_\sigma = 0$$

where the last step follows from the antisymmetricity of the four-dimensional Levi Civita tensor in the summation indices whereas the product of 4-momenta is symmetric in these indices. This property of the Pauli Lubanski vector expresses the translation invariance of $\hat{W}$.

Property (iii) implies that the Pauli Lubanski vector $\hat{W}$ transforms as the components of a 4-vector under homogeneous Lorentz transformations. This may be proved by using the commutation relations

(ii) Further, $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is the scalar product of a 4-vector with itself. Hence, it is the square of the length of the 4-vector which makes it invariant under homogeneous Lorentz transformations.

We shall, now, obtain an explicit expression for the Pauli Lubanski vector in the representation space spanned by the basis which is the orthonormal set of eigenfunctions of the two Casimir operators $\hat{C}_1 \hat{C}_2$, the four components of the momentum operator $\hat{P}_\mu$ and $\hat{W}_0$. The basis vectors are, accordingly, labelled by the corresponding eigenvalues, so that

$$\hat{C}_{1(2)}|c, p, \omega\rangle = c_{1(2)}|c, p, \omega\rangle \tag{115}$$

$$\hat{P}_\mu|c, p, \omega\rangle = P_\mu|c, p, \omega\rangle \tag{116}$$

$$\hat{W}_0|c, p, \omega\rangle = \omega|\mathbf{p}||c, p, \omega\rangle \tag{117}$$

In this basis, the commutator $[\hat{W}^\lambda, \hat{W}^\sigma] = i\varepsilon^{\lambda\sigma\mu\nu}\, \hat{W}_\mu \hat{P}_\nu$ takes the form $[\hat{W}_a, \hat{W}_b] = i\varepsilon_{abc}(p_0 \hat{W}_c - \hat{W}_0 pc]$, $[\hat{W}_0, \hat{W}_a] = i\varepsilon_{abc} p_b \hat{W}_c$. These commutators determine a Lie algebra for every fixed value of p. In fact, as we shall see later, the character of this Lie algebra is determined by the nature of p, being timelike, null or spacelike. Let us introduce the transformation $\hat{R}_{\mu\nu}$ by the eqs. $\hat{W}_\mu \to \hat{W}'_\mu = \hat{R}_{\mu\nu} \hat{W}^\nu$ and $\hat{p}_\mu \to \hat{p}'_\mu = \hat{R}_{\mu\nu}\, \hat{p}^\nu$. The objective of introducing this transformation is to achieve maximal simplification of the commutators of the components of $\hat{W}_\lambda$. For the purpose we set $\hat{R}_{0a} = \hat{R}_{a0} = 0$, $\hat{R}_{00} = 1$ and $\hat{R}_{ab} = -R_{ab}$ where $R_{ab} = \hat{\mathbf{P}}.\hat{\mathbf{n}}\ \delta_{ab} - \varepsilon_{abc}\,\theta_c + \theta_a\,\theta_b\,(1 + \hat{\mathbf{P}}.\hat{\mathbf{n}})^{-1}$, $\theta_a = \varepsilon_{abc}\ \hat{p}_b \hat{n}_c$, $\hat{p}_a\ \dfrac{p_a}{|\mathbf{p}|}$ and $\mathbf{n} \equiv (n_1, n_2, n_3)$ is an arbitrary unit vector. Written explicitly,

$$R_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \hat{\mathbf{p}}.\hat{\mathbf{n}} + \dfrac{(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2)^2}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})} & \begin{matrix}-(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1) \\ + \dfrac{(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2)(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3)}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})}\end{matrix} & \begin{matrix}(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3) \\ + \dfrac{(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2)(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1)}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})}\end{matrix} \\ 0 & \begin{matrix}(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1) \\ + \dfrac{(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2)(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3)}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})}\end{matrix} & \hat{\mathbf{p}}.\hat{\mathbf{n}} + \dfrac{(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3)^2}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})} & \begin{matrix}-(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2) \\ + \dfrac{(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3)(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1)}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})}\end{matrix} \\ 0 & \begin{matrix}-(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3) \\ + \dfrac{(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2)(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1)}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})}\end{matrix} & \begin{matrix}(\hat{p}_2\hat{n}_3 - \hat{p}_3\hat{n}_2) \\ + \dfrac{(\hat{p}_3\hat{n}_1 - \hat{p}_1\hat{n}_3)(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1)}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})}\end{matrix} & \hat{\mathbf{p}}.\hat{\mathbf{n}} + \dfrac{(\hat{p}_1\hat{n}_2 - \hat{p}_2\hat{n}_1)^2}{(1+\hat{\mathbf{p}}.\hat{\mathbf{n}})} \end{pmatrix}$$

Under this transformation, the four vector p transforms as $p'_a = n_a|\mathbf{p}|$ (which confirms that the transformation is a spatial rotation for $\mathbf{p}'^2 = \sum_{a=1}^{3} p'^2_a = \sum_{a=1}^{3} n_a^2\, \mathbf{p}^2 = \mathbf{p}^2$) and the transformed components of the Pauli Lubanski vector obey the commutation relations $[\hat{W}'_a, \hat{W}'_b] = i\varepsilon_{abc}(p_0\,\hat{W}'_c - \hat{n}_c|\mathbf{p}|\hat{W}'_0)$ and $[\hat{W}'_0, \hat{W}'_a] = i|\mathbf{p}|\varepsilon_{abc}\,\hat{n}_b\,\hat{W}'_c$. Writing $\hat{W}'_0 = |\mathbf{p}|\hat{\Omega}_0$ and $\hat{W}'_a = \hat{n}_a\,\hat{\Omega}_0 p_0 + \hat{\Omega}_a$, we obtain

$$[\hat{\Omega}_0, \hat{\Omega}_a] = i\varepsilon_{abc}\,\hat{n}_b\hat{\Omega}_c,\ [\hat{\Omega}_a, \hat{\Omega}_b] = ip_\mu p^\mu\,\varepsilon_{abc}\,\hat{n}_c\hat{\Omega}_0 \tag{118}$$

These sets of commutators determine a Lie algebra whose structure constants depend on the nature of $c_1 \equiv p_\mu p^\mu$ and $\hat{\mathbf{n}}$. Inverting the above set of transformations, we obtain $\hat{W}_0 = \hat{W}'_0 = |\mathbf{p}|\hat{\Omega}_0$ and

$$\hat{W}_a = R^{-1}_{ab}\,\hat{W}'_b = \hat{\Omega}_a + \hat{p}_a\,\hat{\Omega}_a p_0 - \frac{(\hat{p}_a + \hat{n}_a)\hat{\Omega}_b\,\hat{p}_b}{1+\hat{\mathbf{n}}.\hat{\mathbf{p}}} \tag{119}$$

Now, since the scalar product of three vectors is preserved under spatial rotations, we have $\hat{\mathbf{W}}.\hat{\mathbf{P}} = \hat{\mathbf{W}}'.\hat{\mathbf{P}}'$. whence

$$0 = \hat{W}_\mu\,\hat{P}^\mu = \hat{W}_0\,\hat{P}_0 - \hat{\mathbf{W}}.\hat{\mathbf{P}} = \hat{W}_0\,p_0 - \hat{\mathbf{W}}.\mathbf{p} = \hat{W}'_0 p'_0 - \hat{\mathbf{W}}'.\mathbf{p}'$$

$$= \hat{W}_0 p_0 - \hat{\mathbf{W}}'.\hat{\mathbf{n}}|\mathbf{p}| = |\mathbf{p}|\Omega_0\,p_0 - (\hat{\mathbf{n}}\Omega_0\,p_0 + \Omega)\hat{\mathbf{n}}|\mathbf{p}| = -\Omega\hat{\mathbf{n}} \tag{120}$$

This condition implies that this Lie algebra has only three linearly independent elements. The Casimir operators for this algebra are

$$\hat{I}_1 = \hat{\Omega}_0^2\,p_\mu p^\mu + \hat{\Omega}_2^1 + \hat{\Omega}_2^2 + \hat{\Omega}_3^2,\ \hat{I}_2 = e^{2i\pi\hat{\Omega}_0} \tag{121}$$

and their eigenvalues can be used for the labelling of the representations. The Casimir nature of $\hat{I}_1$ follows from

$$[\hat{\Omega}_0,\ \hat{I}_1] = \left[\hat{\Omega}_0, \hat{\Omega}_0^2 c_1 + \sum_{a=1}^{3}\hat{\Omega}_a^2\right] = \sum_{a=1}^{3}\left[\hat{\Omega}_0, \hat{\Omega}_a^2\right]$$

$$= \sum_{a=1}^{3}\left\{\hat{\Omega}_a\left[\hat{\Omega}_0, \hat{\Omega}_a\right] + \left[\hat{\Omega}_0, \hat{\Omega}_a\right]\hat{\Omega}_a\right\} = i\sum_{a=1}^{3}\left\{\varepsilon_{abc}\,\hat{\Omega}_a n_b\,\hat{\Omega}_c + \varepsilon_{abc}\,n_b\,\hat{\Omega}_c\,\hat{\Omega}_a\right\} = 0$$

and $$[\hat{\Omega}_j,\ \hat{I}_1] = \left[\hat{\Omega}_j, c_1\hat{\Omega}_0^2 + \sum_{a=1}^{3}\hat{\Omega}_a^2\right] = [\hat{\Omega}_j,\ c_1\hat{\Omega}_0^2] + \left[\hat{\Omega}_j, \sum_{a=1}^{3}\hat{\Omega}_a^2\right]$$

$$= \hat{\Omega}_0[\hat{\Omega}_j,\ c_1\hat{\Omega}_0] + [\hat{\Omega}_j,\ c_1\hat{\Omega}_0]\,\hat{\Omega}_0 + \sum_{a=1}^{3}\left\{\hat{\Omega}_a\left[\hat{\Omega}_j, \hat{\Omega}_a\right] + \left[\hat{\Omega}_j, \hat{\Omega}_a\right]\hat{\Omega}_a\right\}$$

$$= \sum_{k,l=1}^{3}\left\{-ic_1\,\varepsilon_{jkl}\,n_k\,\hat{\Omega}_0\,\hat{\Omega}_l - ic_1\,\varepsilon_{jkl}\,n_k\,\hat{\Omega}_l\,\hat{\Omega}_0\right\}$$

$$+ \sum_{a,m=1}^{3}\left\{ic_1\,\varepsilon_{jam}\,n_m\,\hat{\Omega}_a\,\hat{\Omega}_0 + ic_1\,\varepsilon_{jam}\,n_m\,\hat{\Omega}_0\,\hat{\Omega}_a\right\} = 0$$

thereby establishing the Casimir nature of $\hat{I}_1$. The Casimir operator $\hat{I}_2$ has eigenvalues equal to ± 1 with the positive sign corresponding to single valued representations and the negative sign to double valued ones.

To further explore the structure of this Lie algebra and simplify the commutators, we introduce another spatial rotation R' that aligns the unit vector $\hat{\mathbf{n}}$ with the x^3 axis so that $\hat{\mathbf{n}} \to \hat{\mathbf{n}}' \equiv (0, 0, 1) = R'\hat{\mathbf{n}}$. We also have $\hat{\Omega}_a \to \hat{\Omega}'_a = R'_{ab}\,\hat{\Omega}_b$. Written out explicitly, this rotation matrix takes the form

$$R' = \begin{pmatrix} \dfrac{n_2^2}{1+n_3} + n_3 & -\dfrac{n_1 n_2}{1+n_3} & -n_1 \\ -\dfrac{n_1 n_2}{1+n_3} & \dfrac{n_1^2}{1+n_3} + n_3 & -n_2 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{122}$$

The commutators then become

$$[\hat{\Omega}_0, \hat{\Omega}'_1] = i\hat{\Omega}'_2,\ [\hat{\Omega}_0, \hat{\Omega}'_2] = -\,i\hat{\Omega}'_1,\ [\hat{\Omega}_1, \hat{\Omega}'_2] = ip_\mu p^\mu\,\hat{\Omega}'_0,\ \hat{\Omega}'_3 = 0 \tag{123}$$

Defining the ladder operators $\hat{\Omega}'_\pm = \hat{\Omega}'_1 \pm i\hat{\Omega}'_2$, we obtain

$$[\hat{\Omega}_0, \hat{\Omega}'_\pm] = \pm\,\hat{\Omega}'_\pm,\ [\hat{\Omega}'_+, \hat{\Omega}'_-] = 2c_1\,\hat{\Omega}_0,\ \hat{\Omega}'_-\,\hat{\Omega}'_+ = \hat{I}_1 - c_1\,\hat{\Omega}_0^2 - c_1\,\hat{\Omega}_0,$$

$$\hat{\Omega}'_+, \hat{\Omega}'_- = \hat{I}_1 - c_1\,\hat{\Omega}_0^2 + c_1\,\hat{\Omega}_0 \tag{124}$$

Using the standard methods of angular momentum, we obtain the complete IRRs of this Lie algebra. For this purpose, we choose, as a basis, a complete set of eigenfunctions of the commuting operators $\hat{C}_1 \equiv \hat{P}_\mu\,\hat{P}^\mu$, $\hat{I}_1 = \hat{\Omega}_0^2 p_\mu p^\mu + \hat{\Omega}_1^2 + \hat{\Omega}_2^2 + \hat{\Omega}_3^2$ and $\hat{\Omega}_0$. The explicit form of the Casimir operators $\hat{I}_1$, $\hat{I}_2$ and $\hat{\Omega}_\mu$ that realize a Hermitian IR of this Lie algebra are given by

$$\hat{P}_\mu\,\hat{P}^\mu|c_1, i_1, \omega\rangle = c_1|c_1, i_1, \omega\rangle \tag{125}$$

$$\hat{\Omega}_0|c_1, i_1, \omega\rangle = \omega|c_1, i_1, \omega\rangle \tag{126}$$

$$\hat{I}_1|c_1, i_1, \omega\rangle = -i_1|c_1, i_1, \omega\rangle \tag{127}$$

and $\hat{I}_1|c_1, i_1, \omega\rangle = e^{2i\pi\hat{\Omega}_0}|c_1, i_1, \omega\rangle = (-1)^{2\omega}|c_1, i_1, \omega\rangle$ because ω can assume only integer or half integer values. We also have

$$\hat{\Omega}_0\,\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle = (\hat{\Omega}'_\pm\,\hat{\Omega}_0 \pm \hat{\Omega}'_\pm)|c_1, i_1, \omega\rangle = (\omega \pm 1)\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle \tag{128}$$

whence $\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle$ are eigenvectors of $\hat{\Omega}_0$ with respective eigenvalues $\omega \pm 1$. It follows that $\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle$ is proportional to $|c_1, i_1, \omega \pm 1\rangle$ respectively so that we can write $\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle = C_\pm|c_1, i_1, \omega \pm 1\rangle$. To determine $C_\pm$, we have $|C_\pm|^2 = |\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle|^2 = \langle c_1, i_1, \omega|\hat{\Omega}'^\dagger_\pm\,\hat{\Omega}'_\pm|c_1, i_1, \omega\rangle$

In addition to the two main Casimir operators viz. $\hat{P}_\mu \hat{P}^\mu$ and $\hat{W}_\mu \hat{W}^\mu$ and the operator $e^{2i\pi \hat{J}_{12}} = e^{2i\pi \hat{J}_{23}} = e^{2i\pi \hat{J}_{31}}$ that has the eigenvalues ± 1 and determines the single/double valuedness of the representation, we also have other Casimir operators that are specific to a particular class of representations. For the moment we denote these class specific Casimir operators by $\hat{C}_\alpha$.

To completely comprehend the relevance of the Pauli Lubanski vector in the representation theory of the Poincare group, we need to study the theory underlying the decomposition of the Poincare group into the relevant factor group by the group of translations in four spacetime T_4, that constitutes the abelian invariant subgroup of the Poincare group. For the purpose, we need to address the issue of the labelling of basis vectors within a given IRR. As mentioned above, the four-dimensional translation group T_4 constitutes an invariant subgroup of the Poincare group. We also know that the basis vectors in a representation of the translation group can be labelled by the eigenvalues of the generators of the translation group. However, the specification of a basis vector using a single index corresponding to the generators of translations is incomplete. For a complete specification of the basis vectors, we introduce a second index m that encompasses all the other degrees of freedom of the group (that are necessary to be considered to ensure the completeness of the basis) but is independent of the translation generators. A complete specification of a basis vector in the representation space (c_1, c_2, c_α) would, thus, consist of five indices, two of them related to the eigenvalues of the main Casimir operators $\hat{C}_1 = \hat{P}_\mu \hat{P}^\mu$ and $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ of the Poincare group and the third to the class specific set of Casimir operators $\hat{C}_\alpha$ for identifying the representation itself and the other two (p, m), where p is the eigenvalue of the momentum operator $\hat{P}_\mu$ and m represents the residual set of degrees of freedom (that are necessary for ensuring the completeness of the basis but are not covered by p), for specifying a basis vector within a representation. Now, all the basis vectors in a given representation space correspond to the same eigenvalue $c_1 \equiv M^2$ of the first Casimir operator. We also have the relativistic mass-energy relation $p_0 = \pm \sqrt{\mathbf{p}^2 + M^2}$. It follows that we can as well use the eigenvalues $\mathbf{p}$ of 3-momentum in lieu of the eigenvalues $p \equiv (p_0, \mathbf{p})$ of 4-momentum for specification of the basis vectors.

Now, $exp\{-i[\hat{L}(\Lambda)\hat{P}_\nu \hat{L}(\Lambda)^{-1}]a^\nu\} = \hat{L}(\Lambda)\, e^{-i\hat{P}_\nu a^\nu}\, \hat{L}(\Lambda)^{-1} = \hat{L}(\Lambda)\, \hat{T}(a)\, \hat{L}(\Lambda)^{-1} = \hat{T}(\Lambda a) = e^{-i\hat{P}_\nu \Lambda^\nu{}_\mu a^\mu}$ whence we get the transformation rules for the covariant generators of translations as $\hat{L}(\Lambda)\hat{P}_\mu\, \hat{L}(\Lambda)^{-1} = \hat{P}_\nu\, \Lambda^\nu{}_\mu$ or equivalently

$$\hat{L}(\Lambda)^{-1}\, \hat{P}_\mu\, \hat{L}(\Lambda) = \hat{P}_\nu (\Lambda^{-1})^\nu{}_\mu \tag{135}$$

Given a basis vector $|\mathbf{p}, m\rangle$, we have $\hat{P}_\mu\, \hat{L}(\Lambda)|\mathbf{p}, m\rangle = \hat{P}_\nu(\Lambda^{-1})^\nu{}_\mu\, \hat{L}(\Lambda)|\mathbf{p}, m\rangle = p_\nu(\Lambda^{-1})^\nu{}_\mu\, \hat{L}(\Lambda)|\mathbf{p}, m\rangle$ thereby showing that $\hat{L}(\Lambda)|\mathbf{p}, m\rangle$ is also an eigenvector of $\hat{P}_\mu$ with the eigenvalue $p_\nu(\Lambda^{-1})^\nu{}_\mu$.

Let us define the operator $\hat{S}(\Lambda)$ by

$$\hat{S}(\Lambda)|\mathbf{p}, m\rangle = |\Lambda\mathbf{p}, m\rangle \tag{136}$$

$\hat{S}(\Lambda)$ is unitary because it leaves the inner product

$$\langle \mathbf{p}, m|\mathbf{p}', m'\rangle = \delta_{mm'}\, \omega_p\, \delta(\mathbf{p} - \mathbf{p}') \tag{137}$$

invariant for $\langle \Lambda\mathbf{p}', m'|\Lambda\mathbf{p}, m\rangle = \langle \mathbf{p}', m'|\hat{S}^\dagger \hat{S}|\mathbf{p}, m\rangle = \delta_{mm'}\, \omega_{\Lambda p}\, \delta(\Lambda\mathbf{p} - \Lambda\mathbf{p}') = \delta_{mm'}\, \omega_p\, \delta(\mathbf{p} - \mathbf{p}') = \langle \mathbf{p}', m'|\mathbf{p}, m\rangle$ where the penultimate step follows from the Lorentz invariance of $\omega_p\, \delta(\mathbf{p} - \mathbf{p}')$.

Further $\hat{S}(\Lambda)\, \hat{S}(\Lambda') = \hat{S}(\Lambda\Lambda')$ implying that $\hat{S}(\Lambda)$ is a unitary representation of the Lorentz group. Also being independent of m, it is diagonal with respect to the corresponding generators.

We also have $\hat{S}(\Lambda)\, \hat{T}(\Lambda^{-1}a)|\mathbf{p}, m\rangle = \hat{S}(\Lambda)\, e^{-i\hat{p}_\mu(\Lambda^{-1}a)^\mu}|\mathbf{p}, m\rangle$

$= \hat{S}(\Lambda)\, e^{-i\hat{p}_\mu(\Lambda^{-1}a)^\mu}|\mathbf{p}, m\rangle = e^{-i\hat{p}_\mu(\Lambda^{-1}a)^\mu}|\Lambda\mathbf{p}, m\rangle = e^{-i(\Lambda^{-1}\hat{p})_\mu(\Lambda^{-1}a)^\mu}|\Lambda\mathbf{p}, m\rangle$

$= \hat{T}(a)|\Lambda\mathbf{p}, m\rangle = \hat{T}(a)\, \hat{S}(\Lambda)|\mathbf{p}, m\rangle$ whence, because of the completeness of the set of basis vectors, we can infer that

$$\hat{T}(\Lambda^{-1}a)\, \hat{S}(\Lambda)^{-1} = \hat{S}(\Lambda)^{-1}\, \hat{T}(a) \tag{138}$$

From eq. (104), we also infer that $\hat{L}(\Lambda)\hat{T}(\Lambda^{-1}a) = \hat{T}(a)\, \hat{L}(\Lambda)$ which, together with eq. (138) gives

$$[\hat{T}(\Lambda)\, \hat{S}(\Lambda)^{-1}, \hat{T}(a)] = 0 \tag{139}$$

Since this equation holds for arbitrary a, it follows that $\hat{L}(\Lambda)\, \hat{S}(\Lambda)^{-1}$ commutes with the translation operator. This implies that it is diagonal with respect to the generators of translation $\hat{P}_\mu$. Writing it as $\hat{Q}(\Lambda, \hat{P})$, we obtain

$$\hat{L}(\Lambda) = \hat{Q}(\Lambda, \hat{P})\, \hat{S}(\Lambda) \tag{140}$$

Further, since both $\hat{L}(\Lambda)$ and $\hat{S}(\Lambda)$, are unitary, it follows that $\hat{Q}(\Lambda, \hat{P})$ is also unitary.

Now, $\hat{Q}(\Lambda\Lambda', \hat{P})\, \hat{S}(\Lambda\Lambda')|\mathbf{p}, m\rangle = \hat{L}(\Lambda\Lambda')|\mathbf{p}, m\rangle = \hat{L}(\Lambda)\, \hat{L}(\Lambda')|\mathbf{p}, m\rangle$

$= \hat{Q}(\Lambda, \hat{P})\, \hat{S}(\Lambda)\, \hat{Q}(\Lambda', \hat{P})\, \hat{S}(\Lambda')|\mathbf{p}, m\rangle = \hat{Q}(\Lambda, \hat{P})\hat{S}(\Lambda)\, \hat{Q}(\Lambda', \hat{P})|\Lambda'\,\mathbf{p}, m\rangle$

$= \hat{Q}(\Lambda, \hat{P})\, \hat{S}(\Lambda)\, \hat{Q}(\Lambda', \Lambda'\hat{P})|\Lambda'\,\mathbf{p}, m\rangle = \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda', \Lambda'\hat{P})\, \hat{S}(\Lambda)|\Lambda'\,\mathbf{p}, m\rangle$

$= \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda', \Lambda'\hat{P})|\Lambda\Lambda'\,\mathbf{p}, m\rangle = \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda', \Lambda^{-1}\hat{P})|\Lambda\Lambda'\,\mathbf{p}, m\rangle$

$= \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda', \Lambda^{-1}\hat{P})\, \hat{S}(\Lambda\Lambda')|\mathbf{p}, m\rangle$ whence

$$\hat{Q}(\Lambda\Lambda', \hat{P}) = \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda', \Lambda^{-1}\hat{P}) \tag{141}$$

Using eq. (141), we obtain

$|\mathbf{p}, m\rangle = \hat{L}(\mathbf{I})|\mathbf{p}, m\rangle = \hat{L}(\Lambda\Lambda^{-1})|\mathbf{p}, m\rangle = \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda^{-1}, \Lambda^{-1}\hat{P})\, \hat{S}(\Lambda\Lambda^{-1})|\mathbf{p}, m\rangle$

$= \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda^{-1}, \Lambda^{-1}\hat{P})|\mathbf{p}, m\rangle = \hat{Q}(\Lambda, \hat{P})\, \hat{Q}(\Lambda, \hat{P})^{-1}|\mathbf{p}, m\rangle$ so that

$$\hat{Q}(\Lambda, \hat{P})^{-1} = \hat{Q}(\Lambda^{-1}, \Lambda^{-1}\hat{P}) \tag{142}$$

each other through suitable Lorentz transformations. Let $\{\eta\}$ be the set of Lorentz transformations that leave q_μ invariant, i.e.,

$$\eta^\mu_\nu\, q^\nu = q^\mu \text{ or } \eta q = q \tag{144}$$

$\{\eta\}$is a subset of the Lorentz group and is called a "little group" of the Lorentz group. Now, since the elements of $\{p\}$ are connected inter se by Lorentz transformation, given a $p_\mu \in \{p\}$, there would exist a Lorentz transformation κ_p such that

$$p = \kappa_p\, q \tag{145}$$

Corresponding to an arbitrary element $q_\mu \in \{p\}$, the Lorentz transformation $\eta_p = \kappa^{-1}_{\Lambda p}\, \Lambda\kappa_p$ or equivalently $\eta_p = \kappa^{-1}_p\, \Lambda\kappa_{\Lambda^{-1}p}$ is an element of the little group $\{\eta\}$ for, we have

$$\eta_p\, q = \kappa^{-1}_{\Lambda p}\, \Lambda\kappa_p\, q = \kappa^{-1}_{\Lambda p}\, \Lambda p = q \text{ or}$$

$$\eta_p\, q = \kappa^{-1}_p\, \Lambda\kappa^{-1}_{\Lambda p}\, q = \kappa^{-1}_p\, \Lambda\Lambda^{-1}\, p = q \tag{146}$$

where we have used eq. (145). Using eqs. (141) and (146), we obtain

$$\hat{Q}(\Lambda,\, \hat{P}) = \hat{Q}(\kappa_{\Lambda p},\, \eta_p\, \kappa^{-1}_p,\, \hat{P}) = \hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\eta_p\, \kappa^{-1}_p,\, \kappa^{-1}_{\Lambda p}\, \hat{P})$$

$$\hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\eta_p,\, \kappa^{-1}_{\Lambda p}\, \hat{P})\; \hat{Q}(\kappa^{-1}_{p,}\, \eta^{-1}_{p},\, \kappa^{-1}_{\Lambda p}\, \hat{P}) \tag{147}$$

Now, operating the right hand side of eq. (147) on the state $|\Lambda\mathbf{p},\, m\rangle$, we obtain

$$\hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\eta_p,\, \kappa^{-1}_{\Lambda p}\, \hat{P})\; \hat{Q}(\kappa^{-1}_{p,}\, \eta^{-1}_{p},\, \kappa^{-1}_{\Lambda p}\, \hat{P})|\Lambda\mathbf{p},\, m\rangle$$

$$\hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\eta_p,\, \kappa^{-1}_{\Lambda p}\, \hat{P})\; \hat{Q}(\kappa^{-1}_{p,}\, \eta^{-1}_{p},\, \kappa^{-1}_{\Lambda p}\, \Lambda p)|\Lambda\mathbf{p},\, m\rangle$$

$$\hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\kappa^{-1}_{p,}\, \eta^{-1}_{p},\, \kappa^{-1}_{\Lambda p}\, \Lambda p)\; \hat{Q}(\eta_p,\, \kappa^{-1}_{\Lambda p}\, \hat{P})|\Lambda\mathbf{p},\, m\rangle$$

$$\hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\kappa^{-1}_{p,}\, \eta^{-1}_{p},\, \kappa^{-1}_{\Lambda p}\, \Lambda p)\; \hat{Q}(\eta_p,\, \kappa^{-1}_{\Lambda p}\, \Lambda p)|\Lambda\mathbf{p},\, m\rangle$$

$$\hat{Q}(\kappa_{\Lambda p,}\, \hat{P})\; \hat{Q}(\eta_{p,}\, q)\; \hat{Q}(\kappa^{-1}_{p,}\, \eta^{-1}_{p},\, q)|\Lambda\mathbf{p},\, m\rangle$$

$$\hat{Q}(\kappa_{\Lambda p},\, \hat{P})\; \hat{Q}(\eta_p,\, q)\; \hat{Q}(\kappa^{-1}_{p,}\, q)|\Lambda\mathbf{p},\, m\rangle$$

$$= \hat{Q}(\kappa_{\Lambda p,}\, \Lambda_p)\; \hat{Q}(\eta_{p,}\, q)\; \hat{Q}(\kappa^{-1}_{p,}\, q)|\Lambda\mathbf{p},\, m\rangle \quad \text{whence}$$

$$\hat{Q}(\Lambda,\, \hat{P}) = \hat{Q}(\kappa_{\Lambda p,}\, \Lambda\hat{P})\; \hat{Q}(\eta_{p,}\, q)\; \hat{Q}(\kappa^{-1}_{p,}\, q) \tag{148}$$

Eq. (148) is identical to the expression $\hat{U}(\Lambda p)^{-1}\; \hat{Q}(\eta_{p,}\, q)\; \hat{U}(p)$ if we identify $\hat{U}(\Lambda p)^{-1}$ $= \hat{Q}(\kappa_{\Lambda p,}\, \Lambda p)$ or $\hat{U}(\Lambda p) = \hat{Q}(\kappa_{\Lambda p,}\, \Lambda p)^{-1} = \hat{Q}(\kappa^{-1}_{\Lambda p},\, \kappa^{-1}_{\Lambda p}\, \Lambda p) = \hat{Q}(\kappa^{-1}_{\Lambda p},\, q)$ and $\hat{U}(p) = \hat{Q}(\kappa^{-1}_{p,}\, q)$.

Since $\hat{U}(p)^{-1}$ is unitary, it follows that $\hat{U}(p)$ is unitary as well. It, therefore, follows from eq. (148) that $\hat{Q}(\eta_{p,}\, q)$ generates a representation of the Poincare group that is unitarily equivalent to $\hat{Q}(\Lambda,\, \hat{P})$. Hence, in lieu of eq. (140), we can write the equivalent expression $\hat{L}(\Lambda) = \hat{Q}(\eta_{p,}\, q)$ $\hat{S}(\Lambda)$ without any loss of generality. Further, using eq. (142), we get

$$\hat{Q}(\kappa_p, \hat{P})^{-1} = \hat{Q}(\kappa_p^{-1}, \kappa^{-1}{}_p \hat{P}) = \hat{Q}(\kappa_p^{-1}, q) \tag{149}$$

Making use of eq. (141), we can also write $\hat{Q}(\eta_p\, \eta'_p, \hat{P}) = \hat{Q}(\eta_p, \hat{P})\ \hat{Q}(\eta'_p\, \eta_p^{-1}, \hat{P})$ and $\hat{Q}(\eta_p\, \eta'_p, q) = \hat{Q}(\eta_p, q)\ \hat{Q}(\eta'_p\, \eta_p^{-1}, q)\ \hat{Q}(\eta_p, q)\ \hat{Q}(\eta'_p, q)$ where $\eta_p = \kappa_{\Lambda p}^{-1} \Lambda \kappa_p$ and $\eta'_p = \kappa_{\Lambda p}^{-1} \Lambda' \kappa_p$.

This shows that $\hat{Q}(\eta_p, q)$ is a unitary representation of the relevant little group $\hat{Q}(\Lambda, \hat{P})$. The operator $\hat{Q}(\eta_p, q)$ is called "Wigner Rotation". It is simple to see that $\hat{Q}(\eta_p, q)\ \hat{S}(\Lambda)$ is a unitary representation of the Lorentz group. Unitarity follows from the unitarity of $\hat{Q}(\eta_p, q)$ and $\hat{S}(\Lambda)$. Further, $\hat{Q}(\eta_p, q)\ \hat{S}(\Lambda)\ \hat{Q}(\eta'_p, q)\ \hat{S}(\Lambda')$.

$\hat{Q}(\eta_p, q)\ \hat{Q}(\eta'_{\Lambda^{-1}p}, q)\ \hat{S}(\Lambda)\ \hat{S}(\Lambda') = \hat{Q}(\eta_p\, \eta'_{\Lambda^{-1}p}, q)\ \hat{S}(\Lambda\Lambda')$ and $\eta_p\, \eta'_{\Lambda^{-1}p} = \kappa_{\Lambda p}^{-1} \Lambda \kappa_p\, (\kappa_p)^{-1} \Lambda' \kappa_{\Lambda^{-1}p} = \kappa_{\Lambda p}^{-1} \Lambda\Lambda'\ \kappa_{\Lambda^{-1}p} = \kappa^{-1}_{\Lambda\Lambda' \cdot \Lambda^{-1}p}\ \Lambda\Lambda'\ \kappa_{\Lambda^{-1}p}$, thereby establishing that $\hat{Q}(\eta_p, q)\ \hat{S}(\Lambda)$ is a representation of the Lorentz group. There is, however, a rider in the sense that since the representations of the Lorentz group are at most double valued, only those $\hat{Q}(\eta_p, q)$ contribute to the representations of the Lorentz group that have at most double valued representations.

It remains to be proved that:

(i) In a unitary representation of the Poincare group $\hat{Q}(\eta_p, q)$, constitutes a unitary irreducible representation of a "little group"; and

(ii) If $\hat{Q}(\eta_p, q)$ is a unitary irreducible representation of a "little group", there exists a corresponding unitary representation of the Poincare group.

To prove (i), let us assume that $\hat{Q}(\eta_p, q)$ is reducible. Now, consider eq. (140). In this eq. $\hat{S}(\Lambda)$ is diagonal in generators relating to the set of eigenvalues collectively represented by m. It follows that the entire dependence of $\hat{L}(\Lambda)$ on 'm' related generators is encompassed in $\hat{Q}(\eta_p, q)$ (which, incidentally, is diagonal in $\hat{p}_\mu$ and hence, independent of the translation operator). Therefore, the reducibility, if any, of $\hat{Q}(\eta_p, q)$ would emanate from the degrees of freedom in 'm'. In other words, if $\hat{Q}(\eta_p, q)$ is to be reducible, such reducibility comes from the freedom associated with 'm'. Since, $\hat{L}(\Lambda)$ is 'm' dependent and the entire such dependence comes from $\hat{Q}(\eta_p, q)$, reducibility of $\hat{Q}(\eta_p, q)$ would manifest itself as reducibility of $\hat{L}(\Lambda)$. Again, since $\hat{T}(a)$ is diagonal with respect to both $\hat{p}_\mu$ and 'm' related degrees of freedom, reducibility of $\hat{L}(\Lambda)$ would imply reducibility of the Poincare group. Consequently, we cannot have a situation wherein the given representation of the Poincare group is irreducible yet the corresponding representation of $\hat{Q}(\eta_p, q)$ is reducible.

To prove (ii), let us assume that $\hat{Q}(\eta_p, q)$ has, at most, double valued representations. Let φ be an element of the Poincare group and let $D(\varphi)$ be its unitary representation consisting of $\hat{L}(\Lambda)$ expressed as $\hat{Q}(\eta_p, q)\ \hat{S}(\Lambda)$ together with the representation of the translation operator $\hat{T}(a)$. Let us assume that this representation is reducible. If the representation is reducible, there will exist state vectors $|\Psi\rangle$ and $|\Phi\rangle$ such that $\langle\Phi|D(\varphi)|\Psi\rangle = 0$. Now, because $D(\varphi)\hat{T}(a)$ is also a representation of

Again, using $\hat{L}(\eta) = \hat{Q}(\kappa_p^{-1}\eta\kappa_{\eta^{-1}p}, q), \hat{S}(\eta)$ and noting that is η an element of the little group so that $\eta q = q$, whence we can write $\hat{L}(\eta) = \hat{Q}(\kappa_p^{-1}\eta\kappa_q, q), \hat{S}(\eta)$ and hence obtain

$$\sum_{m',m''} C_1(\mathbf{p}', m')\, C_2^*(\mathbf{p}'', m'')\, \langle \mathbf{p}'', m''|\hat{L}(\kappa_{p''})\, \hat{L}(\eta)\, \hat{L}(\kappa_{p'}^{-1})|\mathbf{p}', m'\rangle$$

$$= \sum_{m',m''} C_1(\mathbf{p}', m')\, C_2^*(\mathbf{p}'', m'')\, \langle q, m''|\hat{Q}(\kappa_q, q)\, \hat{Q}(\kappa_q^{-1}\eta\, k_q, q)\, \hat{S}(\eta)\, \hat{Q}(\kappa_q^{-1}, q)|q, m'\rangle$$

$$= \sum_{m',m''} C_1(\mathbf{p}', m')\, C_2^*(\mathbf{p}'', m'')\, \langle q, m''|\hat{Q}(\kappa_q, q)\, \hat{Q}(\kappa_q^{-1}\eta\, k_q, q)\, \hat{Q}(\kappa_{\eta^{-1}q}^{-1}, q)\, \hat{S}(\eta)|q, m'\rangle$$

$$= \sum_{m',m''} C_1(\mathbf{p}', m')\, C_2^*(\mathbf{p}'', m'')\, \langle q, m''|\hat{Q}(\kappa_q, q)\, \hat{Q}(\kappa_q^{-1}\eta\, k_q, q)\, \hat{Q}(\kappa_{\eta^{-1}q}^{-1}, q)|\eta q, m'\rangle$$

$$= \sum_{m',m''} C_1(\mathbf{p}', m')\, C_2^*(\mathbf{p}'', m'')\, \langle q, m''|\hat{Q}(\kappa_q, q)\, \hat{Q}(\kappa_q^{-1}\eta\, k_q, q)\, \hat{Q}(\kappa_{\eta^{-1}q}^{-1}, q)|q, m'\rangle$$

$$= \sum_{m',m''} C_1(\mathbf{p}', m')\, C_2^*(\mathbf{p}'', m'')\, \langle q, m''|\hat{Q}(\eta, q)|q, m'\rangle \tag{151}$$

The left hand side of eq. (151) vanishes because of eq. (150). Since η is an arbitrary element of the little group, eq. (151) implies that $\hat{Q}(p, q)$ is a reducible representation of the little group thereby contradicting our assumption. In other words, reducibility of the Poincare group implies reducibility of $\hat{Q}(p, q)$. Conversely, if $\hat{Q}(p, q)$ is irreducible, then the corresponding representation of the Poincare group must necessarily be irreducible. It is reiterated here that the index **p** used for identifying the basis vectors is to be construed as equivalent to the four vector $\mathbf{p} \equiv \left(\sqrt{\mathbf{p}^2 + c_1}, \mathbf{p}\right)$.

It is in the context of the "little group" decomposition of the Poincare group that the Pauli Lubanski vector assumes its physical significance. Given a unit four vector $\{p_\mu\}$, the independent components of $\hat{W}^\mu$ form a Lie algebra on the subspace of $\{p_\mu\}$. This Lie algebra is the Lie algebra of the "little group" of p_μ. Stated otherwise, the generators of the "little group" of p_μ are the independent components of $\hat{W}^\mu$ corresponding to p_μ.

2.4 IRRs CORRESPONDING TO MASSIVE PARTICLES (WIGNER, 1939; GEL'FAND, 1963; NAIMARK, 1964; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994; SOO, 2003; STEFANOVICH, 2007)

Now, for the study of the representations corresponding to the massive $[M_\pm]$ class and the identification of the corresponding "little group" let us define the "standard vector" as

$q = (\text{sgn}(p_0)M, 0, 0, 0)$. We have identified the standard vector in the "rest frame" so that all the spatial components are zero. The factor group of the Poincare group with respect to the subgroup of translations in four dimensions, T_4, (which is abelian) is the homogeneous Lorentz group. The maximal subgroup of the homogeneous Lorentz group that leaves q invariant is the group of three-dimensional rotations, identifiable with $SO(3)$ which, thus, constitutes the "little group" for this class of representations.

This may also be seen by invoking the fact that the generators of the little group of the standard vector, e.g., q are the independent components of the Pauli Lubanski vector $\hat{W} \equiv (\hat{W}^0, \hat{\mathbf{W}}) = \frac{1}{2}\varepsilon^{\lambda\mu\nu\sigma}\,\hat{J}_{\mu\nu}\,\hat{P}_{\sigma} = (\hat{\mathbf{P}}.\hat{\mathbf{J}},\ \hat{P}^0\ \hat{\mathbf{P}} \times \hat{\mathbf{K}})$ corresponding to the standard vector q. The second Casimir operator for massive particles corresponding to q is, therefore, $\hat{W}^{\mu}\,\hat{W}_{\mu} = M\hat{J}^{\mu}\,M\hat{J}_{\mu} = M^2\,\hat{\mathbf{J}}^2$ (as $\mathbf{p} = 0$, $p_0 = M$ for the standard vector). The corresponding Lie algebra takes the form $[\hat{W}^i, \hat{W}^j] = i\varepsilon^{ijk0}\,\hat{W}_k\,\hat{P}_0$ or equivalently $[\hat{J}^i, \hat{J}^j] = i\varepsilon^{ijk}\,\hat{J}_k$ which is the Lie algebra of the group of rotations in three-dimensional space thereby confirming that the little group corresponding to massive representations is $SO(3)$.

As is common in the literature on relativity, we can segregate Lorentz transformation in two types with distinctly identifiable features viz. (i) the θ transformations that constitute spatial rotations about the three spatial axes respectively and do not involve any mixing of spatial and temporal coordinates and (ii) the τ transformations that involve Lorentz boosts along the three spatial axes that are in the nature of spatiotemporal rotations. The coordinate transformation equations under these transformations can be summarized as

$$\left.\begin{aligned} \mathbf{x} &\to \mathbf{x} + \mathbf{x}\times\theta \\ x^0 &\to x^0 \end{aligned}\right\} \quad \theta\ \textit{tranformation} \tag{152}$$

$$\left.\begin{aligned} \mathbf{x} &\to \mathbf{x} - \tau x^0 \\ x^0 &\to x^0 - \tau\,\mathbf{x} \end{aligned}\right\} \quad \tau\ \textit{tranformation} \tag{153}$$

We shall, now, obtain explicit expressions for the infinitesimaWigner rotation for each of these transformations. Corresponding to $q \equiv (\pm M, 0, 0, 0)$, the Lorentz transformation that satisfies the equation $p = \kappa_p\, q$ takes the explicit form

$$\kappa_p = \begin{pmatrix} \omega_p/M & \pm p_1/M & \pm p_2/M & \pm p_3/M \\ \pm p_1/M & 1+\varpi_p\, p_1^2 & \varpi_p\, p_1\, p_2 & \varpi_p\, p_1\, p_3 \\ \pm p_2/M & \varpi_p\, p_2\, p_1 & 1+\varpi_p\, p_2^2 & \varpi_p\, p_2\, p_3 \\ \pm p_3/M & \varpi_p\, p_3\, p_1 & \varpi_p\, p_3\, p_2 & 1+\varpi_p\, p_3^2 \end{pmatrix} \tag{154}$$

where $\varpi_p = \frac{1}{p^2}\left(\frac{\omega_p}{M} - 1\right)$. In terms of the rapidity, $\varsigma = \cosh^{-1}\frac{\omega_p}{M} = \sinh^{-1}\frac{|\mathbf{p}|}{M}$, eq. (154) may be

$$\hat{Q}(\Lambda, p) = \mathbf{I} + \frac{i}{2}\left[\omega_{ij} - \frac{1}{p^0 + M}\left(p_i\,\omega_{j0} - p_j\,\omega_{i0}\right)\right]\hat{M}^{ij} \tag{158}$$

with $p^0 \equiv \sqrt{|\mathbf{p}|^2 + M^2} = \omega_p.$

We can compute the finite Wigner rotations for general Lorentz transformations by decomposing such transformation as $\Lambda = \hat{R}(\psi).\hat{L}(\alpha)$ where the former factor is a pure spatial rotation and the latter is a pure Lorentz boost. Using eq. (141), we then have $\hat{Q}(\Lambda, p) = \hat{Q}(\hat{R}(\psi), \hat{L}(\alpha)p).\hat{Q}(\hat{L}(\alpha).p)$. Now, irrespective of the momenta, Wigner angles are degenerate with ordinary rotation angles when the boost parameters are zero. Therefore, the first factor is just $(i\psi.\hat{\mathbf{J}})$, while the second factor is the Wigner rotation for an arbitrary pure boost which can be written in the form $\hat{Q}(\hat{L}(\tau), p) \equiv \exp(i\phi(\tau).\hat{\mathbf{J}})$.

$$\cos\phi = \frac{[\cosh\tau + \cosh\zeta + \sinh\tau\sinh\zeta(\hat{\tau}.\hat{\mathbf{p}}) + (\cosh\tau - 1)(\cosh\zeta - 1)(\hat{\tau}.\hat{\mathbf{p}})^2]}{[1 + \cosh\tau\cosh\zeta + \sinh\tau\sinh\zeta(\hat{\tau}.\hat{\mathbf{p}})}$$

$$= \frac{[M\cosh\tau + p^0 + \sinh\tau(\hat{\tau}.\hat{\mathbf{p}}) + (\cosh\tau - 1)(p'^0 - M)(\hat{\tau}.\hat{\mathbf{p}})^2]}{[M + p^0\cosh\tau + \sinh\tau(\hat{\tau}.\hat{\mathbf{p}})]}$$

$$(\sin\phi)\,\hat{\phi} = \frac{[\sinh\tau\sinh\zeta + (\cosh\tau - 1)(\cosh\zeta - 1)(\hat{\tau}.\hat{\mathbf{p}})]}{[1 + \cosh\tau\cosh\zeta + \sinh\tau\sinh\zeta(\hat{\tau}.\hat{\mathbf{p}})]}(\hat{\tau} \times \hat{\mathbf{p}})$$

$$= \frac{[|\mathbf{p}|\sinh\tau + (p^0 - M)(\cosh\tau - 1)(\hat{\tau}.\hat{\mathbf{p}})]}{[M + p^0\cosh\tau + p\sinh\tau(\hat{\tau}.\hat{\mathbf{p}})]}(\hat{\tau} \times \hat{\mathbf{p}})$$

The complete expression for the Wigner rotation may, therefore, be written as

$$\hat{Q}(\Lambda, p) = \exp(i\boldsymbol{\theta}_w.\hat{\mathbf{J}}) = \exp(i\psi.\hat{\mathbf{J}}).\exp(i\phi(\boldsymbol{\alpha}).\hat{\mathbf{J}}) \tag{159}$$

with

$$\cos\left(\frac{\theta_{\mathbf{w}}}{2}\right) = \left(\cos\frac{\psi}{2}\right)\left(\cos\frac{\phi}{2}\right) - \left(\sin\frac{\psi}{2}\right)\left(\sin\frac{\phi}{2}\right)(\hat{\psi}.\hat{\phi})$$

$$\sin\left(\frac{\theta_{\mathbf{w}}}{2}\right)\hat{\theta}_W = \left(\cos\frac{\psi}{2}\right)\left(\sin\frac{\phi}{2}\right)\hat{\phi} + \left(\sin\frac{\theta}{2}\right)\left(\cos\frac{\phi}{2}\right)\hat{\psi} + \left(\sin\frac{\psi}{2}\right)\left(\sin\frac{\phi}{2}\right)(\hat{\phi} \times \hat{\psi})$$

Full derivation using the SL(2,C) representation of the Lorentz group is given in a subsequent section.

2.4.1 Construction of the Basis Vectors for $[M_{\pm}]$

The Hilbert space Hof a massive particle can be represented as a direct sum of the eigenspaces H_P of the momentum operator, i.e., $\mathbf{H} = \otimes_{p\,\varepsilon\,\mathbb{C}^3} H_p$. In the sequel, we consider the "positive energy" subspace of the massive, $[M_{\pm}]$, representations i.e. the representation space corresponding to the representation class $[M_{+}]$.

To construct the basis we select the "standard vector" as $p_t^{\mu} = (p^0, \mathbf{p}) = (M, \mathbf{0})$. Since $\mathbf{p} = 0$, this scenario corresponds to the particle being in a state of rest with mass (rest energy) equal to M. The vector p_t^{μ} is invariant under all rotations in the three spatial dimensions. Hence, the "little group" of p_t is the subgroup consisting of rotations in the three dimensional space which is isomorphic to $SO(3)$. The Pauli Lubanski vector that corresponds to the second Casimir operator, in this case, becomes proportional to $\hat{\mathbf{J}}^2$, as we have already shown. Each irreducible unitary representation, labeled by the eigenvalue of $\hat{\mathbf{J}}^2$, of the rotation group $SO(3)$, being the little group of p_t, induces an irreducible unitary representation of the Poincare group. The IRRs are labeled by the eigenvalue $c_1 > 0$ of the Casimir operator $\hat{C}_1 = \hat{P}_{\mu}\hat{P}^{\mu}$ (or equivalently, by M) alongwith the labeling index $s(s + 1)$ corresponding to the operator $\hat{\mathbf{J}}^2$ of the unitary representations of the rotation group in the three spatial dimensions.

Within a representation labeled by (M, s) we shall label the basis vectors of the subspace H_0 corresponding to the rest frame eigenvalues p_t of $\hat{P}^{\mu}$ by $|\mathbf{0}\lambda\rangle$ since $p^0 = M$, $\mathbf{p} = 0$, and λ is the eigenvalue of $\hat{J}_3$. For a particular IRR, M, $s(s + 1)$ remain invariant and hence, are used for labeling the IRRs. In designating the basis vectors of a particular IRR, they may be omitted in the labeling scheme of the basis vectors and we may abbreviate the basis vectors as $|\mathbf{0}\lambda\rangle$. These basis vectors are defined by the following relations:

$$\hat{P}^{\mu}|\mathbf{0}\lambda\rangle = \hat{P}_t^{\mu}|\mathbf{0}\lambda\rangle, \quad \text{where } p_t^{\mu} = (M, \mathbf{0}) \tag{160}$$

$$\hat{\mathbf{J}}^2|\mathbf{0}\lambda\rangle = s(s+1)|\mathbf{0}\lambda\rangle \tag{161}$$

$$\hat{J}_3|\mathbf{0}\lambda\rangle = \lambda|\mathbf{0}\lambda\rangle \tag{162}$$

As mentioned above, this subspace, H_0, is invariant under the action of the little group being the group of rotations in the three spatial dimensions. It has the following properties:

(i) It is invariant with respect to rotations for

$$\hat{\mathbf{P}}(\hat{\mathbf{I}} - i\hat{\mathbf{J}}\theta|\mathbf{0}\lambda\rangle = \hat{\mathbf{P}}|\mathbf{0}\lambda\rangle - i\hat{\mathbf{P}}\hat{\mathbf{J}}\theta|\mathbf{0}\lambda\rangle = 0 \tag{163}$$

showing that the rotated vector has also zero momentum and hence belongs to H_0. In arriving at this result, we have used the commutation relation $[\hat{P}_m, \hat{J}_n] = i\varepsilon^{mnl}\,\hat{P}_l$;

(ii) If we define the relativistic spin operator for positive energy massive particles as

$$\hat{S} = \frac{1}{M}\hat{P}^0\,\hat{\mathbf{J}} - \frac{1}{M}\hat{\mathbf{P}} \times \hat{\mathbf{K}} - \frac{1}{(\hat{P}_0 + M)M}\,\hat{\mathbf{P}}(\hat{\mathbf{P}}.\hat{\mathbf{J}}), \text{ then}$$

$$\hat{S}_3|\mathbf{0}\lambda\rangle = \left[\frac{1}{M}\hat{P}^0\,\hat{J}_3 - \frac{1}{M}(\hat{P}_1\,\hat{K}_2 - \hat{P}_2\,\hat{K}_1) - \frac{1}{(\hat{P}_0 + M)M}\hat{P}_3\,(\hat{\mathbf{P}}.\hat{\mathbf{J}})\right]|\mathbf{0}\lambda\rangle$$

confirming that $|\mathbf{p}\lambda\rangle$ is an eigenstate of the momentum operator with momentum $\mathbf{p}$. Now, consider

$$\hat{S}_3|\mathbf{p}\lambda\rangle = N(\mathbf{p})\ \hat{S}_3\, e^{-i\hat{\mathbf{k}}\,\theta_{\mathbf{p}}}|\mathbf{0}\lambda\rangle = N(\mathbf{p})\ e^{-i\hat{\mathbf{k}}\,\theta_{\mathbf{p}}}\, e^{-i\hat{\mathbf{k}}\,\theta_{\mathbf{p}}}\, \hat{S}_3\, e^{-i\hat{\mathbf{k}}\,\theta_{\mathbf{p}}}|\mathbf{0}\lambda\rangle$$
$$\approx N(\mathbf{p})\ (\hat{\mathbf{I}} - i\hat{\mathbf{K}}\theta_{\mathbf{p}})\ [(\hat{\mathbf{I}} + i\hat{\mathbf{K}}\theta_{\mathbf{p}})\ \hat{S}_3(\hat{\mathbf{I}} - i\hat{\mathbf{K}}\theta_{\mathbf{p}})]|\mathbf{0}\lambda\rangle$$
$$\approx N(\mathbf{p})\ (\hat{\mathbf{I}} - i\hat{\mathbf{K}}\theta_{\mathbf{p}})\ (\hat{S}_3 + i[\hat{\mathbf{K}}\theta_{\mathbf{p}},\ \hat{S}_3])|\mathbf{0}\lambda\rangle \tag{167}$$

We have $\hat{S}_3|0\lambda\rangle = \lambda|\mathbf{0}\lambda\rangle$ by definition of the basis vector, since it is labelled by the eigenvalue of $\hat{S}_3$. To determine $[\hat{\mathbf{K}}\theta_p,\ \hat{S}_3]|\mathbf{0}\lambda\rangle$, we make use of the explicit representation of the spin operator as

$$\hat{S}_3 = \frac{1}{M}\hat{P}^0\hat{J}_3 - \frac{1}{M}\,(\hat{P}_1\ \hat{K}_2 - \hat{P}_2\ \hat{K}_1) - \frac{1}{(\hat{P}_0 + M)M}\ \hat{P}_3(\hat{\mathbf{P}}.\hat{\mathbf{J}}) \tag{168}$$

We, then, have $[\hat{K}_1\ \theta^1_{\mathbf{p}},\ \hat{P}_0\ \hat{J}_3]|\mathbf{0}\lambda\rangle = \{\hat{P}_0[\hat{K}_1\theta^1_{\mathbf{p}},\ \hat{J}_3] + [\hat{K}_1\theta^1_{\mathbf{p}},\ \hat{P}_0]\hat{J}_3\}|\mathbf{0}\lambda\rangle$

$$= (-\,i\hat{P}_0\ \hat{K}_2\ \theta^1_{\mathbf{p}} - i\hat{P}_1\theta^1_{\mathbf{p}}\hat{J}_3)|\mathbf{0}\lambda\rangle = (-i\hat{P}_0\ \hat{K}_2\ \theta^1_{\mathbf{p}}|\mathbf{0}\lambda\rangle = -\,i(i\hat{P}_2 +\ \hat{K}_2\ \hat{P}_0)\theta^1_{\mathbf{p}}|\mathbf{0}\lambda\rangle$$
$$= -iM\hat{K}_2\ \theta^1_{\mathbf{p}}|\mathbf{0}\lambda\rangle$$
$$[\hat{K}_2\ \theta^2_{\mathbf{p}},\ \hat{P}_0\ \hat{J}_3]|\mathbf{0}\lambda\rangle = \{\hat{P}_0[\hat{K}_2\ \theta^2_{\mathbf{p}},\ \hat{J}_3] + [\hat{K}_2\ \theta^2_{\mathbf{p}},\ \hat{P}_0]\hat{J}_3\}|\mathbf{0}\lambda\rangle$$
$$= (i\hat{P}_0\ \hat{K}_1\ \theta^2_{\mathbf{p}} - i\hat{P}_2\ \theta^2_{\mathbf{p}}\ \hat{J}_3)|\mathbf{0}\lambda\rangle = (i\hat{P}_0\ \hat{K}_1\ \theta^2_{\mathbf{p}})|\mathbf{0}\lambda\rangle = i(i\hat{P}_1 + \hat{K}_1\ \hat{P}_0)\theta^2_{\mathbf{p}}|\mathbf{0}\lambda\rangle$$
$$= iM\hat{K}_1\ \theta^2_{\mathbf{p}}|\mathbf{0}\lambda\rangle$$
$$[\hat{K}_3\,\theta^3_{\mathbf{p}},\ \hat{P}_0\ \hat{J}_3]|\mathbf{0}\lambda\rangle = \{\hat{P}_0[\hat{K}_3\ \theta^3_{\mathbf{p}},\ \hat{J}_3] + [\hat{K}_3\ \theta^3_{\mathbf{p}},\ \hat{P}_0]\hat{J}_3\}|\mathbf{0}\lambda\rangle$$
$$= (-i\hat{P}_3\ \theta^3_{\mathbf{p}}\ \hat{J}_3|\mathbf{0}\lambda\rangle = 0 \quad \text{so that } [\hat{K}\theta_{\mathbf{p}},\ \hat{P}_0\hat{J}_3]|\mathbf{0}\lambda\rangle = iM\,[\hat{K}_1\ \theta^2_{\mathbf{p}} - \hat{K}_2\ \theta^1_{\mathbf{p}}]|\mathbf{0}\lambda\rangle.$$

Similarly, $[\hat{K}_1\,\theta^1_{\mathbf{p}},\ \hat{P}_1\ \hat{K}_2]|\mathbf{0}\lambda\rangle = \{\hat{P}_1[\hat{K}_1\ \theta^1_{\mathbf{p}},\ \hat{K}_2] + [\hat{K}_1\ \theta^1_{\mathbf{p}},\ \hat{P}_1]\hat{K}_2\}|\mathbf{0}\lambda\rangle$

$$= (-i\hat{P}_1\hat{J}_3\theta^1_{\mathbf{p}} - i\hat{P}_0\ \theta^1_{\mathbf{p}}\ \hat{K}_2)|\mathbf{0}\lambda\rangle = (-i\hat{P}_0\hat{K}_2\theta^1_{\mathbf{p}})|\mathbf{0}\lambda\rangle$$
$$= -i(i\hat{P}_2 + \hat{K}_2\ \hat{P}_0\,)\theta^1_{\mathbf{p}}|\mathbf{0}\lambda\rangle = -iM\hat{K}_2\ \theta^1_{\mathbf{p}}|\mathbf{0}\lambda\rangle$$
$$[\hat{K}_2\,\theta^2_{\mathbf{p}},\ \hat{P}_1\hat{K}_2]|\mathbf{0}\lambda\rangle = \{\hat{P}_1[\hat{K}_2\ \theta^2_{\mathbf{p}},\ \hat{K}_2] + [\hat{K}_2\ \theta^2_{\mathbf{p}},\ \hat{P}_1]\hat{K}_2\}|\mathbf{0}\lambda\rangle = 0$$
$$[\hat{K}_3\,\theta^3_{\mathbf{p}},\ \hat{P}_1\hat{K}_2]|\mathbf{0}\lambda\rangle = \{\hat{P}_1[\hat{K}_3\ \theta^3_{\mathbf{p}},\ \hat{K}_2] + [\hat{K}_3\ \theta^3_{\mathbf{p}},\ \hat{P}_1]\hat{K}_2\}|\mathbf{0}\lambda\rangle$$
$$= (i\hat{P}_1\hat{J}_1\ \theta^3_{\mathbf{p}})|\mathbf{0}\lambda\rangle = 0$$

so that $\qquad [\hat{K}\theta_{\mathbf{p}},\ \hat{P}_1\hat{K}_2]|\mathbf{0}\lambda\rangle = -iM\hat{K}_2\ \theta^1_{\mathbf{p}}|\mathbf{0}\lambda\rangle.$

Similarly, $\quad [\hat{K}\theta_{\mathbf{p}},\ \hat{P}_2\hat{K}_1]|\mathbf{0}\lambda\rangle = -iM\hat{K}_1\ \theta^2_{\mathbf{p}}|\mathbf{0}\lambda\rangle$ and

$$[\hat{K}\theta_{\mathbf{p}},\ \hat{P}_3(\hat{\mathbf{P}}.\hat{\mathbf{J}})]|\mathbf{0}\lambda\rangle = \{\hat{P}_3[\hat{K}\boldsymbol{\theta}_{\mathbf{p}},\ (\hat{\mathbf{P}}.\hat{\mathbf{J}})] + [\hat{K}\theta_{\mathbf{p}},\ \hat{P}_3](\hat{\mathbf{P}}.\hat{\mathbf{J}})\}|\mathbf{0}\lambda\rangle = 0$$

Putting all these pieces together, we get

$$\hat{S}_3|\mathbf{p}\lambda\rangle = \lambda \mathrm{N}(\mathbf{p})\ (\hat{\mathbf{I}} - i\hat{\mathbf{K}}\theta_{\mathbf{p}})|\mathbf{0}\lambda\rangle = \lambda|\mathbf{p}\lambda\rangle \tag{169}$$

showing that $|\mathbf{p}\lambda\rangle$ is an eigenstate of $\hat{S}_3$ with eigenvalue λ. The effect of various transformations constituting the Poincare group on the basis vectors is summarized thus:

(a) Translations: We have

$$e^{-i\mathbf{P}a}|\mathbf{p}\lambda\rangle = e^{-i\mathbf{p}a}|\mathbf{p}\lambda\rangle,\ e^{i\hat{H}x^0}|\mathbf{p}\lambda\rangle = e^{i\omega_{\mathbf{p}}x^0}|\mathbf{p}\lambda\rangle \tag{170}$$

with $\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + M^2$;

(b) Spatial Rotations: We have $e^{-i\hat{\mathbf{J}}\varphi}|\mathbf{p}\lambda\rangle = N(\mathbf{p})e^{-i\hat{\mathbf{J}}\varphi}\ e^{-i\hat{\mathbf{K}}\theta_{\mathbf{p}}}|\mathbf{0}\lambda\rangle$

$$= N(\mathbf{p})e^{-i\hat{\mathbf{J}}\varphi}\ e^{-i\hat{\mathbf{K}}\theta_{\mathbf{p}}}e^{i\hat{\mathbf{J}}\varphi}e^{-i\hat{\mathbf{J}}\varphi}|\mathbf{0}\lambda\rangle = N(\mathbf{p})e^{-i(R(\varphi)^{-1}\hat{\mathbf{K}})\theta_{\mathbf{p}}} \sum_{\lambda'=-s}^{s} D_{\lambda'\lambda}(\varphi)|\mathbf{0}\lambda'\rangle$$

$$= N(\mathbf{p})e^{-i\hat{\mathbf{K}}R(\varphi)\theta_{\mathbf{p}}} \sum_{\lambda'=-s}^{s} D_{\lambda'\lambda}(\varphi)|\mathbf{0}\lambda'\rangle = \sum_{\lambda'=-s}^{s} D_{\lambda'\lambda}(\varphi)|R(\varphi)\mathbf{p},\ \lambda'\rangle \tag{171}$$

where we have used

$$\hat{U}(R)\ \hat{P}_i|\mathbf{p}\rangle = \hat{U}(R)\ \hat{P}_i\hat{U}(R)^{-1}\ \hat{U}(R)|\mathbf{p}\rangle = \hat{U}(R)\hat{P}_i\ \hat{U}(R)^{-1}|\mathbf{p}'\rangle$$

$$= \hat{U}(R)\ p_i|\mathbf{p}\rangle = p_i|\mathbf{p}'\rangle = \sum_j (R^{-1})_i^j\ p_j'|\mathbf{p}'\rangle = \sum_j (R^{-1})_i^j\ \hat{P}_j|\mathbf{p}'\rangle \text{ whence}$$

$$\hat{U}(R)\hat{P}_i\hat{U}\ (\mathrm{R})^{-1} = \sum_j (R^{-1})_i^j\ \hat{P}_j \tag{172}$$

(c) Lorentz Boosts: Let us apply a Lorentz boost Λ to a basis vector $|\mathbf{p}\lambda\rangle$ to obtain $\Lambda|\mathbf{p}\lambda\rangle = \Lambda N(\mathbf{p})\tau_{\mathbf{p}}|0\lambda\rangle$ where $\tau_{\mathbf{p}}$ transforms a vector of zero momentum to a vector of momentum **p**. Now, the right hand side, being the product of two boosts, is also a Lorentz transformation and hence, can be represented by the product of a spatial rotation followed by a boost, i.e., $\Lambda|\mathbf{p}\lambda\rangle = N(\mathbf{p})\ \Lambda\tau_{\mathbf{p}}|\mathbf{0}\lambda\rangle = N(\mathbf{p})\tau_{\mathbf{p}}.R(\varphi_W(\mathbf{p},\ \Lambda))|0\lambda\rangle$ or equivalently $\tau_{\mathbf{p}}^{-1}.\Lambda\tau_{\mathbf{p}}|0\lambda\rangle = R(\varphi_W(\mathbf{p},\ \Lambda)|0\lambda\rangle$. Now, we have shown above that a rotation keeps invariant the subspace of zero momentum. It follows that the sequence of boosts on the left hand side must return each vector of zero momentum to the subspace of zero momentum. Now, $\tau_{\mathbf{p}}$ transforms a vector of zero momentum to a vector of momentum **p**. Subsequent application of the boost Λ would transform this vector's momentum to Λ**p**. It follows that the boost $\tau_{\mathbf{p}'}$ will transform the vector with momentum Λ**p** back to zero, i.e, $\tau_{\mathbf{p}'} = \tau_{\Lambda\mathbf{p}}$. Therefore,

$$e^{-i\hat{\mathbf{K}}\theta}|\mathbf{p}\lambda\rangle = N(\mathbf{p})\ \Lambda\tau_{\mathbf{p}}|0\lambda\rangle = N(\mathbf{p})\ \tau_{\Lambda\mathbf{p}}\ R(\varphi_W(\mathbf{p}.\Lambda))|0\lambda\rangle$$

$$= N(\mathbf{p})\ \tau_{\Lambda\mathbf{p}} \sum_{\lambda'=-s}^{s} D_{\lambda'\lambda}^{s}(\varphi_W(\mathbf{p},\ \Lambda))|0\lambda'\rangle$$

2.5 TRANSFORMATION RULES FOR THE WAVEFUNCTIONS (MOMENTUM REPRESENTATION)

Two different eigenstates of the momentum operator, being different eigenstates of a Hermitian operator must necessarily be orthogonal, i.e., $\langle \mathbf{p}|\mathbf{p}'\rangle = 0$ if $\mathbf{p} \neq \mathbf{p}'$. Had the spectrum of eigenvalues $Spec\{\mathbf{p}\}$ been discrete, we would have the simple normalization $\langle \mathbf{p}|\mathbf{p}'\rangle = \delta_{\mathbf{pp}'}$. However, such a normalization is obviously not possible for continuous spectra. To obviate this problem and obtain normalizable vectors, we introduce momentum space wavefunctions $\psi(\mathbf{p})$ and write an arbitrary state vector as $|\Psi\rangle = \int d\mathbf{p}\psi(\mathbf{p})|\mathbf{p}\rangle$ with the normalization $\langle \mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p} - \mathbf{p}')$ so $\langle \mathbf{p}|\psi\rangle = \langle \mathbf{p}|\int d\mathbf{p}' \psi(\mathbf{p}')|\mathbf{p}'\rangle = \int d\mathbf{p}' \psi(\mathbf{p}') \langle \mathbf{p}'|\mathbf{p}\rangle = \int d\mathbf{p}' \psi(\mathbf{p}') \delta(\mathbf{p} - \mathbf{p}') = \psi(\mathbf{p})$ where we have, for the time being omitted the spin index and identified the eigenstates only by the momentum eigenvalue.

Transformation properties of the momentum space wavefunction $\psi(\mathbf{p})$ are summarized below:

$$\text{(i)}\quad \hat{P}_\mu \psi(\mathbf{p}) = \langle \mathbf{p}|\hat{P}_\mu|\psi\rangle = \langle \mathbf{p}|\hat{P}_\mu^\dagger|\psi\rangle = p_\mu\langle \mathbf{p}|\psi\rangle = p_\mu\psi(\mathbf{p}) \tag{174}$$

$$\text{(ii)}\quad \hat{J}_1 \psi(\mathbf{p}) = (\hat{x}^2\hat{P}_3 - \hat{x}^3\hat{P}_2)\,\psi(p) = i\left(p_3\frac{d}{dp_2} - p_2\frac{d}{dp_3}\right)\psi(\mathbf{p}) \tag{175}$$

$$\text{(iii)}\quad \hat{K}_1 \psi(\mathbf{p}) = i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}[e^{-i\hat{K}_1\theta}\,\psi(\mathrm{p})] = i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}\,[\langle \mathbf{p}|e^{-i\hat{K}_1\theta}|\Psi\rangle]$$

$$= i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}[\langle \mathbf{p}|e^{(i\hat{K}_1\theta)\dagger}|\Psi\rangle] = i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}\,[\langle \mathbf{p}|(e^{i\hat{K}_1\theta})^\dagger|\Psi\rangle]$$

$$= i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}\left[\sqrt{\frac{\omega_{\Lambda^{-1}p}}{\omega_p}}\left\langle \Lambda^{-1}\mathbf{p},\lambda\,|\,\Psi\right\rangle\right] = i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}\left[\sqrt{\frac{\omega_{\Lambda^{-1}p}}{\omega_p}}\,\psi(\Lambda^{-1}\mathbf{p})\right]$$

$$= i\mathrm{Lim}_{\theta\to 0}\frac{d}{d\theta}\left[\sqrt{\frac{\sqrt{\mathbf{p}^2+M^2}\,\cosh\theta - p_1\sinh\theta}{\sqrt{\mathbf{p}^2+M^2}}}\,\psi\left(p_1\cosh\theta - \sqrt{\mathbf{p}^2+M^2}\,\sinh\theta,\,p_2,\,p_3\right)\right]$$

$$= -\,i\left[\omega_p\frac{d}{dp_1} + \frac{p_1}{2\omega_p}\right]\psi(\mathbf{p}) \tag{176}$$

With explicit spin dependence, the basis vectors $|\mathbf{p}\lambda\rangle$ transform under a Lorentz transformation as $U[\Lambda]|\mathbf{p}\lambda\rangle = \sum_\sigma D_{\sigma\lambda}\;[W(\Lambda,p)]|\Lambda\mathbf{p},\sigma\rangle$ where $D_{\sigma\lambda}$ are the matrix elements of the unitary operator D that corresponds to the Wigner rotation $W(\Lambda, p)$. Let $|\Psi\rangle = \sum_\lambda\int|\,p\lambda\rangle\psi^\lambda\;[\mathbf{p}]\,dp$ be a quantum state, so that the wavefunction $\psi^\lambda(\mathbf{p}) = \langle \mathbf{p}\lambda|\Psi\rangle$ whence

$$|\Psi'\rangle = \hat{U}[\Lambda]\Psi\rangle = \sum_{\sigma,\nu} \int |\Lambda\mathbf{p}, \sigma\rangle\, D^s[W(\Lambda, p)]^{\sigma}_{\nu}\, \psi^{\nu}(\mathbf{p})dp$$

$$\sum_{\sigma,\nu} \int |\mathbf{p}, \sigma\rangle\, D^s\,[W(\Lambda, \Lambda^{-1}\, p)]^{\sigma}_{\nu}\, \psi^{\nu}(\Lambda^{-1}\,\mathbf{p})\, dp = \sum_{\lambda} \int |\mathbf{p}\lambda\rangle\, \psi'^{\lambda}(\mathbf{p})\, dp$$

whence we get

$$\Psi'^{\lambda}(\mathbf{p}) = \sum_{\lambda'} D^s\,[W[\Lambda, \Lambda^{-1}\, p)]^{\lambda}{}_{\lambda'}\, \psi^{\lambda'}\,(\Lambda^{-1}\,\mathbf{p}) \tag{177}$$

which elucidates the rule for the transformation of multicomponent wavefunctions under arbitrary Lorentz transformations.

2.6 TRANSFORMATION OF VECTORS

Consider an operator $\hat{P}$. We have

$$\hat{U}(\Lambda)\ \hat{P}^{\mu}|\mathbf{p}\rangle = \hat{U}(\Lambda)\hat{P}^{\mu}\ \hat{U}(\Lambda)^{-1}\ \hat{U}(\Lambda)|\mathbf{p}\rangle = \hat{U}(\Lambda)\ \hat{P}^{\mu}\hat{U}(\Lambda)^{-1}|\mathbf{p}'\rangle$$
$$= \hat{U}(\Lambda)\, p^{\mu}|\mathbf{p}\rangle = p^{\mu}|\mathbf{p}'\rangle = (\Lambda^{-1})^{\mu}_{\nu}\, p'^{\nu}|\mathbf{p}'\rangle = (\Lambda^{-1})^{\mu}_{\nu}\ \hat{P}^{\nu}|\mathbf{p}'\rangle$$

whence

$$\hat{U}(\Lambda)\ \hat{P}^{\mu}\ \hat{U}(\Lambda)^{-1} = (\Lambda^{-1})^{\mu}_{\nu}\ \hat{P}^{\nu} \tag{178}$$

For a covariant vector, we have $\hat{U}(\Lambda)\ \hat{p}^{\mu}\ \hat{U}(\Lambda)^{-1} = \hat{p}_{\nu}\ \Lambda^{\nu}_{\mu}$. These vectors transform according to the $(u, v) \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$ representation of the proper Lorentz group because under three-dimensional rotations, the time component is unchanged (corresponding to $j_0 = 0$) and the spatial components transform as a 3-vector (corresponding to $j_1 = 1$) whence $u = \frac{j_1 + j_0}{2} = \frac{1}{2}$ and $v = \frac{j_1 - j_0}{2} = \frac{1}{2}$. The representation matrices corresponding to the $(u, v) \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$ representation are the $SL(2)$ matrices.

2.7 TRANSFORMATION OF SECOND RANK ANTISYMMETRIC TENSORS

The generators of the proper Lorentz group $\hat{J}_{\mu\nu}$ are second rank antisymmetric tensors. It is, therefore, important to study their transformation properties.

Let us construct a basis for the space of the second rank tensors in Minkowski space. Let $\{e_{\mu}, \mu = 0, 1, 2, 3\}$ constitute a basis for the Minkowski space. Second rank tensors over this space form a 16-dimensional space that is spanned by $e_{\mu} \otimes e_{\nu}$. Using these basis vectors, we constitute the six antisymmetric tensors $e_{[\mu\nu]} := e_{\mu} \otimes e_{\nu} - e_{\nu} \otimes e_{\mu}$. The set of these six antisymmetric tensors constitutes a basis set for the space of second rank antisymmetric tensors and we can write an arbitrary second rank antisymmetric tensor in terms of these basis vectors as $\hat{T} = T^{\mu\nu} e_{[\mu\nu]}$. Let us designate these six basis vectors as $\{f_i, i = 1, 2, ...6\}$.

Our first step is to prove that the space of rank two antisymmetric tensors is invariant under the action of Lorentz transformations. For the purpose we write $\hat{T}^{\mu}_{\nu} = \dfrac{T^{\mu}_{\nu} - T^{\nu}_{\mu}}{2}$ so that $\hat{T}'^{\mu}_{\nu} = [\hat{U}(\Lambda)\hat{T}\hat{U}(\Lambda)^{-1}]^{\mu}_{\nu}$

$$= \frac{1}{2}\{\hat{U}(\Lambda)^{\mu}_{\lambda}\, T^{\lambda}_{\kappa}\, [\hat{U}(\Lambda)^{-1}]^{\kappa}_{\nu} - \hat{U}(\Lambda)^{\nu}_{\rho}\, T^{\rho}_{\sigma}\, [\hat{U}(\Lambda)^{-1}]^{\sigma}_{\mu}\} = \frac{1}{2}(T'^{\mu}_{\nu} - T'^{\nu}_{\mu})$$

which is antisymmetric so that the subspace of antisymmetric second rank tensors is invariant under Lorentz transformations. Having established the invariance of the relevant subspace we shall now show that it admits the representation $(1, 0) \otimes (0, 1)$ by explicitly decomposing this space into a direct sum of minimal invariant subspaces.

We, now, examine the behaviour of the basis tensors $e_{[\mu\nu]}$ under various Lorentz transformations.

We first consider pure rotations. Let us examine the purely spatial components of $e_{[\mu\nu]}$ viz. $e_{[12]}, e_{[23]}, e_{[31]}$. Let us write $e_i = \frac{1}{2}\varepsilon^{ijk} e_{jk}$. Its image under an $SO(3)$ transformation will be $e'_k = \frac{1}{2}\varepsilon^{ij}_k\, e'_{ij} = \frac{1}{2}\varepsilon^{ij}_k\, R^l_i\, e_{\mathrm{lm}}\, R^m_j$. Multiplying the right hand side by $\delta^n_k = R^n_a R^a_k$ (which follows from the orthogonality of the rotation matrices), renaming $k \to n$ and then summing over n, we get $e'_k = \frac{1}{2}\varepsilon^{ij}_n\, R^l_i\, e_{lm}\, R^m_j\, \delta^n_k = \frac{1}{2}\varepsilon^{ij}_n\, R^l_i\, e_{lm}\, R^m_j R^n_a\, R^a_k$. Noting that for a rotation matrix we have $R^i_l\, R^j_m\, R^k_n\, \varepsilon^{lmn}\, \varepsilon^{ijk}$ whence

$R^l_i\, R^m_j\, R^n_a\, \varepsilon^{ij}_n = R^l_i\, R^m_j\, R^n_a\, \varepsilon^{ija}\, \delta_{an} = \varepsilon^{lmn}\, \delta_{an}$ so that $e'_k = \frac{1}{2}\varepsilon^{\mathrm{lm}}_a\, e_{lm}\, R^a_k = e_a R^a_k$ which is the transformation law for 3-vectors with spin one. The other three basis vectors viz. $e_{[01]} = e_{[02]}, e_{[03]}$ also transform among each other as vectors since $e_{[0i]} = e_0 e_i - e_i e_0$ and e_0, being the time component, is invariant under rotations and, therefore, a scalar. The above analysis implies that we can decompose the space of second rank antisymmetric tensors into a direct sum of two invariant spin one subspaces insofar as rotations is concerned.

Let us, now, consider Lorentz boosts. To study the effect of boosts on these basis vectors, we need to ascertain the eigenvectors and eigenvalues of the operators $\hat{M}_3 = \dfrac{(\hat{J}_3 + i\hat{K}_3)}{2}$ and $\hat{N}_3 = \dfrac{(\hat{J}_3 - i\hat{K}_3)}{2}$. Let us start with boosts along the x^3 direction. We have

(iv) $ie_{[01]} + e_{[02]} + e_{[23]} - ie_{[31]}$ (Eigenvalue –1)

(v) $e_{[01]} + ie_{[02]} - ie_{[23]} + e_{[31]}$ (Eigenvalue +1)

(vi) $e_{[03]} + e_{[12]}$ (Eigenvalue 0)

It can easily be seen by explicit calculations that for each of the above eigenvectors (i) – (vi) (designated $|\phi\rangle$), we have $\hat{N}_3\, \hat{M}_3|\phi\rangle = \hat{M}_3\, \hat{N}_3|\phi\rangle = (\hat{J}_3^2 + \hat{K}_3^2)|\phi\rangle = 0$ thereby showing that the eigenvectors of $\hat{M}_3$ are annihilated by $\hat{N}_3$ and vice versa. This establishes our result that the space of second rank antisymmetric tensors can be written as the direct sum of two subspaces that form the (1, 0) ⊗ (0, 1) representation of the proper Lorentz group.

The generators $\hat{J}_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ of infinitesimal homogeneous Lorentz transformations are antisymmetric tensors of rank two defined in terms of the equation

$$\Lambda(\delta\omega) = \mathbf{I} - \frac{i}{2}\hat{J}_{\mu\nu}\, \delta\omega^{\mu\nu} \tag{179}$$

where $\delta\omega^{\mu\nu}$ are antisymmetric infinitesimal parameters representing infinitesimal angular displacements in the (μ, ν) plane.

The transformation of $\Lambda(\omega)$ under the effect of an arbitrary Lorentz transformation is investigated as

$$\Omega\Lambda(\omega)\Omega^{-1} = \Omega e^{-i\hat{J}_{\mu\nu}\omega^{\mu\nu}}\,\Omega^{-1} = \Omega\left[\sum_i \frac{1}{i!}(-i\hat{J}_{\mu\nu}\,\omega^{\mu\nu})^i\right]\Omega^{-1} = \left[\sum_i \frac{1}{i!}(-i\Omega\hat{J}_{\mu\nu}\omega^{\mu\nu}\,\Omega^{-1})^i\right]$$

$$= \left[\sum_i \frac{1}{i!}(-i\Omega\hat{J}_{\mu\nu}\,\Omega^{-1}\,\Omega\omega^{\mu\nu}\,\Omega^{-1})^i\right] = \left[\sum_i \frac{1}{i!}(-i\hat{J}'_{\mu\nu}\,\omega'^{\mu\nu})^i\right] = e^{-i\hat{J}_{\mu\nu}\omega^{\mu\nu}} \tag{180}$$

where $\omega'^{\mu\nu} = \Omega^{\mu}_{\lambda}\,\omega^{\lambda\sigma}\,\Omega^{\nu}_{\sigma}$ and $\hat{J}'_{\mu\nu} = \Omega\hat{J}_{\mu\nu}\,\Omega^{-1} = \Omega^{\lambda}_{\mu}\,\hat{J}_{\lambda\sigma}\,\Omega^{\sigma}_{\nu}.$

2.8 TRANSFORMATION OF TENSOR TRACE

Consider a second rank tensor **T**. Given a unitary representation of the proper Lorentz group $\hat{L}(\Lambda)$, **T** would transform as $\mathbf{T}' = \hat{L}(\Lambda)\,\mathbf{T}\hat{L}(\Lambda)^{-1}$. Then, $Tr\,\mathbf{T}' = Tr\,[\hat{L}(\Lambda)\,\mathbf{T}\hat{L}(\Lambda)^{-1}] = Tr\,\mathbf{T}$ showing that the tensor trace is invariant under proper Lorentz transformation i.e. it is a Lorentz scalar and transforms as the (0, 0) representation of the Lorentz group.

2.9 TRANSFORMATION OF THE SYMMETRIC SECOND RANK TENSOR

We have so far established the following:

(i) The Lorentz vectors transform as the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation;

(ii) Therefore, second rank tensors in Minkowski space transform under the reducible $\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right)$ representation.

(iii) The $\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right)$ representation can be decomposed as: $\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right)$ $= (0 \oplus 1) \otimes (0 \oplus 1) = (0 \otimes 0) \oplus (1 \otimes 0 \oplus 0 \otimes 1) \oplus (1 \otimes 1)$.

(iv) The trace of the tensor, being a Lorentz scalar, transforms as the (0, 0) representation.

(v) The antisymmetric part transforms as the $(1 \otimes 0) \otimes (0 \otimes 1)$ representation.

It, follows, therefore, that the traceless symmetric part transforms as the $(1 \otimes 1)$ representation of the proper Lorentz group.

2.10 THE POINCARE GROUP & RELATIVISTIC WAVE EQUATION (WIGNER, 1939; BARGMANN, 1948; FOLDY, 1956; SHIROKOV, 1958; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994)

As is well known, quantum mechanical dynamics are described by wave equations whose solutions represent physical states of definite mass and spin. A generic wave equation can be represented in the form

$$\hat{\Pi}(M, i\partial)^{\alpha}_{\beta}\, \hat{\Psi}^{\beta}(x) = 0 \tag{181}$$

where $\hat{\Pi}$ is a linear differential operator and $\hat{\Psi}(x)$ is an operator that depends on the space variables as well. It is emphasized that single particle quantum theory can be developed either in terms of the c number wavefunctions $\psi(x)$ (that are single component or multicomponent single column vectors) or the corresponding space dependent operators $\hat{\Psi}(x)$. For many body systems, e.g., quantum field theory, these c number wavefunctions $\psi(x)$ take the form of operators $\hat{\Psi}(x)$ that operate on the vector space of physical states. For the common wave equations $\hat{\Pi}$ is of the first or second degree in ∂_{μ}. A solution of this operator valued equation can be obtained by taking the Fourier transform to get

$$\hat{\Psi}^{\alpha}(x) = \frac{1}{(2\pi)^3} \int dp\, \hat{\Phi}^{\alpha}(p)\, e^{-ipx} \tag{182}$$

where $\hat{\Phi}^{\alpha}(p)$ satisfies

$$\hat{\Pi}(M, p)^{\alpha}_{\beta}\, \hat{\Phi}^{\beta}(p) = 0 \tag{183}$$

which is easily seen by substituting the latter eq. in the former. Since, e^{-ipx} is a Lorentz scalar it follows that the transformation properties of $\hat{\Psi}(x)$ and $\hat{\Phi}(p)$ under Lorentz transformations would be identical.

We can easily obtain the transformation properties of $\hat{\Psi}(x)$ by exploiting its relationship with the c number wavefunction $\psi(x)$. Given an arbitrary single particle state $|\psi\rangle$, the c number

Since any occurrence of p^0 in $\hat{\Pi}$ can be replaced by $(\mathbf{p}^2 + M^2)^{1/2}$, if we focus only on the positive energy solutions, as is the case hereafter, it follows then that $\hat{\Pi}$ can be considered a function of the 3-vector $\mathbf{p}$ in lieu of the 4-vector p.

(c) Let us assume that the field operator $\hat{\Phi}(p)$ transforms according to the (u, v) representation of the Lorentz group. Then, we have $|u - v| \leq j \leq (u + v)$. Now, if $u \neq 0$, $v \neq 0$ then j will have multiple values implying that the $\hat{\Phi}(p)$ has multiple spin states. However, in order that our relativistic equation $\hat{\Pi}(M, p)^{\alpha}_{\beta}\, \hat{\Phi}^{\beta}(p) = 0$ be physically acceptable and the solutions thereof correspond to particles with unique spin, $\hat{\Pi}(M. p)$ must necessarily act as a projection operator, which acting on the field operator $\hat{\Phi}(p)$ would project out the appropriate spin components corresponding to the spin characteristic of the particle being described.

Let us now assume that we are describing a particle of spin s. Since λ can take $(2s + 1)$ viz., $\lambda = -s, -s + 1, ...0, ...s - 1, s$ values, the matrix equation $\hat{\Pi}(M, \pm p)\hat{\Phi}\pm(p) = 0$ must necessarily have $(2s + 1)$ independent solutions. Considering the particle in the rest frame $p \equiv pt = (p^0 = M, \mathbf{p} = 0)$, we write these independent solutions as $u(\mathbf{0}\lambda)$ so that $\hat{\Pi}(M, p_t)^{\alpha}_{\beta}\, u^{\beta}(\mathbf{0}\lambda) = 0$ for $\lambda = -s,...s$. The solutions corresponding to an arbitrary momentum vector $\mathbf{p}$ can be obtained as

$$u^{\alpha}(\mathbf{p}\lambda) = D[\varphi_W(p, \Lambda)]^{\alpha}_{\beta}\, u^{\beta}(\mathbf{0}\lambda) \tag{189}$$

where $\varphi_W(p, \Lambda)$ is a Wigner rotation that brings the rest frame momentum vector $p_t \equiv (M, \mathbf{0})$ to the desired momentum $p \equiv (p^0, \mathbf{p})$. We also have

$$\begin{aligned} \hat{\Pi}(M, p)\, u(\mathbf{p}\lambda) &= D[\varphi_W(p, \Lambda)]\; \hat{\Pi}(M, p_t)\; D[\varphi_W(p, \Lambda)]^{-1}\; u(\mathbf{p}\lambda) \\ &= D[\varphi_W(p, \Lambda)]\; \hat{\Pi}(M, p_t)\; u(\mathbf{0}\lambda) = 0 \end{aligned} \tag{190}$$

thereby confirming that $u(\mathbf{p}\lambda)$ is a solution of the wave equation. $u(\mathbf{p}\lambda)$ constitute concrete realizations of the linear momentum basis states $\{|\mathbf{p}\lambda\rangle\}$ of the time like irreducible representations of the Poincare group. These realizations are identified by (i) the IRR representation labels (M, s) (ii) the representation labels (u, v) for identifying the representation of the proper Lorentz group (iii) the set of Poincare labels $(\mathbf{p}, \lambda)$ for identifying the basis vectors where the range of λ is determined by the label s, i.e., $\lambda = -s, ...,0,..., +s$ and (iv) the set of Lorentz labels collectively indicated by α whose range is determined by (u, v). To summarize, then, we have the following situation:

(i) If the field operator $\hat{\Phi}(p)$ has multiple spin content characterized by $|u - v| \leq j \leq (u + v)$, then the differential operator $\hat{\Pi}(M, p)$ acting on $\hat{\Phi}(p)$ projects out the desired spin components corresponding to the particle states being described.

(ii) If the spin of the particle being described is designated s (which is projected out by the action of $\hat{\Pi}(M, p)$ on $\hat{\Phi}(p)$), then corresponding to this s, there would exist $(2s + 1)$ independent solutions to the wave equation $\hat{\Pi}(M, \pm p)\; \hat{\Phi}_{\pm}(p) = 0$ identified with $\lambda = -s$, ...0, ...+s.

(iii) The general solution to the wave equation can be written as

$$\hat{\Psi}^a(x) = \sum_\lambda \int \widetilde{d}p[b(\mathbf{p}, \lambda)u^a(\mathbf{p}, \lambda)e^{-ipx}] + \text{negative energy term.} \tag{191}$$

The coefficients $b(\mathbf{p}\lambda)$ carry the operator property on the right hand side of the above equation and constitute the annihilation operators for the particle states. The adjoints $b^\dagger(\mathbf{p}\lambda)$ are the creation operators and can be used to write the basis vectors $\{|\mathbf{p}\lambda\rangle\}$ as $|\mathbf{p}\lambda\rangle = b^\dagger(\mathbf{p}, \lambda)|0\rangle$. Since the basis vectors $\{|\mathbf{p}\lambda\rangle\}$ transform under a proper Lorentz transformation as $\Lambda|\mathbf{p}\lambda\rangle = D^s[\varphi_W(\Lambda, p)]^{\lambda'}_{\lambda}|\Lambda\mathbf{p}, \lambda'\rangle$, we have

$$\hat{U}(\Lambda)|\mathbf{p}\lambda\rangle = \hat{U}(\Lambda)b^\dagger(\mathbf{p}, \lambda)\,\hat{U}(\Lambda)^{-1}\,\hat{U}(\Lambda)|0\rangle$$

$= \hat{U}(\Lambda)b^\dagger(\mathbf{p}, \lambda)\,\hat{U}(\Lambda)^{-1}|0\rangle = D^s[\varphi_W(\Lambda, p)]^{\lambda'}{}_{\lambda}\,b^\dagger(\Lambda\mathbf{p}, \lambda')|0\rangle$ whence we get the transformation law for $b^\dagger(\mathbf{p}\lambda)$ as

$$\hat{U}(\Lambda)b^\dagger(\mathbf{p}, \lambda)\,\hat{U}(\Lambda)^{-1} = D^s[\varphi_W(\Lambda, p)]^{\lambda'}_{\lambda}b^\dagger(\Lambda\mathbf{p}, \lambda') \tag{192}$$

The Lorentz transformation law for the annihilation operators follows by conjugation as

$$\hat{U}(\Lambda)b(\mathbf{p}, \lambda)\,\hat{U}(\Lambda)^{-1} = D^s[\varphi_W(\Lambda, p)^{-1}]^{\lambda}_{\lambda'}\;b(\Lambda p, \lambda') \tag{193}$$

Making use of the orthonormality of the momentum spin basis states viz. $\langle\mathbf{p}'\,\lambda'|\mathbf{p}\lambda\rangle = \delta^{\lambda}_{\lambda'}\,\delta(\mathbf{p}-\mathbf{p}')$, we obtain $u^\alpha(\mathbf{p}\lambda)\,e^{-ipx} = \langle 0|\hat{\Psi}^\alpha(x)|\mathbf{p}\lambda\rangle$. Noting that $\langle 0|b(\mathbf{p}', \lambda') = \langle\mathbf{p}'\,\lambda'|$, $b^\dagger(\mathbf{p}, \lambda)|0\rangle = |\mathbf{p}\lambda\rangle$, we get $\langle 0|b(\mathbf{p}', \lambda')b^\dagger(\mathbf{p}, \lambda)|0\rangle = \langle\mathbf{p}', \lambda'|\mathbf{p}\lambda\rangle = \delta^{\lambda'}_{\lambda}\,\delta(\mathbf{p}-\mathbf{p}')$. Let us, now, obtain the transformation law for the c number wavefunctions $u^\alpha(\mathbf{p}, \lambda)$. For the purpose, we have $\hat{U}(\Lambda)\hat{\Psi}^\alpha(x)\,\hat{U}(\Lambda^{-1}) = D[\Lambda^{-1}]^\alpha_\beta\,\hat{\Psi}^\beta(\Lambda x)$

$$= D[\Lambda^{-1}]^\alpha_\beta \sum_\lambda \int \widetilde{d}q\,[b(\mathbf{q}, \lambda)u^\beta(\mathbf{q}, \lambda)e^{-iq\Lambda x} + \text{negative energy term}]$$

$$= D[\Lambda^{-1}]^\alpha_\beta \sum_\lambda \int \widetilde{d}p\,[b(\Lambda\mathbf{p}, \lambda)u^\beta(\Lambda\mathbf{p}, \lambda)e^{-iqx} + \text{negative energy term}]$$

where in the last step we have changed the integration variable from q to $p = \Lambda^{-1}q$. The infinitesimal integration volume element is invariant under Lorentz transformations. We also have

$$\hat{U}(\Lambda)\sum_{\lambda,}\int \hat{d}p\,[b(\mathbf{p}, \lambda)\,u^\alpha(\mathbf{p}, \lambda)e^{-ipx}] + \text{negative energy term}\}\;\hat{U}(\Lambda^{-1})$$

$$= \sum_{\lambda,\lambda'}\int \hat{d}p\;\{D^s[\varphi_W(\Lambda, p)^{-1}]^{\lambda}_{\lambda'}\,b(\Lambda\mathbf{p}, \lambda')\,u^\alpha(\mathbf{p}, \lambda)e^{-ipx} + \text{negative energy term}\}$$

where we have used $\hat{U}(\Lambda)\,b(\mathbf{p}, \lambda)\,\hat{U}(\Lambda)^{-1} = D^s[\varphi_W(\Lambda, p)^{-1}]^{\lambda}_{\lambda'}\,b(\Lambda\mathbf{p}, \lambda')$. Interchanging $\lambda \leftrightarrow \lambda'$ and then comparing the two equations, we get

$$D[\Lambda^{-1}]^\alpha_\beta\,u^\beta(\Lambda\mathbf{p}, \lambda) = D^{(s)}[\varphi_W(\Lambda, p)^{-1}]^{\lambda'}_{\lambda}\,u^\alpha(\mathbf{p}, \lambda') \tag{194}$$

or equivalently

$$D[\Lambda]^\alpha_\beta\,u^\beta(\mathbf{p}, \lambda) = u^\alpha(\Lambda\mathbf{p}, \lambda')\,D^{(s)}[\varphi_W(\Lambda, p)]^{\lambda'}_{\lambda} \tag{195}$$

The Pauli Lubanski vector $\hat{W}^{\mu} = \frac{1}{2}\,\varepsilon^{\mu}_{\nu\rho\sigma}\,\hat{J}^{\nu\rho}\,\hat{P}^{\sigma}\,(\hat{W}^0 = \hat{\mathbf{J}}.\hat{\mathbf{P}},\ \hat{\mathbf{W}} = \hat{\mathbf{J}}.\hat{P}^0 - \hat{\mathbf{K}} \times \hat{\mathbf{P}})$(satisfies the following commutators:

$$[\hat{W}^{\lambda}, \hat{W}^{\sigma}] = i\varepsilon^{\lambda\sigma\mu\nu}\,\hat{W}_{\mu}\,\hat{P}_{\nu} \tag{200}$$

$$[\hat{W}^{\lambda},\ \hat{J}^{\mu\nu}] = i(\hat{W}^{\nu}\,g^{\mu\lambda} - \hat{W}^{\mu}\,g^{\lambda\nu}] \tag{201}$$

$$[\hat{W}^{\mu}, \hat{P}^{\sigma}] = 0 \tag{202}$$

For the massive case under consideration, the little group is the three-dimensional spatial rotation group $SO(3)$ so that the independent components of $\hat{W}^{\mu}$ coincide with the generators of spatial three-dimensional rotations that satisfy $[\hat{S}^i, \hat{S}^j] = i\varepsilon^{ijk}\,\hat{S}^k$. We follow the method of Shirokov to obtain the Foldy-Shirokov realizations of the Poincare group. Introducing the operator $\hat{g}$ by

$$\hat{g}^{\mu} = \hat{J}^{\mu}_{\nu}\,\hat{P}^{\nu} \tag{203}$$

so that

$$\hat{g}^0 = \hat{J}^0_{\nu}\,\hat{P}^{\nu} = \hat{\mathbf{K}}.\hat{\mathbf{P}},\ \hat{\mathbf{g}} = \hat{\mathbf{K}}\,\hat{P}^0 + \hat{\mathbf{J}} \times \hat{\mathbf{P}} \tag{204}$$

Since we are working in a linear momentum basis in which the momentum operators are diagonal, we have

$$\hat{g}^{\mu}\,\hat{P}_{\mu} = 0 \tag{205}$$

Using eqs. (200)-(202) and the definition (203), we arrive at the following commutators:

$$[\hat{g}^{\mu}, \hat{W}^{\sigma}] = i\hat{W}^{\mu}\,\hat{P}^{\sigma} \tag{206}$$

$$[\hat{g}^{\mu}, \hat{P}^{\sigma}] = i(\hat{P}^{\mu}\,\hat{P}^{\nu} - g^{\mu\nu}\,\hat{P}^{\sigma}\,\hat{P}_{\sigma}) \tag{207}$$

$$[\hat{g}^{\mu},\ \hat{g}^{\nu}] = i(\hat{g}^{\mu}\,\hat{P}^{\nu} - \hat{g}^{\nu}\,\hat{P}^{\mu} - \varepsilon^{\mu\nu}_{\rho\sigma}\,\hat{W}^{\rho}\,\hat{P}^{\sigma}) \tag{208}$$

Using the definition of the Pauli Lubanski vector and eqs. (202), (204), we obtain:

$$M^2\,\hat{\mathbf{J}} = \hat{\mathbf{g}} \times \hat{\mathbf{P}} + \hat{\mathbf{W}}\,\hat{P}^0 - \hat{W}^0\,\hat{\mathbf{P}} \tag{209}$$

$$M^2\,\hat{\mathbf{K}} = -\,\hat{\mathbf{W}} \times \hat{\mathbf{P}} + \hat{\mathbf{g}}\,\hat{P}^0 - \hat{g}^0\,\hat{\mathbf{P}} \tag{210}$$

or equivalently, in a linear momentum basis,

$$\hat{J}^{\mu\nu} = \frac{\hat{g}^{\mu}\,\hat{P}^{\nu} - \hat{g}^{\nu}\,\hat{P}^{\mu} - \varepsilon^{\mu\nu}_{\rho\sigma}\,\hat{W}^{\rho}\,\hat{P}^{\sigma}}{\hat{P}^{\xi}\,\hat{P}_{\xi}} \tag{211}$$

In the linear momentum basis, $\hat{W}(p)$ is the restriction of the Pauli Lubanski vector ($\hat{W}^0 = \hat{\mathbf{J}}.\hat{\mathbf{P}}$, $\hat{\mathbf{W}} = \hat{\mathbf{J}}.\hat{P}^0 - \hat{\mathbf{K}} \times \hat{\mathbf{P}}$) to the eigenspace of $\hat{P}^{\mu}$ corresponding to the eigenvalue p. In the rest frame $p_R \equiv (M, 0)$, we have the corresponding restriction ($\hat{W}^0_R = 0, \hat{\mathbf{W}}_R = M\hat{\mathbf{J}}_R = M\hat{\mathbf{S}}_R$). Using eq. (200), it is seen that the components of $\hat{\mathbf{S}}_R$ satisfy $[\hat{S}^i, \hat{S}^j] = i\varepsilon^{ijk}\,\hat{S}^k$ and hence, constitute a three-dimensional spatial rotation structure. To obtain the restriction of $\hat{W}$ corresponding to an arbitrary p, we impart a pure Lorentz boost L_p to the system that transforms the particle of mass M from rest to a state of momentum $\mathbf{p}$. The matrix representation of L_p is $(L_p)^0_0$, $\frac{p^0}{M}$, $p^0 = (|\mathbf{p}|^2 + M^2)^{1/2}$, $(L_p)^i_0 = \frac{p^i}{M}$, $(L_p)^0_i = -\frac{p_i}{M}$, $(L_p)^i_j = \delta^i_j - \frac{p^i p_j}{M(p^0 + M)}$. We know from eqs. (201) & (202) that the components of $\hat{W}$ transform as a vector under Lorentz transformation so that $\hat{W}^0 = \hat{\mathbf{S}}.\mathbf{p}$, $\hat{\mathbf{W}} = \mathbf{M}\hat{\mathbf{S}} + \frac{\hat{\mathbf{S}}.\mathbf{p}}{p^0 + M}\mathbf{p}$ where we have replaced the momentum operators with the corresponding eigenvalues which is permitted since we are working in a basis in which the momentum operators are diagonal. Our next step is to obtain explicit expressions for the components of $\hat{g}$.

From eq. (207), we obtain $[\hat{g}^0, \hat{P}^i] = i\hat{P}^0\hat{P}^i$ whence we can write $\hat{g}^0 = i\hat{P}^0\hat{O}$ where the operator $\hat{O}$ satisfies $[\hat{O}, \hat{P}^i] = \hat{P}^i$ which has the solution

$$\hat{O} = \frac{\partial}{\partial \mathbf{p}}.\mathbf{p} = \mathbf{p}.\frac{\partial}{\partial \mathbf{p}} + 3 \tag{212}$$

whence we get

$$\hat{g}^0 = ip^0\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + 3ip^0 \tag{213}$$

where we have replaced the momentum operators by their respective eigenvalues. Since $\hat{g}^\mu p_\mu = 0$, we have

$$\hat{\mathbf{g}}.\mathbf{p} = \hat{g}^0 p^0 = ip^0\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right)p^0 + 3i(p^0)^2 = ip^0\left[p^0\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + \frac{|\mathbf{p}|^2}{p^0}\right]3i(p^0)^2$$

$$= i|\mathbf{p}|^2\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + iM^2\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + i|\mathbf{p}|^2 + 3i(p^0)^2$$

$$= i\mathbf{p}.\left[\mathbf{p}\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + \mathbf{p}\right] + iM^2\left[\left(\frac{\partial}{\partial \mathbf{p}}.\mathbf{p}\right) - 3\right] + 3i(p^0)^2$$

$$= i\mathbf{p}.\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right)\mathbf{p} + iM^2\left(\frac{\partial}{\partial \mathbf{p}}.\mathbf{p}\right) + 3i\mathbf{p}.\mathbf{p} = i\left[\mathbf{p}\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + M^2\frac{\partial}{\partial \mathbf{p}} + 3\mathbf{p}\right].\mathbf{p}$$

whence

$$\hat{\mathbf{g}} = i\left[\mathbf{p}\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + M^2\frac{\partial}{\partial \mathbf{p}} + 3\mathbf{p}\right] + (\hat{O} \times \mathbf{p}) \tag{214}$$

Using eqs. (206) together with $\hat{W}^0 = \hat{\mathbf{S}}.\mathbf{p}$ we obtain $[g, \hat{W}^0] = i\mathbf{W}p^0$ or

$$\left[\mathbf{p}\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + M^2 \frac{\partial}{\partial \mathbf{p}} + 3\mathbf{p} - i(\hat{O} \times p), \hat{\mathbf{S}}.\mathbf{p}\right] = p^0 \left(M\hat{\mathbf{S}} + \frac{\hat{\mathbf{S}}.\mathbf{p}}{p^0 + M}\mathbf{p}\right) \text{ or}$$

$$\mathbf{p}(\hat{\mathbf{S}}.\mathbf{p}) + M^2 \hat{\mathbf{S}} - i\,[\hat{O} \times \mathbf{p},\, \hat{\mathbf{S}}.\mathbf{p}] = p^0 \left(M\hat{\mathbf{S}} + \frac{\hat{\mathbf{S}}.\mathbf{p}}{p^0 + M}\mathbf{p}\right) \text{ or}$$

$$[\hat{O} \times \mathbf{p},\, \hat{\mathbf{S}}.\mathbf{p}] = \frac{iM}{p^0 + M}[|\mathbf{p}|^2\, \hat{\mathbf{S}} - (\hat{\mathbf{S}}.\mathbf{p})\mathbf{p}] \text{ whence } \hat{O} = \frac{M}{p^0 + M}\, \hat{\mathbf{S}} \text{ giving}$$

$$\hat{\mathbf{g}} = i\mathbf{p}\left(\mathbf{p}.\frac{\partial}{\partial \mathbf{p}}\right) + \mathrm{i}\mathrm{M}^2 \frac{\partial}{\partial \mathbf{p}} + 3i\mathbf{p} + \frac{M}{p^0 + M}(\hat{\mathbf{S}} \times \mathbf{p}) \tag{215}$$

Using eqs. (209), (210), (213), (214) and $\hat{\mathbf{W}} = M\hat{S} + \dfrac{\hat{\mathbf{S}}.\mathbf{p}}{P^0 + M}\mathbf{p}$, $\hat{W}^0 = \hat{\mathbf{S}}.\mathbf{p}$, we, finally, obtain

$$\hat{\mathbf{J}} = -\, i\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}} + \hat{\mathbf{S}} \tag{216}$$

$$\hat{\mathbf{K}} = ip^0 \frac{\partial}{\partial \mathbf{p}} - \frac{\hat{\mathbf{S}} \times \mathbf{p}}{P^0 + M} \tag{217}$$

which constitute the Shirokov form of the realizations of the Poincare group generators.

2.12 PARTICLE SPIN IN RELATIVISTIC QUANTUM MECHANICS

A particle is said to possess "spin s" if the squared angular momentum operator $\mathbf{J}^2$ acting on the states of the particle in its rest frame returns eigenvalues of the form $s(s + 1)$. We note from the above that $J^2|\mathbf{0}\lambda\rangle = s(s + 1)|\mathbf{0}\lambda\rangle$ implying that the index s can be identified with the spin of the particle. We have already shown above that M corresponds to particle's mass and $\mathbf{p}$ to its 3-momentum. To obtain an explicit physical interpretation of λ, we consider an arbitrary state and consider its motion in a chosen frame of reference such that the x^3 axis coincides with the direction of motion of the particle. We designate the state with reference to the chosen frame as $|p\hat{\mathbf{z}}, \lambda\rangle$ so that $P_1|p\hat{\mathbf{z}}, \lambda\rangle = P_2|p\hat{\mathbf{z}}, \lambda\rangle = 0$, $\mathbf{P}|p\hat{\mathbf{z}}, \lambda\rangle = P_3|p\hat{\mathbf{z}}, \lambda\rangle = p|p\hat{\mathbf{z}}, \lambda\rangle$. Since the operators $\mathbf{P}$ and $\mathbf{J}.\mathbf{P}$ commute with each other, they possess simultaneous eigenvectors—let us assume that $|p\hat{\mathbf{z}}, \lambda\rangle$ is a simultaneous eigenvector of $\mathbf{P}$ and $\mathbf{J}.\mathbf{P}$. Then $\mathbf{J}.\mathbf{P}|p\hat{\mathbf{z}}, \lambda\rangle\, J_3\, P_3|p\hat{\mathbf{z}}, \lambda\rangle, \lambda p|p\hat{\mathbf{z}}, \lambda\rangle$ whence

$$\frac{\mathbf{J}.\mathbf{P}}{p}|p\hat{\mathbf{z}}, \lambda\rangle = J_3|p\hat{\mathbf{z}}, \lambda\rangle = \lambda|p\hat{\mathbf{z}}, \lambda\rangle \tag{218}$$

As there can be no orbital angular momentum along the direction of motion, we can interpret the index λ as the eigenvalue of the total angular momentum $\mathbf{J}$ along the direction of linear motion of the particle.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ 0 & \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ 0 & -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ \cos\alpha\sin\beta\sinh\xi & \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta\cosh\xi \\ \sin\alpha\sin\beta\sinh\xi & \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta\cosh\xi \\ \cos\beta\sinh\xi & -\sin\beta & 0 & \cos\beta\cosh\xi \end{pmatrix} \tag{222}$$

so that

$$p^0 = H(p)^0_0\, p^0_t = M\cosh\xi,\; p^1 = H(p)^1_0\, p^0 = M\cos\alpha\sin\beta\sinh\xi$$
$$p^2 = H(p)^2_0\, p^0 = M\sin\alpha\sin\beta\sinh\xi \text{ and } p^3 = H(P)^3_0\, p^0 = M\cos\beta\sinh\xi \tag{223}$$

Obviously, $\mathbf{p}^2 = p_1^2 + p_2^2 + p_3^2 = M^2 \sinh^2\xi$ so that $|\mathbf{p}| = M\sinh\xi$.

(iv) As an immediate corollary to (1), we have

$$T(b)|\mathbf{p}\lambda\rangle = e^{-ib^{\mu}p_{\mu}}|\mathbf{p}\lambda\rangle \tag{224}$$

(v) We have

$$\Lambda|\mathbf{p}\lambda\rangle = \Lambda H(p)|\mathbf{0}\lambda\rangle = H(p')\,[H(p')^{-1}\,\Lambda H(p)]|\mathbf{0}\lambda\rangle$$
$$= D^s[R(\Lambda, p)]^{\lambda'}_{\lambda}|p'\,\lambda'\rangle \tag{225}$$

where $p'^{\mu} = \Lambda^{\mu}_{\ \nu}\, p^{\nu}$ and $R(\Lambda, p) \equiv H(p')^{-1}\,\Lambda H(p)$, and $D^s\,[R]$ is the representation matrix of the group of three-dimensional rotations, $SO(3)$ corresponding to the angular momentum s.

(vi) To confirm that $R(\Lambda, p) \equiv H(p')^{-1}\,\Lambda H(p)$ is a rotation in three dimensions, we examine the sequence-wise effect of each of the three operations on $|\mathbf{0}\lambda\rangle$. We have $H\,(p)|\mathbf{0}\lambda\rangle = |\mathbf{p}\,\lambda\rangle$, $\Lambda|\mathbf{p}\lambda\rangle = D^s\,[R]|\mathbf{p}'\,\lambda'\,\rangle$ and $H(p')^{-1}D^s[R]|\mathbf{p}'\,\lambda'\rangle = D^s[R]|\mathbf{0}\lambda\rangle$, and whence the effect of the combined Lorentz transformation embodied in $R(\mathrm{L}, p) \equiv H(p')^{-1}\,\Lambda H(p)$ is to leave the standard vector invariant. This implies that $R(\Lambda, p)$ is a member of the little group and hence, a rotation in the three-dimensional space.

(vii) In view of (ii) and (iii), we conclude that the vector space spanned by the vectors $\{\mathbf{p}\lambda\}$ is invariant under the transformations of the Poincare group.

(viii) The irreducibility of the representation follows from the fact that all the basis vectors are generated from the action of group transformation generators $\hat{J}$ and group transformations $H(p)$ on a single "standard vector".

(ix) The unitarity of the representation follows from: (i) generators of the group transformations being realized as Hermitian operators and (ii) the representation matrices of the translation operations $e^{-ib^{\mu}p_{\mu}}$ and the $D^s[R]$ matrices representing rotations are both unitary.

2.14 NULL VECTOR BASED REPRESENTATIONS OF POINCARE GROUP (WIGNER, 1939; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994)

Consider the null vector designated as $p_n^\mu = 0$. This vector is invariant under all homogeneous Lorentz transformations for $\Lambda^\mu_\nu p_n^\nu = 0$ for all homogeneous Lorentz transformations Λ. Hence, the "little group" of p_n is the homogeneous Lorentz group $\widetilde{L}_+$. Each irreducible unitary representation of the homogeneous Lorentz group, therefore, induces an irreducible unitary representation of the Poincare group. These IRRs are labelled by the eigenvalue $c_1 = 0$ of the Casimir operator C_1 alongwith the labelling indices (j_0, ν) of the unitary representations of the homogeneous Lorentz group, viz. $\widetilde{L}_+$.

The basis vectors in a particular representation are taken as the simultaneous eigenvectors of the three commuting operators $(P^\mu, \mathbf{J}^2, J_3)$ with the corresponding eigenvalues $(p_n^\mu = 0, j, m)$, respectively. The basis vectors satisfy the equations

$$T(b)|0\, jm\rangle = |0\, jm\rangle \tag{226}$$

$$\Lambda|0\, jm\rangle = D^{j_0\nu}\,[\Lambda]^{j'm'}_{jm}|\mathbf{0}\, j'm'\rangle \tag{227}$$

where $D^{j_0\nu}[\Lambda]$ are identified as the unitary representation matrices of the homogeneous Lorentz group.

In addition to the usual Casimir operator $\hat{C}_1 = \hat{P}^\mu \hat{P}_\mu$ mentioned above, this class of representations also admits the Casimir operators $\hat{C}_{n_1} = \hat{J}_{\mu\nu}\hat{J}^{\mu\nu}$ and $\hat{C}_{n_2} = \frac{1}{8}\,\varepsilon_{\mu\nu\rho\sigma}\,\hat{J}^{\nu\rho}\,\hat{J}^{\mu\sigma}$, being the Casimir operators of the homogeneous Lorentz group (which is the little group for this class of representations). Besides, we also have the universal Casimir operator $\hat{C}_3 = \exp(2i\pi\hat{J}_{12}) = \exp(2i\pi\hat{J}_{23})\exp(2i\pi\hat{J}_{31})$ that has eigenvalues of ± 1 corresponding to the representation being single valued or double valued respectively.

2.15 LIGHT LIKE REPRESENTATIONS OF POINCARE GROUP (WIGNER, 1939; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994)

The light like case is characterized by $c_1 = 0$, $\mathbf{p} \neq 0$, the former condition ($c_1 = 0$) implies that $p^\mu p_\mu = 0$, whence $|p^0| = |\mathbf{p}|$. Hence, in this case, a "standard vector" can be picked to be of the form $p_l^\mu = (\omega_0, 0, 0, \omega_0)$. A general state of momentum $p^\mu = (\omega, \mathbf{p}) \equiv (\omega, \omega, \hat{\mathbf{p}})$, where $\omega = p^0 = |\mathbf{p}| = \dfrac{\mathbf{p}}{\hat{\mathbf{p}}}$, can be generated from the standard vector by

(i) Applying a Lorentz boost $L_3(\xi)$ to the standard vector to transform the energy from ω_0 to ω.

(ii) Thereafter applying a rotation $R(\phi, \theta, 0)$ (where the unit vector $\hat{\mathbf{p}}$ is characterized by the angles (θ, ϕ) to bring the x^3 axis in the direction of unit vector $\hat{\mathbf{p}}$.

We denote this composite transformation from p_l^μ to p^μ by $H(p)$ so that, we have $p^\mu = H(p)^\mu_\nu\, p_l^\nu = [R(\phi, \theta, 0)L_3(\xi)]^\mu_\nu\, p_l^\nu$.

(i) By applying a Lorentz boost $L_3(\xi)$ to the standard vector we transform the energy from ω_0 to ω.

(ii) Thereafter by applying a rotation $R(\phi, \theta, 0)$ (where the unit vector $\hat{\mathbf{p}}$ is characterized by the angles (θ, ϕ)), we bring the x^3 axis in the direction of unit vector $\hat{\mathbf{p}}$.

The above also follows from the fact that in order to generate a complete basis of this irreducible representation consisting of general eigenvectors of $\hat{P}^\mu$, we shall operate on the basis vector $|\mathbf{p}_l\, \lambda\rangle$ by the remaining transformations of the factor group. We note that:

(i) The factor group of the Poincare group with respect to the full translation group is the homogeneous Lorentz group.

(ii) The homogeneous Lorentz group comprises rotations in the three spatial coordinates and Lorentz "boosts" that involve the time axis.

(iii) The little group of the standard vector i.e. the group comprising Lorentz transformations that leave the subspace of the standard vector invariant is the two-dimensional Euclidean group with the generators $\hat{W}^1$, $\hat{W}^2$, $\hat{J}_3$. In other words, we can say that the transformations induced by these generators shall simply move one basis vector into another and will not result in the generation of new basis vectors. As a corollary, it follows to generate new basis vectors, we need to operate on the standard vector by the residual transformations of the factor group. Now, in terms of the components of the Pauli Lubanski vector for the standard vector, $(\omega_0, 0, 0, \omega_0)$, we can write the generators of the homogeneous Lorentz group as $\hat{W}^1 = \omega_0\ (\hat{J}_1 + \hat{K}_2)$, $\hat{W}^2 = \omega_0(\hat{J}_2 - \hat{K}_1)$, $\hat{J}_3$, $\hat{K}_3$, $\hat{Z}^1$. $= \omega_0\ (\hat{J}_1 - \hat{K}_2)$, $\hat{Z}^2 = \omega_0(\hat{J}_2 + \hat{K}_1)$.

Out of these, the first three generators relate to the little group of the standard vector, $(\omega_0, 0, 0, \omega_0)$, as mentioned earlier. Hence, operation with these generators will not generate any new basis vectors. The other three generators when operating on the standard vector will generate new basis vectors.

Hence, we need consider here only the impact of a Lorentz boost $L_3(\xi)$ followed by a rotation $R(\phi, \theta, 0)$. Let us symbolize

$$|\mathbf{p}\lambda\rangle = H(p)|\mathbf{p}_l\, \lambda\rangle = R(\phi, \theta, 0)|p\hat{\mathbf{z}}\lambda\rangle = R(\phi, \theta, 0)\ L_3(\xi)|\mathbf{p}_l\, \lambda\rangle \tag{234}$$

$p = |\mathbf{p}| = \omega_0\, e^\xi$ being the magnitude of the 3 momentum and θ, ϕ, the polar and azimuthal angles of the momentum vector $\mathbf{p}$. The Lorentz transformation $R(\theta, \phi, 0)\ L_3(\xi) \equiv H(p)$ transforms our "standard vector" corresponding to $(\omega_0, 0, 0, \omega_0)$, to a general vector p^μ, thereby completing the basis.

These basis vectors $\{|\mathbf{p}\lambda\rangle\}$ transform as follows under the group transformation operations of the Poincare group:

$$T(b)|\mathbf{p}\lambda\rangle = e^{-ib^\mu p_\mu}|\mathbf{p}\lambda\rangle \tag{235}$$

$$\Lambda|\mathbf{p}\lambda\rangle = e^{-i\lambda\theta(\Lambda,\, p)}|\Lambda\mathbf{p}\lambda\rangle \tag{236}$$

where $\theta(\Lambda, p)$ is determined from $e^{-i\lambda\theta(\Lambda,\, p)} = \langle \mathbf{p}_l\, \lambda|\, H^{-1}(\Lambda\mathbf{p})\ \Lambda H(p)\, |\mathbf{p}_l\, \lambda\rangle$. The first equation follows exactly as in the case of timelike representations. For the second equation, we have

$$\Lambda|\mathbf{p}\lambda\rangle = \Lambda H(p)|\mathbf{p}_l\, \lambda\rangle = H(\Lambda p)\ [H(\Lambda p)^{-1}\ \Lambda H(p)]|\mathbf{p}_l\, \lambda\rangle$$

$$= H(\Lambda p)|\mathbf{p}_l\, \lambda\rangle\, \langle \mathbf{p}_l\, \lambda|[H(\Lambda p)^{-1}\, \Lambda H(p)]|\mathbf{p}_l\, \lambda\rangle$$
$$= |\Lambda \mathbf{p}\lambda\rangle\, \langle \mathbf{p}_l\, \lambda|[H(\Lambda p)^{-1}\, \Lambda H(p)]|\mathbf{p}_l\, \lambda\rangle$$
$$= e^{-i\lambda\theta(\Lambda,\, p)}|\Lambda \mathbf{p}\lambda\rangle$$

Following are some important features that characterize $M = 0$ or light like representations:

(i) For integral $\lambda \in \mathbb{C}$, the representations are single valued whereas for λ odd half integer, the representations are double valued. This is due to the fact that on taking the limit $M \to 0$ of $c_1^2 = M^2 > 0$, the little group of the $c_1^2 = M^2 > 0$ representation, viz., $SO(3)$ picks the single and double valued representations of the restricted little group $SO(2)$ (that is the relevant little group for the $M = 0$ case) from its multivalued representations.

(ii) The basis vectors in lightlike case $\{|\mathbf{p}\lambda\rangle\}$ are labelled by the momentum $\mathbf{p}$ and the helicity λ. This helicity state can take $(2s + 1)$ values. For timelike cases this index can transform among all the possible $(2s + 1)$ values under Lorentz transformation as is seen from the transformation equation for the basis states for massive particles viz. $\Lambda|\mathbf{p}\lambda\rangle = |\mathbf{p}'\, \lambda'\rangle\, D^s\, [R(\Lambda, p)]^{\lambda}_{\lambda'}$ which mixes the helicity states. However, this helicity index is invariant under Lorentz transformations for massless states as is seen from the transformation equation $\Lambda|\mathbf{p}\lambda\rangle = e^{-i\lambda\theta(\Lambda,\, p)}|\Lambda\mathbf{p}\lambda\rangle$.

2.16 SPACE LIKE REPRESENTATIONS OF POINCARE GROUP (WIGNER, 1939; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994)

The space like representations are characterized by $c_1 = p_\mu\, p^\mu = M^2 < 0$. We write $Q^2 = -M^2 = -\, c_1 > 0$ and represent the "standard vector" by $p_s^\mu = (0, 0, 0, Q)$. The components of Pauli Lubanski vector $\hat{W}^\lambda = \dfrac{1}{2}\, \varepsilon_{\lambda\mu\nu\sigma}\, \hat{J}^{\mu\nu}\, p^\sigma$ in the case of subspace corresponding to the standard vector $p_s^\mu = (0, 0, 0, Q)$ become $\hat{W}^0 = Q\hat{J}_3$, $\hat{W}^1 = -\, Q\hat{K}_2$, $\hat{W}^3 = Q\hat{K}_1$, $\hat{W}^2 = 0$. Using these values, the second Casimir operator is calculated as $\hat{C}_2 = Q^2\, (\hat{J}_3^2 - \hat{K}_1^2 - \hat{K}_2^2)$ and the general Lie algebra for the Pauli Lubanski vector becomes, in the spacelike case, $[\hat{K}_2, \hat{J}_3] = i\hat{K}_1$, $[\hat{K}_1, \hat{J}_3] = -\, i\hat{K}_2$, and $[\hat{K}_1, \hat{K}_2] = -\, i\hat{J}_3$.

This Lie algebra is similar to that of $SO(3)$ with $\hat{K}_1$ and $\hat{K}_2$ replacing $\hat{J}_1$ and $\hat{J}_2$, with a negative sign in the last commutator. The presence of this negative sign implies that the little group is not $SO(3)$ but $SO(2, 1)$. Now, $SO(2, 1)$ is a non-compact group and so all its unitary irreducible representations are infinite dimensional.

We obtain the unitary IRRs of the $SO(2, 1)$ corresponding to the Lie algebra $[\hat{K}_2, \hat{J}_3] = i\hat{K}_1$, $[\hat{K}_1, \hat{J}_3] = -\, i\hat{K}_2$ and $[\hat{K}_1, \hat{K}_2] = i\hat{J}_3$. For this purpose, we define the raising and lowering operators in the usual manner as $\hat{K}_\pm = \hat{K}_1 \pm i\hat{K}_2$ in terms of which the Lie algebra takes the form $[\hat{J}_3, \hat{K}_\pm] = \pm\, \hat{K}_\pm$ and $[\hat{K}_+, \hat{K}_-] = -\, 2\hat{J}_3$. We write the second Casimir operator as

$$\hat{C}_2 = \frac{1}{p_s^2}\, \hat{W}^2 = \frac{1}{-Q^2}\, Q^2\, (\hat{J}_2^3 - \hat{K}_1^2 - \hat{K}_2^2) = (\hat{K}_1^2 + \hat{K}_2^2 - \hat{J}_3^2) = (\hat{K}_+\, \hat{K}_- + \hat{J}_3 - \hat{J}_3^2) \quad (237)$$

instead of $\hat{C}_2 = Q^2\, (\hat{J}_3^2 - \hat{K}_1^2 - \hat{K}_2^2)$ because $Q^2 = -\, c_1$ is the eigenvalue of the first Casimir operator

$$\to p_s^{\prime\prime\mu}\,(Q \sinh \xi \cosh \zeta,\ Q \sinh \xi \sinh \zeta,\ 0,\ Q \cosh \xi)$$

as seen in

$$p_s^{\prime\prime\mu} = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Q\sinh\xi \\ 0 \\ 0 \\ Q\cosh\xi \end{pmatrix} = \begin{pmatrix} Q\sinh\xi\cosh\zeta \\ Q\sinh\xi\sinh\zeta \\ 0 \\ Q\cosh\xi \end{pmatrix}$$

$R_3(\phi)p_s^{\prime\prime\mu}\,(Q \sinh \xi \cosh \zeta, Q \sinh \xi \sinh \zeta, 0, Q \cosh \xi) \to p^\mu(Q \sinh \xi \cosh \zeta, Q \sinh \xi \sinh \zeta \cos \phi, Q \sinh \xi \sinh \zeta \sin \phi, Q \cosh \xi)$ as follows:

$$p^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Q\sinh\xi\cosh\zeta \\ Q\sinh\xi\sinh\zeta \\ 0 \\ Q\cosh\xi \end{pmatrix} = \begin{pmatrix} Q\sinh\xi\cosh\zeta \\ Q\sinh\xi\sinh\zeta\cos\phi \\ Q\sinh\xi\sinh\zeta\sin\phi \\ Q\cosh\xi \end{pmatrix}$$

The parameters (ξ, ζ, ϕ) of the aforesaid sequence of transformations will be determined with the identification $p^\mu(Q \sinh \xi \cosh \zeta, Q \sinh \xi \sinh \zeta \cos \phi, Q \sinh \xi \sinh \zeta \sin \phi, Q \cosh \xi) \equiv (p^0, p^1, p^2, p^3)$

The general vector may be expressed in the usual notation as $|\mathbf{p}\lambda\rangle$ so that

$$|\mathbf{p}\lambda\rangle = H(p)|p_s\,\lambda\rangle \tag{239}$$

where $H(p) = R_3(\phi)\, L_1(\zeta)\, L_3(\xi)$.

The effect of the group operations on the general vector may be expressed, as in the preceding cases, by

$$T(b)|\mathbf{p}\lambda\rangle = e^{-ib_\mu p_\mu}|\mathbf{p}\lambda\rangle,\ \Lambda|\mathbf{p}\lambda\rangle = |\Lambda\mathbf{p}\lambda'\rangle = D^{c_2}\,[H^{-1}(\Lambda p)\,\Lambda H(p)]^{\lambda}_{\lambda'} \tag{240}$$

where D^{c2} symbolizes the representation matrix for the little group $SO(2,1)$ corresponding to the eigenvalue c_2 of the Casimir operator $\hat{C}_2$.

2.17 UNITARITY & NORMALIZATION OF REPRESENTATIONS OF POINCARE GROUP(WIGNER, 1939; OHNUKI, 1976; TUNG, 1984; FUSHCHICH, 1994)

In constructing unitary representations by the induced representation method, all generators are mandated to be Hermitian operators and the eigenvectors of a maximal set of commuting operators are designed to constitute the set of basis vectors. Since, the eigenvectors of Hermitian operators are invariably orthogonal, orthogonality of our basis is achieved by the very process of their construction.

We now consider the issue of their normalization. In constructing representations by the induced representation method, all the basis vectors are obtained from a "standard vector" say $\tilde{p} \equiv p_n, p_t, p_l, p_s$ by a unitary transformation, i.e., $|\mathbf{p}\lambda\rangle = H(p)|\tilde{p}\lambda\rangle$. Now, if we desire that the scalar product $\langle \mathbf{p}'\,\lambda'|\mathbf{p}\lambda\rangle$ is to be Lorentz invariant, then, we must have $\langle p'\,\lambda'|\mathbf{p}\lambda\rangle = \langle \mathbf{p}'\,\lambda'|\Lambda^\dagger\,\Lambda|\mathbf{p}\lambda\rangle$ which, in the case of timelike representations, becomes

$$\langle \mathbf{p}' \lambda' | \Lambda^\dagger \Lambda | \mathbf{p}\lambda \rangle = D^{s\dagger} [R(\Lambda, p')]^{\lambda'}_{\sigma'} \langle \Lambda\mathbf{p}' \sigma' | \Lambda\mathbf{p}\sigma \rangle D^s [R(\Lambda, p)]^{\sigma}_{\lambda} \tag{241}$$

with similar expressions for the other cases. Now, as mentioned above, the basis vectors that are eigenvectors of Hermitian operators with different eigenvalues, must necessarily be orthogonal so that $\langle \mathbf{p}' \lambda' | \mathbf{p}\lambda \rangle = N(p)\delta^3 (\mathbf{p} - \mathbf{p}')\delta^{\lambda'}_{\lambda}$, where $N(p)$ is a normalization constant.

Combining these two equations and using the unitarity of the D^s matrices, we obtain

$$\begin{aligned} N(p)\delta^3 (\mathbf{p} - \mathbf{p}')\delta^{\lambda'}_{\lambda} = \langle \mathbf{p}' \lambda' | \mathbf{p}\lambda \rangle &= \langle \mathbf{p}' \lambda' | \Lambda^\dagger \Lambda | \mathbf{p}\lambda \rangle \\ &= D^{s\dagger} [R(\Lambda, p')]^{\lambda'}_{\sigma'} \langle \Lambda\mathbf{p}' \sigma' | \Lambda\mathbf{p}\sigma \rangle D^s [R(\Lambda, p)]^{\sigma}_{\lambda} \\ &= N(p)\delta^3 (\Lambda\mathbf{p} - \Lambda\mathbf{p}')\delta^{\lambda'}_{\lambda} \end{aligned} \tag{242}$$

i.e. $N(p)\delta^3 (\mathbf{p} - \mathbf{p}') = N(\Lambda p)\delta^3 (\Lambda\mathbf{p} - \Lambda\mathbf{p}')$ as the normalization condition.

Since, in determining the normalization constant, we need to sum over continuous index $\{p^i\}$, we must define a suitable integration measure. For this purpose, we expand an arbitrary state vector $|\psi\rangle$ in the basis $\{|\mathbf{p}\lambda\rangle\}$. We have, $|\psi\rangle = \sum_\lambda \int |\mathbf{p}\lambda\rangle\, \psi^\lambda(p)\widetilde{d}p$. Since our basis is orthogonal, we also have $\psi^{\lambda'}(p') = (\mathbf{p}' \lambda' | \psi\rangle$. We, therefore, have

$$\psi^{\lambda'}(p') = \sum_\lambda \int \langle \mathbf{p}' \lambda' | \mathbf{p}\lambda \rangle \psi^\lambda(p)\widetilde{d}p$$

$$= \int N(p)\delta^3 (\mathbf{p} - \mathbf{p}')\delta^{\lambda'}_{\lambda}\psi^\lambda (p)\widetilde{d}p = \int \psi^{\lambda'} (p)\, N(p)\delta^3 (\mathbf{p} - \mathbf{p}')\, \widetilde{d}p \tag{243}$$

Since $|\psi\rangle$ is an arbitrary state, we must have $\widetilde{d}p = \frac{1}{N(p)}\, d^3 p = \frac{1}{N(p)}\, dp^1\, dp^2\, dp^3$. Now, if $N(p)$ is covariant with respect to any homogeneous Lorentz transformation, i.e., $N(p)$ satisfies $N(p)\delta^3(\mathbf{p} - \mathbf{p}') = N(\Lambda p)\delta^3 (\Lambda\mathbf{p} - \Lambda\mathbf{p}')$, then $\widetilde{d}p = \widetilde{d}\Lambda p$ so that the measure $\widetilde{d}p$ is Lorentz invariant. To determine $N(p)$ explicitly, we note that $d^4 (\Lambda p) = (\det \Lambda)d^4 p = d^4 p$ since the Lorentz transformation has unit determinant and so the 4-volume element is invariant under a Lorentz transformation. Since, the first Casimir operator $\hat{C}_1 = \hat{P}_\mu \hat{P}^\mu$, we have, in terms of eigenvalues $c_1 = p^2 = (p^0)^2 - \mathbf{p}^2$, whence $p^0 = \pm (\mathbf{p}^2 + c_1)^{1/2}$. The positive root is relevant to the timelike and lightlike cases which have physical applications whence an appropriate expression for $\widetilde{d}p$ is $\widetilde{d}p = \frac{1}{N}\theta (p^0)\, \delta(c_1 - p^2)d^4 p = \frac{1}{N}\, \frac{d^3 p}{2p^0} = \frac{1}{N}\, \frac{d^3 p}{2(c_1 + \mathbf{p}^2)^{1/2}}$. The presence of $\theta(p^0)$ mandates that we consider only positive values of $p^0 = (\mathbf{p}^2 + c_1)^{1/2}$ whereas the δ term ensures that $c_1 = p^2 = (p^0)^2 - \mathbf{p}^2$. Further, $p^0 = + (\mathbf{p}^2 + c_1)^{1/2}$ implies $d^4 p \equiv dp^0\, d^3p = \frac{1}{2p^0}\, d^3p$. Conventionally, N is chosen as $N \equiv (2\pi)^3$ so that $\widetilde{d}p = \frac{1}{(2\pi)^3}\frac{d^3 p}{2p^0}$ and the normalization condition reads

$$\langle \mathbf{p}' \lambda' | \mathbf{p}\lambda \rangle = (2\pi)^3\, 2p^0\, \delta^3(\mathbf{p} - \mathbf{p}')\delta^{\lambda'}_{\lambda} \equiv \widetilde{\delta}(p - p')\delta^{\lambda'}_{\lambda} \tag{244}$$

2.18 SL(2,C) (SPINOR) REPRESENTATION OF THE LORENTZ GROUP & EXPLICIT CALCULATION OF WIGNER ANGLE (HALPERN, 1968; OHNUKI, 1976)

The three Pauli matrices $\{\sigma_i\ i = 1, 2, 3\}$ constitute the basic instruments for obtaining the spinor representation. To this set, we append the matrix $\sigma_0 = \mathbf{I}_{2\times 2}$. Together, this set of four matrices constitutes a basis for the four-dimensional space of 2×2 matrices so that any such matrix can be expressed as a linear combination of these four matrices i.e. given a 2×2 matrix **M**, we can write it as $M = \sum_{\mu=0}^{3} c^{\mu}\sigma_{\mu}$ where $c^{\mu} = \frac{1}{2} Tr\,(\sigma_{\mu}{}^{\dagger}\,\mathbf{M}) = \frac{1}{2} Tr\,(\sigma_{\mu}\,\mathbf{M})$. It can be explicitly verified that

$$\mathbf{M}\sigma_2\,\mathbf{M}^T\sigma_2 = \det(\mathbf{M}) \tag{245}$$

Further, we can write a 4-vector with components x^{μ}, $\mu = 0, 1, 2, 3$ in the above basis as a 2×2 matrix as $x \equiv \mathbf{X} = x^{\mu}\,\sigma_{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$. The matrix **X** is Hermitian if x is real. Also $\det(\mathbf{X}) = (x^0)^2 - \mathbf{x}^2 = x^2$. Let us assume that **X** is transformed by a matrix of the $SL(2, \mathbb{C})$ group as $\mathbf{X}' = \mathbf{AXA}^{\dagger}$ where $\det(\mathbf{A}) = 1$. Then, $\det(\mathbf{X}') = \det(\mathbf{X})$ and the components of x are given by $x^{\mu} = \frac{1}{2} Tr(\sigma_{\mu}\ \mathbf{X}')$. Writing $x' = x'^{\mu}\,\sigma_{\mu}$, we obtain $x'^{\mu} = \frac{1}{2} Tr(\sigma_{\mu}\mathbf{X}') = \frac{1}{2} Tr(\sigma_{\mu}\ \mathbf{AXA}^{\dagger})$ $= \frac{1}{2}\,Tr(\sigma_{\mu}\,\mathbf{A}\sigma_{\nu}x^{\nu}\,\mathbf{A}^{\dagger}) = \frac{1}{2}\,Tr(\sigma_{\mu}\,\mathbf{A}\sigma_{\nu}\,\mathbf{A}^{\dagger})x^{\nu} = \Lambda^{\mu}_{\nu}(\mathbf{A})x^{\nu}$ where $\Lambda^{\mu}_{\nu}(\mathbf{A}) = \frac{1}{2}\,Tr(\sigma_{\mu}\,\mathbf{A}\sigma_{\nu}\,\mathbf{A}^{\dagger})$ is a Lorentz transformation because $x^2 = \det(\mathbf{X}) = \det(\mathbf{X}') = x'^2$. Since $\Lambda^{\mu}_{\nu}(\mathbf{A}) = \frac{1}{2}\ Tr(\sigma_{\mu}\ \mathbf{A}\sigma_{\nu}\ \mathbf{A}^{\dagger})$, it follows that we can write $\mathbf{A}\sigma_{\mu}\,\mathbf{A}^{\dagger} = \sum_{\nu}\Lambda^{\nu}_{\mu}\ \sigma_{\mu}$. We also have the following properties for Λ^{ν}_{μ}:

(i) We have $\Lambda^{\lambda}_{\nu}(\mathbf{AB}) = \frac{1}{2} Tr[\sigma_{\lambda}\ \mathbf{AB}\sigma_{\nu}\ (\mathbf{AB})^{\dagger}] = \frac{1}{2} Tr[\sigma_{\lambda}\ \mathbf{AB}\sigma_{\nu}\ \mathbf{B}^{\dagger}\mathbf{A}^{\dagger}]$

$$= \frac{1}{2} Tr[\mathbf{A}^{\dagger}\ \sigma_{\lambda}\ \mathbf{AB}\sigma_{\nu}\ \mathbf{B}^{\dagger}] = \frac{1}{2} Tr[\mathbf{A}^{\dagger}\ \sigma_{\lambda}\ \mathbf{A}\sigma_{\mu}\,\sigma_{\mu}\ \mathbf{B}\sigma_{\nu}{}^{\dagger}\ \mathbf{B}^{\dagger}]$$

$$= \sum_{\mu}\frac{1}{4} Tr\ [\mathbf{A}^{\dagger}\sigma_{\lambda}\mathbf{A}\sigma_{\mu}]\ Tr\ [\sigma_{\mu}\mathbf{B}\sigma_{\nu}\mathbf{B}^{\dagger}]$$

$$= \sum_{\mu}\frac{1}{4} Tr\ [\sigma_{\lambda}A\sigma_{\mu}\mathbf{A}^{\dagger}]\ Tr\ [\sigma_{\mu}\mathbf{B}\sigma_{\nu}\mathbf{B}^{\dagger}] = \sum_{\mu}\Lambda^{\lambda}_{\mu}\,(\mathbf{A})\Lambda^{\mu}{}_{\nu}(\mathbf{B})$$

$$= [\Lambda(\mathbf{A})\Lambda(\mathbf{B})]^{\lambda}{}_{\nu}\ \text{ whence } \Lambda(\mathbf{AB}) = \Lambda(\mathbf{A})\ \Lambda(\mathbf{B})$$

$$= \frac{2}{\det(\sigma_\mu)\det(\sigma_\nu)}\left\{\sum_\lambda [\det(\sigma_\lambda)(\Lambda^\lambda_\mu)(\Lambda^\lambda_\nu)]\right\}$$

$$= \frac{2}{\det(\sigma_\mu)\det(\sigma_\nu)}\left\{\sum_\lambda [\det(\sigma_\lambda)(\Lambda^\mu_\lambda)(\Lambda^\lambda_\nu)]\right\}$$

whence $\delta_{\mu\nu} = \left\{\sum_\lambda [\det(\sigma_\lambda)(\Lambda^\lambda_\mu)(\Lambda^\lambda_\nu)]\right\}$. This equation also gives $(\Lambda^0_0)^2 \geq 1$ which reads with $\Lambda^0_0\ (\mathbf{A}) = \frac{1}{2} Tr\ (\sigma_0 \mathbf{A} \sigma_0 \mathbf{A}^\dagger) = \frac{1}{2} Tr\ (\mathbf{A}\mathbf{A}^\dagger) \geq 0$ yields $\Lambda^0_0 \geq 1$.

(v) To obtain the determinant of Λ, we introduce a transformation of the basis vector $\mathbf{I} = \sigma_0 \to i\sigma_0 = \sigma_4$. In terms of this new set of basis vectors $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} = i\sigma_0$, we have the following relations:

(a) Any 2×2 matrix $\mathbf{M}$ may be expressed as $\mathbf{M} = \sum_{\mu=1}^{4} b^\mu \sigma_\mu$ where $b^\mu = \frac{1}{2} Tr\ (\sigma^\dagger_\mu \mathbf{M})$;

(b) $\det(\mathbf{M}) = \mathbf{M}\sigma_2 \mathbf{M}^T \sigma_2$;

(c) For any $\mathbf{B} \in SL(2, \mathbb{C})$, we can write $\mathbf{B}\sigma_\mu \mathbf{B}^\dagger = \sum_{\nu=1}^{4} \Lambda^\nu_\mu \sigma_\nu$ where $\Lambda^\nu_\mu \frac{1}{2}\ Tr\ (\sigma^\dagger_\nu \mathbf{B} \sigma_\mu \mathbf{B}^\dagger)$;

(d) Λ^i_j, Λ^4_4 are all real and Λ^i_4, Λ^4_j are purely imaginary;

(e) $\mathbf{B}^{-1\dagger}\ \sigma^\dagger_\mu \mathbf{B}^{-1} = \Lambda^\nu_\mu \sigma^\dagger_\nu$;

(f) $\delta_{\mu\nu} = \sum_\lambda [(\Lambda^\lambda_\mu)(\Lambda^\lambda_\nu)]$

(g) To compute the determinant of Λ, we have, in the new basis

$$2 = Tr\ (\sigma_1 \sigma^\dagger_2 \sigma_3 \sigma^\dagger_4) = Tr\ (\mathbf{A}\sigma_1 \sigma^\dagger_2 \sigma_3 \sigma^\dagger_4 \mathbf{A}^{-1})$$
$$= Tr\ [(\mathbf{A}\sigma_1 \mathbf{A}^\dagger)\ (\mathbf{A}^{\dagger-1}\sigma^\dagger_2 \mathbf{A}^{-1})(\mathbf{A}\sigma_3 \mathbf{A}^\dagger])\ (\mathbf{A}^{\dagger-1}\sigma^\dagger_4 \mathbf{A}^{-1})]$$

$$= Tr\left(\sum_{\mu,\nu,\lambda,\rho=1}^{4} \Lambda^\mu_1 \Lambda^\nu_2 \Lambda^\lambda_3 \Lambda^\rho_4 \sigma_\mu \sigma^\dagger_\nu \sigma_\lambda \sigma^\dagger_\rho\right)$$

$$= \sum_{\mu,\nu,\lambda,\rho=1}^{4} Tr(\Lambda^\mu_1 \Lambda^\nu_2 \Lambda^\lambda_3 \Lambda^\rho_4 \sigma_\mu \sigma^\dagger_\nu \sigma_\lambda \sigma^\dagger_\rho)$$

$$= \sum_{\mu,\nu,\lambda,\rho=1}^{4} \Lambda^\mu_1 \Lambda^\nu_2 \Lambda^\lambda_3 \Lambda^\rho_4\ Tr\ (\sigma_\mu \sigma^\dagger_\nu \sigma_\lambda \sigma^\dagger \rho)$$

$$= 2 \sum_{\mu,\nu,\lambda,\rho=1}^{4} (\Lambda_1^\mu \Lambda_2^\nu \Lambda_3^\lambda \Lambda_4^\rho) \times (\delta_\nu^\mu \delta_\rho^\lambda - \delta_\lambda^\mu \delta_\rho^\nu + \delta_\rho^\mu \delta_\lambda^\nu + \varepsilon_{\mu\nu\lambda\rho})$$

$$= 2 \sum_{\mu,\nu,\lambda,\rho=1}^{4} \varepsilon_{\mu\nu\lambda\rho}(\Lambda_1^\mu \Lambda_2^\nu \Lambda_3^\lambda \Lambda_4^\rho) = 2\det(\Lambda)$$

whence det $(\Lambda) = 1$. The above properties sindicate the fact that Λ is a Lorentz matrix.

It follows from all the above properties that there exists a homomorphism from the $SL(2,\mathbb{C})$ group to the Lorentz group. Corresponding to each element **A** of the $SL(2,\mathbb{C})$ group, there exists an element $\Lambda(\mathbf{A})$ of the Lorentz group and this correspondence is preserved under the group operation of multiplication, i.e., $\Lambda(\mathbf{AB}) = \Lambda(\mathbf{A})\Lambda(\mathbf{B})$. Now, since the equation $\Lambda_\nu^\mu(\mathbf{A}) = \frac{1}{2} Tr(\sigma_\mu \mathbf{A} \sigma_\nu \mathbf{A}^\dagger)$ is quadratic in the elements of **A** so that $\Lambda(\mathbf{A})$ and $\Lambda(-\mathbf{A})$ are equal.

We now examine whether any elements of $SL(2,\mathbb{C})$ other than **A** and $-$**A** correspond to $\Lambda(\mathbf{A})$. For this purpose, we identify the elements of $SL(2,\mathbb{C})$ that correspond to the identity of the Lorentz group. We revert to our original basis $\{\sigma_0 = \mathbf{I}, \sigma_i, i = 1, 2, 3\}$ and let $\mathbf{A}_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ correspond to the identity element of the Lorentz group. Then, we must have $\delta_{\mu\nu} = \frac{1}{2} Tr(\sigma_\mu \mathbf{A}_0 \sigma_\nu \mathbf{A}_0^\dagger)$. This represents a set of 16 equations for the four elements of $\mathbf{A}_0$. We consider the equations corresponding to $\mu = \nu = 0, 1, 3$ that are $a^*d + cb^* + c^*b + d^*a = 2$, $aa^* - bb^* - cc^* + dd^* = 2$ and $aa^* + bb^* + cc^* + dd^* = 2$. From the last pair of equations we obtain $aa^* + dd^* = 2$ and $b = c = 0$. Hence, from the first equation, we get $a^*d + ad^* = 2$ whence $(a^* - d^*)(a - d) = 0$ or $|a - d|^2 = 0$ giving $a = d$. It, thus, follows that the matrix $\mathbf{A}_0$ is a constant multiple of the identity matrix. Further, since $\mathbf{A}_0 \in SL(2,\mathbb{C})$, we must have $\det(\mathbf{A}_0) = 1$, whence the constant multiple can only take the values ± 1. Now, if the matrix $\mathbf{A} \in SL(2,\mathbb{C})$ corresponds to the Lorentz transformation $\Lambda(\mathbf{A})$ and $\mathbf{A}^{-1}$ to $\Lambda(\mathbf{A}^{-1})$, then $\Lambda(\mathbf{A})\Lambda(\mathbf{A}^{-1}) = \Lambda(\mathbf{AA}^{-1}) = \Lambda(\mathbf{I}) = \mathbf{I}$ implying that $\Lambda(\mathbf{A}^{-1}) = \Lambda^{-1}(\mathbf{A})$. Again, if $\Lambda(\mathbf{A}) = \Lambda(\mathbf{B})$ then $\Lambda(\mathbf{AB}^{-1}) = \Lambda(\mathbf{A})\Lambda(\mathbf{B}^{-1}) = \Lambda(\mathbf{A})\Lambda^{-1}(\mathbf{B}) = \Lambda(\mathbf{B})\Lambda^{-1}(\mathbf{B}) = \mathbf{I}$ whence, in line with the above argument regarding $\mathbf{A}_0$, we must have $\mathbf{AB}^{-1} = \pm\mathbf{I}$, i.e., $\mathbf{A} = \pm\mathbf{B}$. This analysis, thus, establishes two things (i) each element $\mathbf{A} \in SL(2,\mathbb{C})$ corresponds to an element $\Lambda(\mathbf{A})$ of the Lorentz group and (ii) $\Lambda(\mathbf{A}) \neq \Lambda(\mathbf{B})$ unless $\mathbf{B} = \pm\mathbf{A}$.

What remains to be shown in completing the correspondence between the $SL(2,\mathbb{C})$ group and the Lorentz group is that given a Lorentz transformation, Λ, does there exist an $\mathbf{A} \in SL(2,\mathbb{C})$ such that $\Lambda = \Lambda(\mathbf{A})$. To establish this correspondence, we make use of the fact that any Lorentz transformation can be factored into a rotation followed by a boost and again another rotation. It would therefore, suffice to prove here that there exist $SL(2,\mathbb{C})$ matrices that represent pure rotations and pure boosts. We, shall establish this in the following steps.

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, $a, b, c, d \in \mathbb{C}$ be arbitrary. We can, then, write

$$\mathbf{M}' = \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}(\sigma.\mathbf{n})\right] [x^0 + \boldsymbol{\sigma}.\mathbf{x}] \left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(\sigma.\mathbf{n})\right]$$

$$= x^0 + \cos^2\frac{\theta}{2}(\boldsymbol{\sigma}.\mathbf{x}) + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}[(\boldsymbol{\sigma}.\mathbf{n})(\boldsymbol{\sigma}.\mathbf{x}) - (\boldsymbol{\sigma}.\mathbf{x})(\boldsymbol{\sigma}.\mathbf{n})]$$

$$+ \sin^2\frac{\theta}{2}(\boldsymbol{\sigma}.\mathbf{n})(\boldsymbol{\sigma}.\mathbf{x})(\boldsymbol{\sigma}.\mathbf{n})$$

$$= x^0 + \sigma.\{(\mathbf{n.x})\mathbf{n} + \cos\theta[\mathbf{x} - (\mathbf{n.x}).\mathbf{n}] + \sin\theta(\mathbf{x}\times\mathbf{n})\}.$$

Since, $\mathbf{M}' = x'^{\mu}\sigma_{\mu} = x'^0 + \boldsymbol{\sigma}\mathbf{x}'$, we have $x'^0 = x^0$ and $\mathbf{x}' = (\mathbf{n.x})\mathbf{n} + \cos\theta\,[\mathbf{x} - (\mathbf{n.x}).\mathbf{n}] + \sin\theta\,(\mathbf{x}\times\mathbf{n})$. The fact that the above transformation represents a rotation is clearly evident from (i) $x'^0 = x^0$ and (ii) $\mathbf{x}'.\mathbf{x}' = \mathbf{x}\mathbf{x}$. Actually, the above transformation is a rotation with the vector **n** as the axis through an angle θ in the clockwise direction when we look towards the origin along **n**. In the expression for the transformed vector $\mathbf{x}'$, the component $(\mathbf{n.x})\mathbf{n}$ represents the unaltered component of **x** in the direction of **n**; the component $[\mathbf{x} - (\mathbf{n.x}).\mathbf{n}]$ is the component perpendicular to **n** - it is scaled by cos θ as a result of the rotation and the third component $(\mathbf{x}\times\mathbf{n})$ is perpendicular to both **n** and **x** and its coefficient gets rescaled from 0 to sin θ due to the rotation. In conclusion, we note that any rotation is specified by the axis of rotation **n** and the angle of rotation θ and the corresponding pair of unitary $SL(2,\mathbb{C})$ matrices is given by $\pm\mathbf{A} = \cos\theta + i\sin\theta\boldsymbol{\sigma}.\mathbf{n}$ with **n** & θ both real.

We, now, address the issue of representation of Lorentz boosts by $SL(2,\mathbb{C})$ matrices. For the purpose, we consider Hermitian $SL(2,\mathbb{C})$ matrices. The condition of Hermiticity manifests itself as $\mathbf{A}^{\dagger} = \mathbf{A}$ or $\cos^*\theta - i\sin^*\theta(\boldsymbol{\sigma}.\mathbf{n})^{\dagger} = \cos\theta - i\sin\theta(\boldsymbol{\sigma}.\mathbf{n})$ thereby implying that **n** must necessarily be real and θ purely imaginary so that we can write such a matrix as $\cosh\theta + \sinh\theta(\boldsymbol{\sigma}.\mathbf{n})$. The transformation of a 4-vector by such a matrix is evaluated as in the case of rotations and we have

$$\mathbf{M}' = \left[\cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}(\sigma.\mathbf{n})\right] [x^0 + \boldsymbol{\sigma}.\mathbf{x}] \left[\cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}(\sigma.\mathbf{n})\right]$$

$$= \left(\cosh^2\frac{\alpha}{2} + \sinh^2\frac{\alpha}{2}\right)x^0 + \sinh\frac{\alpha}{2}\cosh\frac{\alpha}{2}[(\boldsymbol{\sigma}.\mathbf{n})(\boldsymbol{\sigma}.\mathbf{x}) + (\boldsymbol{\sigma}.\mathbf{x})(\boldsymbol{\sigma}.\mathbf{n})]$$

$$+ 2\sinh\frac{\alpha}{2}\cosh\frac{\alpha}{2}(\boldsymbol{\sigma}.\mathbf{n})\,x^0 + \cosh^2\frac{\alpha}{2}(\boldsymbol{\sigma}.\mathbf{x}) + \sinh^2\frac{\alpha}{2}(\boldsymbol{\sigma}.\mathbf{n})(\boldsymbol{\sigma}.\mathbf{x})(\boldsymbol{\sigma}.\mathbf{n})$$

$$= \cosh\alpha x^0 + \sinh\alpha(\mathbf{n.x}) + \boldsymbol{\sigma}.[\sinh\alpha x^0\mathbf{n} + \mathbf{x} + (\cosh\alpha - 1)(\mathbf{n.x})\mathbf{n}]$$

whence $x'^0 = \cosh\alpha x^0 + \sin\alpha(\mathbf{n.x})$ and $\mathbf{x}' = [\mathbf{x} - (\mathbf{n.x})\mathbf{n}] + \mathbf{n}[\cosh\alpha(\mathbf{n.x}) + \sinh\alpha x^0]$. It can easily be verified that $(x'^0)^2 - (\mathbf{x}')^2 = (x^0)^2 - (\mathbf{x})^2$ thereby confirming that $x \to x'$ is indeed a Lorentz transformation. It is instructive to resolve $\mathbf{x}'$ into components parallel and perpendicular to **n**. We have $x'^0 = \cosh\alpha x'^0 + \sinh\alpha(\mathbf{n.x})$, $(\mathbf{n.x}') = \sinh\alpha x^0 + \cosh\alpha(\mathbf{n.x})$ and $x' - (\mathbf{n.x}')\mathbf{n} = \mathbf{x} - (\mathbf{n.x})\mathbf{n}$. Thus, the component of **x** perpendicular to **n** is left unchanged by such a transformation while the time coordinate and the component of **x** parallel to **n** undergo the usual two-dimensional Lorentz boost. We have, thus, established that any Lorentz boost is specified by the axis of the boost **n** and the hyperbolic angle α (called rapidity) and the corresponding pair of Hermitian $SL(2, \mathbb{C})$ matrices is given by ±**A** = cosh α + sinh α**σ.n** with **n** & α both real (which may also be written in the usual form as ±**A** = cos θ + *i*sinθ**σ.n** with **n** real and θ purely imaginary).

Since every Lorentz transformation with $\Lambda_0^0 \geq +1$ can be written as a composition of rotations and boosts and, as shown above, there exist a pair of $SL(2, \mathbb{C})$ matrices corresponding to any rotation or boost and further, that the set of all $SL(2, \mathbb{C})$ matrices forms a group, it immediately follows that given any Lorentz transformation, we can associate a pair of $SL(2, \mathbb{C})$ matrices with such transformation.

Having established the existence of homomorphism between the Lorentz group and the $SL(2, \mathbb{C})$ group, we make use of this connection to obtain the explicit expression for the Wigner rotation $R(\varphi_W, \mathbf{p}) = L^{-1}(p')\,\Lambda L(p)$ with $\Lambda p = p'$. For the purpose, we write

$$\Lambda = \cosh\frac{\alpha}{2} + \sin\frac{\alpha}{2}(\boldsymbol{\sigma}.\mathbf{e}) \tag{247}$$

$$L(p) = \cosh\frac{\beta}{2} + \sinh\frac{\beta}{2}(\boldsymbol{\sigma}.\mathbf{f})$$

$$= \left(\frac{\omega_p + M}{2M}\right)^{1/2} + \left(\frac{\omega_p + M}{2M}\right)^{1/2} \frac{\boldsymbol{\sigma}.\mathbf{p}}{|\mathbf{p}|} \tag{248}$$

where $\omega_p = M \cosh\beta$ and $\mathbf{p} = M \sinh \beta \mathbf{f}$ and

$$L(p') = \cosh\frac{\gamma}{2} + \sinh\frac{\gamma}{2}(\boldsymbol{\sigma}.\mathbf{g}) = \left(\frac{\omega_{p'} + M}{2M}\right)^{1/2} + \left(\frac{\omega_{p'} - M}{2M}\right)^{1/2} \frac{\boldsymbol{\sigma}.\mathbf{p}'}{|\mathbf{p}'|} \tag{249}$$

with $\omega_{p'} = M\cosh\gamma$, $\mathbf{p}' = M\sinh\gamma\mathbf{g}$, $\omega_{p'} = \omega_p \cosh\alpha + \mathbf{p.e}\sinh\alpha$ and $\mathbf{p}' = [\mathbf{p} - (\mathbf{p.e})\mathbf{e}] + [\omega_p \sinh\alpha + \mathbf{p.e}\cosh\alpha]\mathbf{e}$. We, then, have

$$2MR(\varphi_W, \mathbf{p}) = [(\omega_p + M)(\omega_{p'} + M)]^{1/2}\left[\cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}(\sigma.\mathbf{e})\right]$$

$$- \frac{1}{|\mathbf{p}'|}[(\omega_p + M)(\omega_{p'} - M)]^{1/2}\left\{\cosh\frac{\alpha}{2}(\sigma.\mathbf{p}') + \sinh\frac{\alpha}{2}[\mathbf{e.p}' + i\sigma.(\mathbf{p}' \times e)]\right\}$$

$$+ \frac{1}{|\mathbf{p}|}[(\omega_p - M)(\omega_{p'} + M)]^{1/2}\left\{\cosh\frac{\alpha}{2}(\sigma.\mathbf{p}) + \sinh\frac{\alpha}{2}[\mathbf{e.p} + i\sigma.(e \times \mathbf{p})]\right\}$$

$$- \frac{1}{|\mathbf{p}||\mathbf{p}'|}[(\omega_p - M)(\omega_{p'} - M)]^{1/2}\left\{\begin{matrix}\cosh\frac{\alpha}{2}[\mathbf{p.p}' + i\sigma.(\mathbf{p}' \times \mathbf{p})] + \sinh\frac{\alpha}{2} \\ \times\begin{bmatrix} i(\mathbf{p}' \times \mathrm{e}).\mathbf{p} + (\sigma.\mathrm{p})(\mathbf{p}'.\mathrm{e}) \\ -(\sigma.\mathrm{e})(\mathbf{p.p}') + (\sigma.\mathbf{p}')(\mathbf{p}.\mathrm{e})\end{bmatrix}\end{matrix}\right\}$$

$$= \frac{1}{[(\omega_p + M)(\omega_{p'} + M)]^{1/2}} \left[\cosh\frac{\alpha}{2}(2M^2 + 2M\,\omega_p) + 2M\,(\mathbf{p.e})\sinh\frac{\alpha}{2}\right]$$

$$= \frac{2M}{[(\omega_p + M)(\omega_{p'} + M)]^{1/2}} \left[\cosh\frac{\alpha}{2}(\omega_p + M) + \sinh\frac{\alpha}{2}(\mathrm{p.e})\right] \tag{250}$$

$$B = -\frac{1}{|\mathbf{p}'|}[(\omega_p + M)(\omega_{p'} - M)]^{1/2}\cosh\frac{\alpha}{2} + \frac{1}{|\mathbf{p}|}[(\omega_p - M)(\omega_{p'} + M)]^{1/2}$$

$$\times \cosh\frac{\alpha}{2} - \frac{1}{|\mathbf{p}||\mathbf{p}'|}[(\omega_p + M)(\omega_{p'} - M)]^{1/2}\sinh\frac{\alpha}{2}$$

$$\times [\omega_p \sinh\alpha + (\mathbf{p.e})\cosh\alpha + (\mathbf{p.e})]$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \left\{\begin{matrix} -(\omega_p + M)\cosh\frac{\alpha}{2} + [\omega_p\cosh\alpha + (\mathbf{p.e})\sinh\alpha + M] \\ \times\cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2}[\omega_p\sinh\alpha + (\mathbf{p.e})\cosh\alpha + (\mathbf{p.e})] \end{matrix}\right\}$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \left\{-(\omega_p + M)\cosh\frac{\alpha}{2} + \omega_p\cosh\frac{\alpha}{2} + M\cosh\frac{\alpha}{2}\right\} = 0 \tag{251}$$

$$C = [\omega_p + M)(\omega_{p'} + M)]^{1/2}\sinh\frac{\alpha}{2} - \frac{1}{|\mathbf{p}'|}[(\omega_p + M)(\omega_{p'} - M)]^{1/2}$$

$$\times \cosh\frac{\alpha}{2}[(\mathbf{p.e})(\cosh\alpha - 1) + \omega_p\sinh\alpha] - \frac{1}{|\mathbf{p}||\mathbf{p}'|}[(\omega_p - M)(\omega_{p'} - M)]^{1/2}$$

$$\times \sinh\frac{\alpha}{2}\begin{bmatrix} -\mathbf{p}^2 + (\mathbf{p.e})^2 - \omega_p(\mathbf{p.e})\sinh\alpha - (\mathbf{p.e})^2\cosh\alpha \\ -(\mathbf{p.e})^2 + \omega_p(\mathbf{p.e})\sinh\alpha + (\mathbf{p.e})^2\cosh\alpha \end{bmatrix}$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \left\{\begin{matrix} (\omega_p + M)[\omega_p\cosh\alpha + (\mathbf{p.e})\sinh\alpha + M]\sinh\frac{\alpha}{2} \\ -(\omega_p + M)\cosh\frac{\alpha}{2}[(\mathbf{p.e})(\cosh\alpha - 1) + \omega_p\sinh\alpha] \\ +\sinh\frac{\alpha}{2}\mathbf{p}^2 \end{matrix}\right\}$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \begin{Bmatrix} -(\omega_p + M)\omega_p \sinh\frac{a}{2} + (\mathbf{p}.\mathbf{e})(\omega_p + M) \\ \times\left(\cosh\frac{\alpha}{2} - \cosh\frac{\alpha}{2}\right) + [(\omega_p + M)M + \mathbf{p}^2]\sinh\frac{\alpha}{2} \end{Bmatrix}$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \sinh\frac{\alpha}{2} \begin{Bmatrix} -\mathbf{p}^2 - M^2 - M\omega_p \\ +M\omega_p + M^2 + \mathbf{p}^2 \end{Bmatrix} = 0 \tag{252}$$

$$D = -\frac{1}{|\mathbf{p}'|}[(\omega_p + M)(\omega_{p'} - M)]^{1/2} \sinh\frac{\alpha}{2} - \frac{1}{|\mathbf{p}|}[(\omega_p - M)(\omega_{p'} + M)]^{1/2}$$

$$\times \sinh\frac{\alpha}{2} + \frac{1}{|\mathbf{p}||\mathbf{p}'|}[(\omega_p - M)(\omega_{p'} - M)]^{1/2}$$

$$\times \cosh\frac{\alpha}{2}[\omega_p \sinh\alpha + (\mathbf{p}.\mathbf{e})(\cosh\alpha - 1)]$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \begin{Bmatrix} -[\omega_p + M + \omega_p\cosh\alpha + (\mathbf{p}.\mathbf{e})\sinh\alpha + M]\sinh\frac{\alpha}{2} \\ +[\omega_p \sinh\alpha + (\mathbf{p}.\mathbf{e})(\cosh\alpha - 1)]\cosh\frac{\alpha}{2} \end{Bmatrix}$$

$$= \frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \begin{Bmatrix} -\omega_p\left(\sinh\frac{\alpha}{2} - \sinh\frac{\alpha}{2}\right) - 2M\sinh\frac{\alpha}{2} \\ -(\mathbf{p}.\mathbf{e})\left(\cosh\frac{\alpha}{2} - \cosh\frac{\alpha}{2}\right) \end{Bmatrix}$$

$$= -2M\frac{(\omega_p + M)^{-1/2}}{(\omega_{p'} + M)^{1/2}} \sinh\frac{\alpha}{2} \tag{253}$$

Substituting the values of A, B, C, D on the expression for the Wigner rotation $R(\varphi_W, p)$, we obtain $R(\varphi_W, p) = \dfrac{(\omega_p + M)}{(\omega_{p'} + M)^{1/2}}$

$$\times \left\{\cosh\frac{\alpha}{2}(\omega_p + M) + \sinh\frac{\alpha}{2}(\mathbf{p}.\mathbf{e}) - i\sinh\frac{\alpha}{2}\sigma.(\mathbf{p}\times\mathbf{e})\right\}. \tag{254}$$

We can also express the Wigner rotation in terms of the rapidity angles α, β, γ as follows. We have $\omega_p = M\cosh\beta$, $\mathbf{p} = M\sinh\beta\mathbf{f}$, $\omega_{p'} = M\cosh\gamma$ and $\mathbf{p}' = M\sinh\boldsymbol{\gamma}\mathbf{g}$ so that

$$[J_j, S_k] = i \sum_l \varepsilon_{jkl} S_l \tag{257}$$

(ii) Since, **S** is an angular momentum, it must satisfy the same commutators among its components as the total and orbital angular momenta, i.e.,

$$[S_j, S_k] = i \sum_l \varepsilon_{jkl} S_l \tag{258}$$

(iii) It should be possible to measure spin and linear momentum simultaneously so that the corresponding operators must commute, i.e.,

$$[P_i, S_j] - 0 \tag{259}$$

(iv) It should be possible to measure spin and position simultaneously so that the corresponding operators must commute, i.e.,

$$[R_i, S_j] - 0 \tag{260}$$

(v) It must be expressible as some combination of the complete set of generators of the Poincare group, i.e., P^0, **P, J, K**

(vi) It must behave as a 3-vector under spatial rotations whence it must be linear in the generators **J, K** so that we can express it as

$\mathbf{S} = f_1\, \mathbf{J} + f_2\, \mathbf{K} + f_3\, \mathbf{P} + f_4(\mathbf{P.J})\mathbf{P} + f_5(\mathbf{P.K})\mathbf{P} + f_6(\mathbf{P} \times \mathbf{J}) + f_7(\mathbf{P} \times \mathbf{K})$ where each $f_i \equiv f_i\,(\mathbf{P}^2, P^0)$ (because each of these coefficients must be three-dimensional scalars). In view of (iii), we get $f_2 = -f_5 \mathbf{P}^2$ and $0 = [S^i, P^i] = i(f_1 + f_7 P^0)\varepsilon^{ijk}\, P^k + i(f_2 P^0 + f_6 \mathbf{P}^2)\delta^{ij} + i(f_5 P^0 - f_6) P^i P_J$ whence $f_1 + f_7 P^0 = 0$, $f_2\, P^0 + f_6\, \mathbf{P}^2 = 0$ and $f_5\, P^0 = f_6 = -f_2\, P^0\, \mathbf{P}^{-2}$. Again, using (ii), we get

$$f_1^2 \left\{ \left[\mathrm{J} - \frac{1}{P^0}(\mathrm{P} \times \mathrm{K}) \right]^k - \frac{1}{(P^0)^2} P^k\, (\mathrm{P.J}) \right\}$$

$$+ f_1 f_2 \left\{ -\left[\mathrm{K} - \frac{1}{\mathbf{P}^2}(\mathbf{P.K})\mathbf{P} - \frac{P^0}{\mathrm{P}^2}(\mathbf{P} \times \mathbf{J}) \right]^k - \frac{1}{P^0}(\mathbf{P} \times \mathbf{J})^k \right\}$$

$$+ f_2 f_4\, \mathbf{P}^2 \left[\mathbf{K} - \frac{1}{\mathbf{P}^2}(\mathbf{P.K})\mathbf{P} - \frac{P^0}{\mathbf{P}^2}(\mathbf{P} \times \mathbf{J}) \right]^k$$

$$+ f_1 f_4 \left\{ \mathbf{P}^2 \left[\mathbf{J} - \frac{1}{P^0}(\mathbf{P} \times \mathbf{K}) \right]^k - \left[\mathbf{J.P} + \frac{1}{P^0}(\mathbf{P} \times \mathbf{K}).\mathbf{P} \right] P^k \right\}$$

$$+ f_2^2 \left[2(\mathbf{K} - \mathbf{J}) \left(1 + \frac{M^2}{\mathbf{P}^2} \right) + 2(\mathbf{P} \times \mathbf{J}) \frac{P^0}{\mathbf{P}^2} + \mathbf{P}(\mathbf{P.J}) \frac{M^2}{\mathbf{P}^4} \right]^k$$

$$= f_1\left[\mathbf{J} - \frac{1}{P^0}(\mathbf{P}\times\mathbf{K})\right]^k + f_2\left[\mathbf{K} - \frac{1}{\mathbf{P}^2}(\mathbf{P.K})\mathbf{P} - \frac{P^0}{\mathbf{P}^2}(\mathbf{P}\times\mathbf{J})\right]^k$$

$+ f_3\, P^k + f_4\, (\mathbf{P.J})P^k$ whence taking product with P^k and summing over k, we get

$$f_1^2\,\frac{M^2}{(P^0)^2}(\mathbf{P.J}) + f_2^2\left[2\left(1+\frac{M^2}{\mathbf{P}^2}\right)(\mathbf{P.K}) - \left(2+\frac{M^2}{\mathbf{P}^2}\right)(\mathbf{P.J})\right]$$

$= f_1(\mathbf{P.J}) + f_3\,\mathbf{P}^2 + f_4(\mathbf{P.J})\mathbf{P}^2$, Since $f_i \equiv f_i(\mathbf{P}^2, P^0)$ the above expression mandates $f_2 = f_3 = 0$ and $f_1^2\frac{M^2}{(P^0)^2} = f_1 + f_4\,\mathbf{P}^2$. Substituting these values, we get $S^k = f_1\left[\mathbf{J} - \frac{1}{P^0}(\mathbf{P}\times\mathbf{K})\right]^k + f_4(\mathbf{P.J})P^k$. Setting this value of S^k and comparing the left and right hand sides, we also obtain $f_1^2 + f_1 f_4\,\mathbf{P}^2 = f_1$ and $f_1^2 + f_1 f_4(P^0)^2 = -f_4(P^0)^2$ so that $f_1 = \pm\frac{P^0}{M}$ and $f_4 = -\frac{1}{M(M \pm P^0)}$ thereby providing a unique expression (upto time reversal) for the relativistic spin operator as:

$$\mathbf{S} = \frac{\mathbf{W}}{M} - \frac{W_0\,\mathbf{P}}{M(M+P_0)}$$

$$= \frac{P^0\,\mathbf{J}}{M} - \frac{\mathbf{P}\times\mathbf{K}}{M} - \frac{\mathbf{P}(\mathbf{P.J})}{M(M+P_0)} \quad (261)$$

This operator meets all the above requirements of a relativistic spin operator for positive massive particles. We have

(i) Making use of the properties of the Pauli Lubanski vector viz. $[\mathbf{J}, W_0] = 0$,

$$[J_j, W_k] = i\sum_{l=1}^{3}\varepsilon_{jkl}\,W_l\,,\ [J_j, P_k] = i\sum_{l=1}^{3}\varepsilon_{jkl}\,P_l\,, \text{ we obtain}$$

$$[J_j, S_k] = \frac{1}{M}[J_j, W_k] - \frac{1}{M(M+P_0)}[J_j, W_0\,P_k]$$

$$= \frac{1}{M}[J_j, W_k] - \frac{1}{M(M+P_0)}\{W_0[J_j, P_k] + [J_j, W_0]P_k\}$$

$$= \frac{i}{M}\sum_{l=1}^{3}\varepsilon_{jkl}\,W_l - \frac{iW_0}{M(M+P_0)}\sum_{l=1}^{3}\varepsilon_{jkl}\,P_l = i\sum_{l=1}^{3}\varepsilon_{jkl}\,\mathrm{S}_l \quad (262)$$

(ii) Similarly, to prove (258), we make use of the definition of the Pauli Lubanski vector as $\mathbf{W}^0 = (\mathbf{P.\,J})$, $\mathbf{W} = P^0.\mathbf{J} - (\mathbf{P}\times\mathbf{K})$ so that $\mathbf{P.W} = P^0(\mathbf{P.J}) = P^0\,W^0$. We also write $F = -\frac{1}{M(M+P_0)}$

and use the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B})$ to get

$$[S_1,\ S_2] = \left[FW_0\,P_1 + \frac{W_1}{M}, FW_0\,P_2 + \frac{W_2}{M}\right]$$

$$= i\left[-\frac{FP_1\,(\mathbf{P}\times\mathbf{W})_2}{M} + \frac{FP_2\,(\mathbf{P}\times\mathbf{W})_1}{M} + \frac{P_0\,W_3 - W_0\,P_3}{M^2}\right]$$

$$= i\left[-\frac{F(\mathbf{p}\times(\mathbf{P}\times\mathbf{W}))_3}{M} + \frac{P_0\,W_3 - W_0\,P_3}{M^2}\right]$$

$$= i\left[-\frac{F(P_3(\mathbf{P.W}) - W_3\,\mathbf{P}^2)}{M} + \frac{P_0\,W_3 - W_0\,P_3}{M^2}\right]$$

$$= i\left[-\frac{F(P_3\,P^0\,W^0 - W_3\,\mathbf{P}^2)}{M} + \frac{P_0\,W_3 - W_0\,P_3}{M^2}\right]$$

$$= iW_3\left(\frac{\mathbf{P}^2\,F}{M} + \frac{P_0}{M^2}\right) + iP_3\,W^0\left(-\frac{P_0\,F}{M} - \frac{1}{M^2}\right). \tag{263}$$

Now, $$\frac{\mathbf{P}^2\,F}{M} + \frac{P_0}{M^2} = -\frac{\mathbf{P}^2}{M^2\,(M + P^0)} + \frac{P_0}{M^2}$$

$$= \frac{P_0\,(M + P_0) - \mathbf{P}^2}{M^2\,(M + P_0)} = \frac{P_0\,(M + P_0) - (M + p_0)(P_0 - M)}{M^2\,(M + P_0)} = \frac{1}{M}$$

and

$$-\frac{P_0\,F}{M} - \frac{1}{M^2} = \frac{P_0}{M^2\,(M + P_0)} - \frac{1}{M^2} = \frac{P_0 - (M + P_0)}{M^2\,(M + P_0)} = -\frac{1}{M(M + P_0)} = F$$

whence from eq. (263) we get

$$[S_1,\ S_2] = i\left(\frac{W_3}{M}\,FW^0\,P_3\right) = iS_3 \tag{264}$$

We can, similarly prove commutation relations among other combinations of the spin operator components.

(iii) We also have

$$[P_\mu,\ S_k] = \frac{1}{M}\,[P_\mu,\ W_k] - \frac{1}{M\,(M + P_0)}\,[P_\mu,\ W_0,\ P_k]$$

(i) We, then, have

$$[J_1, R_2] = -\frac{1}{2}[J_1, P_0^{-1} K_2] - \frac{1}{2}[J_1, K_2 P_0^{-1}] - \left[J_1, \frac{(P_3 S_1 - P_3 S_1)}{P_0 (P_0 + M)}\right]$$

$$= -\frac{i}{2}(P_0^{-1} K_3 + K_3 P_0^{-1}) + \frac{i}{P_0 (P_0 + M)} (P_2 S_1 - P_1 S_2) = iR_3 \tag{270}$$

with similar commutators for the other components.

(ii) We also have

$$[R_1, P_1] = -\frac{1}{2}[(P_0^{-1} K_1 + K_1 P_0^{-1}), P_1] = \frac{i}{2} (P_0^{-1} P_0 + K_0 P_0^{-1}) = i \tag{271}$$

$$[R_1, P_2] = -\frac{1}{2}[(P_0^{-1} K_1 + K_1 P_0^{-1}), P_2] = 0 \tag{272}$$

as desired.

(iii) We now prove that $\mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{S}$. To do so, we make use of the fact that $[K_1 P_2 - K_2 P_1, P_0] = -i(P_1 P_2 - P_2 P_1) = 0$ so that $\mathbf{K} \times \mathbf{P}$ commutes with P_0 and the operator $P_0^{-1} \mathbf{K} \times \mathbf{P}$ is Hermitian.

Furthermore, we also use $\mathbf{P.S} = \frac{\mathbf{P}.P_0 \mathbf{J}}{M} - \frac{\mathbf{P}^2 \mathbf{P.J} (P_0 - M)}{\mathbf{P}^2 M} = \mathbf{P.J}$. We then have

$$\mathbf{J} - \mathbf{R} \times \mathbf{P} = \mathbf{J} + P_0^{-1} \mathbf{K} \times \mathbf{P} + \frac{(\mathbf{P} \times \mathbf{S}) \times \mathbf{P}}{P_0 (M + P_0)}$$

$$= \mathbf{J} + P_0^{-1} \mathbf{K} \times \mathbf{P} - \frac{1}{P_0 (M + P_0)} (\mathbf{P}(\mathbf{P.S}) - \mathbf{SP}^2)$$

$$= \mathbf{J} + P_0^{-1} \mathbf{K} \times \mathbf{P} - \frac{1}{P_0 (M + P_0)} [(\mathbf{P}(\mathbf{P.S}) - \mathbf{S}(P_0 - M)(P_0 + M)]$$

$$= \mathbf{J} + P_0^{-1} \mathbf{K} \times \mathbf{P} - \frac{\mathbf{P}(\mathbf{P.S})}{P_0 (M + P_0)} - \frac{M}{P_0}\mathbf{S} + \mathbf{S} = \mathbf{S} \tag{273}$$

thereby showing that the total angular momentum is the aggregate of the orbital angular momentum and the spin angular momentum just as in classical physics.

(iv) We also have

$$[S_1, R_1] = [J_1 - (\mathbf{R} \times \mathbf{P})_1, R_1] = [J_1, R_1] - [(\mathbf{R} \times \mathbf{P})_1, R_1]$$

$$= \left[J_1, -\frac{1}{2}(P_0^{-1} K_1 + K_1 P_0^{-1}) - \frac{(\mathbf{P} \times \mathbf{S})_1}{P_0 (P_0 + M)}\right] - [(R_2 P_3 - R_3 P_2), R_1]$$

$$= \left[J_1, -\frac{1}{2}(P_0^{-1} K_1 + K_1 P_0^{-1}) - \frac{(P_2 S_3 - P_3 S_2)}{P_0 (P_0 + M)} \right] - [(R_2 P_3 - R_3 P_2), R_1]$$

$$= \left[J_1, -\frac{(P_2 S_3 - P_3 S_2)}{P_0 (P_0 + M)} \right] = -\frac{1}{P_0 (P_0 + M)} \{[J_1, P_2 S_3] - [J_1, P_3 S_2]\}$$

$$= -\frac{1}{P_0 (P_0 + M)} \{P_2[J_1, S_3] + [J_1, P_2] S_3 - P_3[J_1, S_2] - [J_1, P_3]S_2\}$$

$$= -\frac{1}{P_0 (P_0 + M)} \{- iP_2 S_2 + iP_3 S_3 - iP_3 S_3 + iP_2 S_2\} = 0 \tag{274}$$

and

$$[S_1, R_2] = [J_1 - (\boldsymbol{R} \times \mathbf{P})_1, R_2] = [J_1, R_2] - [(\mathbf{R} \times \mathbf{P})_1, R_2]$$
$$= iR_3 - [(R_2 P_3 - R_3 P_2), R_2] = iR_3 - [R_2, P_3, R_2] + [R_3 P_2, R_2]$$
$$= iR_3 - R_2[P_3, R_2] - [R_2, R_2]P_3 + R_3[P_2, R_2] + [R_3, R_2]P_2 = iR_3 - iR_3 = 0$$

where we have assumed the commutativity of the various components of the position operator, which is proved below.

(v) We now prove that all components of the position operator commute with each other, i.e., $[R_i, R_j] = 0$. We have

$$[P_0 R_1, P_0 R_2] = [P_0 R_1, P_0]R_2 + P_0\{P_0 R_1, R_2]$$
$$= P_0[R_1, P_0]R_2 + P_0[P_0, R_2]R_1 + P_0^2[R_1, R_2]$$
$$= i(P_1 R_2 - P_2 R_1) + P_0^2 [R_1, R_2]$$
$$= i(\mathbf{P} \times \mathbf{R})_3 + P_0^2 [R_1, R_2] = -iJ_3 + iS_3 + P_0^2[R_1, R_2] \tag{275}$$

We also have, on using eq. (269)

$$[P_0 R_1, P_0 R_2] = \left[-K_1 - \frac{i}{2}\frac{P_1}{P_0} + F(\mathbf{P} \times \mathbf{W})_{1,} -K_2 - \frac{i}{2}\frac{P_2}{P_0} + F(\mathbf{P} \times \mathbf{W})_2 \right] \tag{276}$$

wherein we have

$$[-K_1, -K_2] = [K_1, K_2] = -iJ_3 \tag{277}$$

$$\left[-\frac{i}{2}\frac{P_1}{P_0}, -K_2 \right] = -\frac{i}{2}\left[K_2, \frac{P_1}{P_0} \right] = -\frac{i}{2}\left[K_2, \frac{P_0 P_1}{P_0^2} \right]$$

$$= -\frac{i}{2} \left\{ P_0\left[K_2, \frac{P_1}{P_0^2} \right] + \left[K_2, \frac{P_0}{P_0^2} \right] P_1 \right\} = -\frac{i}{2} \left(-i\frac{P_2 P_1}{P_0^2} \right) \tag{278}$$

$$= -\,iJ_3 + i\left(-\frac{P_3 W_0}{M(P_0 + M)} + \frac{W_3}{M}\right) = -\,iJ_3 + iS_3 \tag{284}$$

whence, we have, on using eq. (275)

$$P_0^2[R_1, R_2] = 0 \quad \Rightarrow [R_1, R_2] = 0 \tag{285}$$

(vi) From the commutation relation $[R_i, P_j] = i\delta_{ij}$, we have

$$[R_i, P_i^n] = inP_i^{n-1} \tag{286}$$

and, in general, given any function $f(P_1)$,

$$[R_i, f(P_i)] = i\,\frac{\partial f(P_i)}{\partial P_i} \tag{287}$$

Further,

$$[\mathbf{R}, P_0] = \left[\mathbf{R}, \sqrt{\mathbf{P}^2 + M^2}\right] = i\frac{\partial\sqrt{\mathbf{P}^2 + M^2}}{\partial \mathbf{P}} = i\frac{\mathbf{P}}{p_0} = i\mathbf{V} \tag{288}$$

(vi) Time evolution of the Position Operator is given by;

$$\mathbf{R}(x^0) = \exp(ip_0\, x^0)\ \mathbf{R}\ \exp(-ip_0\, x^0)$$

$$= \mathbf{R} + i[P_0, \mathbf{R}]x^0 = \mathbf{R} + \mathbf{V}x^0 \tag{289}$$

(vii) The transformation of the position operator under a "boost" transformation is derived as follows:

For simplicity, we consider the case of a particle without spin, whence the position operator takes the form

$$\mathbf{R} = -\frac{1}{2}(\mathbf{K}P_0^{-1} + P_0^{-1}\mathbf{K}) \tag{290}$$

Now, we have

$$K_2(\theta) = e^{-iK_1\theta}\, K_2\, e^{iK_1\theta}$$

$$= K_2 - i[K_1, K_2] + \frac{\theta^2}{2!}[K_1, [K_1, K_2]] - \frac{i}{3!}[K_1, [K_1,[K_1, K_2]]] + \ldots$$

$$= K_1 - \theta J_3 + \frac{\theta^2}{2!}K_1 - \frac{\theta^3}{3!}J_3 + \ldots$$

$$= K_1 \cosh\theta - J_3 \sinh\theta$$

Hence, the various components of the position operator in the frame O' that is boosted with a velocity $v = \tanh\theta$ along the x^1 axis are given by

$$R_2(\theta) = e^{-iK_1\theta}\, R_2\, e^{iK_1\theta} = -\frac{1}{2}e^{-iK_1\theta}\,(K_2\, P_0^{-1} + P_0^{-1}\, K_2)e^{iK_1\theta}$$

$$= -\frac{1}{2}(K_2 \cosh\theta - J_3 \sinh\theta)\,(P_0 \cosh\theta - P_1 \sinh\theta)^{-1}$$

$$-\frac{1}{2}(P_0 \cosh\theta - P_1 \sinh\theta)^{-1}\,(K_2 \cosh\theta - J_3 \sinh\theta)$$

$$= -\frac{1}{2}K_2(1 - K_2^{-1} J_3 \tanh\theta)\,P_0^{-1}(1 - P_0^{-1} P_1 \tanh\theta)^{-1}$$

$$-\frac{1}{2}\,P_0^{-1}(1 - P_0^{-1} P_1 \tanh\theta)^{-1} K_2(1 - K_2^{-1} J_3 \tanh\theta)$$

$$= -\frac{1}{2}K_2(1 - K_2^{-1} J_3 v)\,P_0^{-1}(1 - V_1 v)^{-1} - \frac{1}{2}P_0^{-1}(1 - V_1 v)^{-1} K_2(1 - K_2^{-1} J_3 v)$$

$$= -\frac{1}{2}(1 - V_1 v)^{-1}\,[(K_2 P_0^{-1} + P_0^{-1} K_2) + (J_3 P_0^{-1} + P_0^{-1} J_3)v] = \beta(R_1 + J_3 P_0^{-1} v) \quad (291)$$

$$R_1(\theta) = -\frac{1}{2}K_1(P_0 \cosh\theta - P_1 \sinh\theta)^{-1} - \frac{1}{2}(P_0 \cosh\theta - P_1 \sinh\theta)^{-1} K_1 = \beta\frac{R_1}{\cosh\theta} \quad (292)$$

$$R_3(\theta) = -\frac{1}{2}(K_3 \cosh\theta + J_2 \sinh\theta)\,(P_0 \cosh\theta - P_1 \sinh\theta)^{-1}$$

$$-\frac{1}{2}(P_0 \cosh\theta - P_1 \sinh\theta)^{-1}\,(K_3 \cosh\theta + J_y \sinh\theta) = \beta(R_3 - J_2 P_0^{-1} v) \quad (293)$$

Eqs. (291) – (293) give the components of the position operator in the boosted frame as functions of the relative velocity between the two frames. In other words, $R_i(\theta)$ will give us the position of the particle with reference to the frame O' at an instant of time (as measured in the rest frame O) when the coordinates of the particle in O are R_i. The time dependence of the position operator in the boosted frame O' is obtained as follows:

We have

$$P_0' = e^{-iK_1\theta} P_0\, e^{iK_1\theta} \text{ so that}$$

$$\mathbf{R}(\theta, x'^0) = e^{iP'_0 x'^0}\,\mathbf{R}(\theta)\,e^{-iP_0 x'^0} = e^{iP'_0 x'^0}\,e^{-iK_1\theta}\,\mathbf{R}e^{iK_1\theta}\,e^{-iP'_0 x'^0}$$

$$= (e^{-iK_1\theta} e^{iP_0 x'^0} e^{iK_1\theta})\,e^{-iK_1\theta}\mathbf{R}e^{iK_1\theta}\,(e^{-iK_1\theta} e^{-iP_0 x'^0} e^{iK_1\theta})$$

$$= e^{-iK_1\theta}\,e^{iP_0 x'^0}\mathbf{R}e^{-iP_0 x'^0}\,e^{iK_1\theta} = e^{-iK_1\theta}\,(\mathbf{R} + \mathbf{V}x'^0)e^{iK_1\theta} = -\,\mathbf{R}(\theta) + \mathbf{V}(\theta)x'^0 \quad (294)$$

where we have used eq. (289). The components of $\mathbf{R}(\theta)$ are given by eqs. (291) – (293). $\mathbf{V} = \mathbf{P}P_0^{-1}$ is the velocity operator with $\mathbf{V}(\theta)$ being the corresponding operator in the moving frame that may be expressed as

$$V_1(\theta) = e^{-iK_1\theta}\,\frac{P_1}{P_0}\,e^{-iK_1\theta} = \frac{P_1 \cosh\theta - P_0 \sinh\theta}{P_0 \cosh\theta - P_1 \sinh\theta} = \frac{P_1 P_0^{-1} - \tanh\theta}{1 - P_1 P_0^{-1} \tanh\theta} = \beta(V_1 - v) \quad (295)$$

$$V_2(\theta) = e^{-iK_1\theta}\,\frac{P_2}{P_0}\,e^{iK_1\theta} = \frac{P_2}{P_0 \cosh\theta - P_1 \sinh\theta} = \frac{P_2 P_0^{-1}}{(1 - P_1 P_0^{-1} \tanh\theta)} = \frac{\beta V_2}{\cosh\theta} \quad (296)$$

2.19.3 Consistency of the Position Operator with Lorentz Transformations of Special Relativity

For simplicity, we consider a quantum particle moving along the x^1 axis. We also assume that the relative motion between the unprimed and the primed frame is also along the same axis so that the other spatial coordinates can be ignored in our analysis. In such a situation, the spacetime Lorentz transformation equations are given by

$$x'^1 = x^1 \cosh \theta - x^0 \sinh \theta, x'^0 = x^0 \cosh \theta - x^1 \sinh \theta \tag{303}$$

so that, we have

$$R_1(\theta, x'^0) = R_1(0, x^0) \cosh \theta - x^0 \sinh \theta = (R_1 + V_1 x^0) \cosh \theta - x^0 \sinh \theta \tag{304}$$

where

$$x'^0 = x^0 \cosh \theta - R_1(0 + x^0) \sinh \theta = x^0 \cosh \theta - (R_1 + V_1 x^0) \sinh \theta \tag{305}$$

Taking the difference between eqs. (300) & (304) and using eq. (305)

$$\frac{\beta R_1}{\cosh\theta} + (V_1 - v)\, \beta[x^0 \cosh \theta - (R_1 + V_1 x^0)\sinh \theta]$$

$$- (R_1 + V_1 x^0) \cosh \theta + x^0 \sinh \theta = \beta R_1 \left(\frac{1}{\cosh\theta} - \sinh\theta (V_1 - v) - \beta^{-1} \cosh\theta \right)$$

$$+ \beta x^0 [(V_1 - v) \cosh \theta - (V_1 - v)V_1 \sinh \theta - \beta^{-1} V_1 \cosh \theta + \beta^{-1} \sinh \theta] = 0$$

where we have used $\beta = (1 - V_1 v)^{-1}$, $\cosh \theta = (1 - v^2)^{-1/2}$, $\sinh \theta = v (1 - v^2)^{-1/2}$. This establishes the consistency of the position operator with the Lorentz transformations.

2.19.4 Uniqueness of the Spin Operator

We now establish that the relativistic spin and position operators as defined in the preceding sections are unique. However, to do so, we need to introduce the concepts of "canonical form" and "power" of operators.

It is obvious that any operator constituted by the generators of the Poincare group viz. P_μ, J_i, K_l with, $\mu = 0, 1, 2, 3$, i, $l = 1, 2, 3$ can be expressed in several equivalent forms using the commutators between these generators. We define the canonical form as the form in which the constituent generators appear in the sequence $C(P_0, P_1, P_2, P_3)$, J_1, J_2, J_3, K_1, K_2, K_3 from left to right where $C(P_0, P_1, P_2, P_3)$ are the functions exclusively of the translation generators. Thus, every such operator admits a unique representation in the canonical form as

$$O = C^{00} + \sum_{i=1}^{3} C_i^{10}\, J_i + \sum_{i=1}^{3} C_i^{10}\, K_i + \sum_{i,j=1}^{3} C_i^{11}\, J_i K_j + \sum_{i,j=1}^{3} C_i^{20} J_i J_j + \sum_{i,j=1}^{3} C_i^{02} K_i K_j + \ldots \tag{306}$$

where $C(P_0, P_1, P_2, P_3)$ are functions of the translation generators.

Now, $[\mathbf{S}', P] = 0 \quad \Rightarrow \beta(M, P^2) = 0$. Further,

$$[S_1', S_2'] = [\alpha S_1 + \chi(\mathbf{S.P})P_1, \alpha S_2 + \chi(\mathbf{S.P})P_2]$$
$$= \alpha^2[S_1, S_2] - i\alpha\chi\, P_1(\mathbf{S} \times \mathbf{P})_2 + i\alpha\chi\, P_2\, (\mathbf{S} \times \mathbf{P})_1$$
$$= i\alpha^2\, S_3 - i\alpha\chi\, P_1(\mathbf{P} \times (\mathbf{S} \times \mathbf{P}))_3$$
$$= i\alpha^2\, S_3 - i\alpha\chi\mathbf{P}^2\, S_3 + i\alpha\chi(\mathbf{S.P})P = iS'_3$$
$$= i\alpha S_3 + i\chi(\mathbf{S.P})P_3$$

whence

$\alpha^2 - \alpha\chi\, \mathbf{P}^2 = \alpha$ & $\alpha\chi = \chi$ giving $\alpha = 1$ & $\chi = 0$ so that $\mathbf{S}' = \mathbf{S}$.

2.19.5 Uniqueness of the Position Operator

Let us assume that, in addition to the Newton-Wigner position operator, there exists another position operator having the requisite physical properties. In view of the commutator $[R_i, P_j] = i\delta_{ij}$ and the property that $Pow\,([A, B]) = Pow\,(A) + Pow\,(B) - 1$ with $Pow\,(P_j) = 0$, we have $Pow\,(R_j) = 1$. Additionally, the position operator must be a vector. The most general vector of power unity is

$$\mathbf{R}' = a(P^2, M)\mathbf{R} + b(P^2, M)\ \mathbf{S} \times \mathbf{P} + c(P^2, M)\mathbf{P} \tag{309}$$

Now,

$$0 = [R_1', S_2] = b[S_2\, P_3 - S_3\, P_2, S_2] = ibS_1S_2 \Rightarrow b = 0 \tag{310}$$

Also,

$$i = [R_1', P_1] = a[R_1, P_1] = ia \Rightarrow a = 1 \tag{311}$$

whence

$$\mathbf{R}' = \mathbf{R} + c\ (P^2, M)\mathbf{p} \tag{312}$$

To establish that $c\ (P^2, M) = 0$, we need to examine the consistency of eq. (310) with the transformations of special relativity by retracing the steps of eqs. (290)-(305). We, shall, however, for simplicity, consider a quantum particle moving along the x^1 axis. We also assume that the relative motion between the unprimed and the primed frames is also along the same axis so that the other spatial coordinates can be ignored in our analysis. In such a situation, the spacetime Lorentz transformation equations are given by eqs. (303)-(305). We also have, for the x^1 component of $\mathbf{R}'$

$$R_1'\ (\theta) = R_1(\theta) + ce^{-iK_1\theta}\, P_1 e^{-iK_1\theta} = \beta\frac{R_1}{\cosh\theta} + (P_1 \cosh\theta - P_0 \sinh\theta) \tag{313}$$

$$R_1'(\theta, x'^0) = e^{ip'_0x'^0}\, R_1'(\theta)e^{-iP'_0x'^0} = e^{iP'_0x'^0}\, [R_1(\theta) + ce^{-iK_1\theta}\, P_1 e^{-iK_1\theta}]\, e^{-iP'_0x'^0}$$
$$= (e^{-iK_1\theta}\, e^{-iP_0x'^0}\, e^{iK_1\theta})\, e^{-iK_1\theta}\, (R_1 + cP_1)\, e^{iK_1\theta}\, (e^{iK_1\theta}\, e^{-iP_0x'^0}\, e^{iK_1\theta})$$
$$= e^{-iK_1\theta}\, e^{iP_0x'^0}\, R_1\, e^{-iP_0x'^0}\, e^{iK_1\theta} + ce^{-iK_1\theta}\, e^{iP_0x'^0}\, P_1\, e^{-iP_0x'^0}\, e^{iK_1\theta}$$
$$= e^{-iK_1\theta}\, (R_1 + V_1\, x'^0)e^{iK_1\theta} + ce^{-iK_1\theta}\, P_1\, e^{iK_1\theta}$$
$$= R_1(\theta) + V_1(\theta)x'^0 + c(P_1 \cosh\theta - P_0 \sinh\theta)$$

$$= \beta\left[\frac{R_1}{\cosh\theta} + (V_1 - v)x'^0\right] + c(P_1 \cosh\theta - P_0 \sinh\theta) \tag{314}$$

Taking the difference between eqs. (314) & (304) and using eq. (305), we conclude that must necessarily equal zero in order that this difference vanishes and hence, implies the consistency of the position operator with special relativity.