# Chapter 2

# Signal Analysis

1.General problems in signal theory and communication:

Write down the expression for an amplitude modulated signal when the carrier frequency is $\omega_c$, the message signal is $m(t)$ and the modulation index is $\mu$. Determine the Fourier transform of this signal. Assuming that the message signal is wide sense stationary, calculate the time averaged autocorrelation function of the amplitude modulated signal.

Ans:

$$x(t) = A_c(1 + \mu.m(t)).cos(\omega_c t)$$

$$X(j\omega) = \pi.A_x(\delta(\omega - \omega_c) + \delta(\omega + \omega_c)) + \frac{\mu}{2}(M(j(\omega - \omega_c)) + M(j(\omega + \omega_c)))$$

For the second part,

$$< R_{xx}(t + \tau, t) >= \frac{A_c^2}{2}cos(\omega_c\tau) + A_c^2\mu^2 R_{mm}(\tau)cos(\omega_c\tau)$$

Note that

$$< R_{xx}(t + \tau, t) >= lim_{T\to\infty}T^{-1}\int_0^T R_{xx}(t + \tau, t)dt$$

2.Estimating the parameters of a nonlinear system using the LMS algorithm. Stochastic differential equation approach for. The true system has input-output equation

$$y(t) = g(x(t), \phi) + w(t)$$

where $\phi$ is a parameter to be estimated and $w(t) = \sigma.dB(t)/dt$ with $B(.)$ as standard Brownian motion. Let $\theta(t)$ be the parameter estimate at time $t$ based on the LMS algorithm. Then,

$$\theta(t + dt) - \theta(t) = -\mu.\frac{\partial}{\partial\theta(t)}(y(t) - g(x(t), \theta(t))^2$$

This proves $C(s)$ is a stoptime. We next wish to show that $A(t)$ is an $\mathcal{F}_{C(s)}$ stoptime for all $t$. To this end, we must show that $\{A(t) \le u\} \in \mathcal{F}_{C(u)}$, or equivalently,

$$\{A(t) \le u\} \cap \{C(u) \le s\} \in \mathcal{F}_s$$

To prove this, we shall first show that $A(t) = inf(s : C(s) > t)$. First note that $C(.)$ is right continuous and nondecreasing. To prove right continuity, we use the definition that

$$C(s+\epsilon) = inf(t : A(t) > s+\epsilon), C(s) = inf(t : A(t) > s)$$

This implies that

$$C(s+\epsilon) > C(s)$$

Physically, the first equation implies that it takes at least a time $C(s+\epsilon)$ for $A(t)$ to cross the level $s+\epsilon$, while the second equation implies that it takes at least a time $C(s)$ for $A(t)$ to cross the level $s$. Hence since $A(.)$ is non-decreasing, it follows that $C(s+\epsilon) \ge C(s)$ from which it follows that $C(s+) \ge C(s)$. On the other hand, let $\delta > 0$ be arbitrary. Then by definition of infimum, $A(C(s+\epsilon)-\delta) \le s+\epsilon$. Letting $\epsilon \to 0$ gives us $A(C(s+)-\delta) \le s$. It follows that $C(s+) - \delta < C(s) + 1/n$ for all $n = 1, 2, ...$ and letting $n \to \infty$ and then $\delta \to 0$ gives us $C(s+) \le C(s)$. This proves that $C(s+) = C(s)$, ie, $C(.)$ is a right continuous process. We shall now prove that $A(t) = inf(s : C(s) > t)$. If $C(s) > t$, then by definition, $A(t) \le s$. It follows that

$$A(t) \le inf\{s : C(s) > t\}$$

On the other hand, suppose $A(t) < s$. Then by definition of $C(s)$, we have that $C(s) > t$. It follows that

$$\{s : A(t) < s\} \subset \{s : C(s) > t\}$$

It follows that

$$A(t) = inf\{s : A(t) < s\} \ge inf\{s : C(s) > t\}$$

which proves the claim. Now we shall show that

$$\{A(t) \le u\} \cap \{C(u) \le s\} \in \mathcal{F}_s$$

Observe that $A(t) < u + 1/n$ implies that $C(u+1/n) > t$. This means that

$$\{A(t) \le u\} \subset \bigcap_{n=1}^{\infty} \{C(u+1/n) > t\}$$

On the other hand $C(u+1/n) > t$ for all $n$ implies that $A(t) \le u + 1/n$ and hence taking intersections, we get

$$\bigcap_{n=1}^{\infty} \{C(u+1/n) > t\} \subset \{A(t) \le u\}$$

We have thus proved that

$$\{A(t) \le u\} = \bigcap_{n=1}^{\infty} \{C(u+1/n) > t\}$$

Thus,

$$\{A(t) \leq u\} \cap \{C(u) \leq s\} = \bigcap_n \{C(u) \leq s\} \cap \{C(u+1/n) > t\}$$

If $t \leq s$ then since $C(x)$ is a stop time for all $x$, it follows that the right side is an element of $\mathcal{F}_s$. If $t > s$, then the right side is empty because $\bigcap_n \{C(u+1/n) > t\} \subset \bigcap_n \{C(u) \geq t\}$ by right continuity of $C(.)$. This completes the proof that $A(t)$ is an $\mathcal{F}_{C(s)}$ stoptime.

Ref:Daniel Revuz and Marc Yor; Continuous Martingales and Brownian Motion, Springer.

6. Least mean square algorithm applied to Frequency modulation. Assume that the FM wave is given by

$$x(t,\beta) = A.cos(\omega_c t + .\beta.sin(\omega_c t)) + w(t) = f(t,\beta) + w(t)$$

$w(t) = \sigma.dB(t)/dt$ with $B(.)$ as Brownian motion. The modulation index $\beta$ is assumed to be an unknown parameter to be estimated. The estimate at time $t$ is given by

$$\hat{x}(t,\beta(t)) = A.cos(\omega_c t + \beta(t).sin(\omega_c t))$$

The continuous time version of the LMS algorithm has the form

$$\beta(t+dt) = \beta(t) - \mu.dt.\frac{d}{d\beta(t)}(x(t,\beta) - \hat{x}(t,\beta(t)))^2$$

This gives

$$d\beta(t) = 2\mu.dt.(x(t,\beta) - \hat{x}(t,\beta(t))).\frac{\partial \hat{x}(t,\beta(t))}{\partial \beta(t)}$$

$$= 2\mu(f(t,\beta) - \hat{x}(t,\beta(t)) + \sigma)\frac{\partial \hat{x}(t,\beta(t))}{\partial \beta(t)} + 2\mu.\sigma.\frac{\partial \hat{x}(t,\beta(t))}{\partial \beta(t)}dB(t)$$

The convergence analysis proceeds by expanding $\hat{x}(t,\beta(t))$ around $\beta$ and deriving differential equations for $\mathbb{E}(\beta(t)-\beta)^m, m = 1,2,....$ For $m = 1,2$, this research work has been carried out by Dr.Tarun, colleague of the author.

7. Determine the impulse response of a second order filter defined by the transfer function

$$H(z) = \frac{1}{(1 - r.exp(i\alpha)z^{-1})(1 - r.exp(-i\alpha)z^{-1})}$$

Use the method of partial fractions.

8. Finite register effects in a second order system. The difference equation governing the system dynamics is

$$y[n] + ay[n-1] + by[n-2] = x[n]$$

Let $\epsilon_1[n]$ be the quantization noise induced by the first multiplier $a$ and $\epsilon_2[n]$ the quantization noise induced by the second multiplier $b$. Then if $f[n]$ is the ouput noise, we have

$$f[n] + af[n-1] + bf[n-2] = -(\epsilon_1[n] + \epsilon_2[n])$$

Assuming $\epsilon_i, i = 1,2$ to be independent white noise processes, with mean zero and variance $\sigma_e^2$, it follow that the output noise power is given by

$$\mathbb{E}f[n]^2 = 2\sigma_e^2 \sum_n h[n]^2 = \frac{\sigma_e^2}{\pi} \int_{-\pi}^{\pi} |H(j\omega)|^2 d\omega$$

where

$$H(j\omega) = \frac{1}{1 + az^{-1} + bz^{-2}}$$

Reference:Oppenheim and Schaffer, Digital Signal Processing.

9. Consider a two dimensional discrete space system defined by the difference equation

$$y[n,m] = a[1]y[n-1,m] + a_2 y[n,m-1] + a_3 y[n-1,m-1] + x[n,m]$$

Show that the two dimensional transfer function of this system is given by

$$H(\omega_1, \omega_2) = \frac{1}{1 - a[1]exp(-j\omega_1) - a[2]exp(-j\omega_2) - a[3]exp(-j(\omega_1 + \omega_2))}$$

Show that if each multiplier introduces a white quantization noise $\epsilon_i[n,m], i = 1,2,3$ each one have zero mean and covariance $\sigma^2\delta[n,m]$, then the quantization noise power in the output is given by

$$\sigma_f^2 = (3\sigma^2/(2\pi)^2) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H(\omega_1, \omega_2|^2 d\omega_1 d\omega_2$$

Show that we can write

$$\sigma_f^2 = (3\sigma^2) \sum_{n,m=-\infty}^{\infty} |h[n,m]|^2$$

Show that $h[n,m]$ is given by the coefficient of $exp(-j(\omega_1 n + \omega_2 m))$ in the expansion

$$\sum_{r=0}^{\infty} (a[1]exp(-j\omega_1) + a[2]exp(-j\omega_2) + a[3]exp(-j(\omega_1 + \omega_2)))^r$$

Using the trinomial theorem, determine a series expansion for $h[n,m]$.

10. Least mean phase algorithm. The input is a complex vector signal $\mathbf{x}(t)$ and the output is given by a linear combination of these signals plus noise, ie,

$$d(t) = \mathbf{w}^T \mathbf{x}(t) + n(t)$$

where $\mathbf{w}$ is a fixed complex weight and $n(t) = \sigma.dB(t)/dt$ with $B(.)$ as standard Brownian motion. The estimated signal at time $t$ is given by

$$\hat{d}(t) = \mathbf{w}(t)^T \mathbf{x}(t)$$

where $\mathbf{w}(t)$ is a complex time varying weight. We are interested only in obtaining the correct phase information about the output. So we match the phase of $d(t)$ with that of $\hat{d}(t)$ by the least mean square method. The continuous time version of the resulting algorithm, which we call the least mean phase algorithm reads

$$d\mathbf{w}(t) = -dt.\nabla_{\mathbf{w}(t)}(Arg(d(t)) - Arg(\hat{d}(t)))^2$$

Now observe that

$$Arg(\hat{d}(t)) = Arg(w(t)^T x(t)) = Im.log(w(t)^T x(t))$$

so that

$$\nabla_{w(t)} Arg(\hat{d}(t)) = Im(x(t)/w(t)^x(t)) = Im(x(t)/\hat{d}(t))$$

We can verify this formula directly too. Write

$$w(t) = (w_i(t)), w_i(t) = w_{i1}(t) + j.w_{i2}(t),$$

$$x(t) = (x_i(t)), x_i(t) = x_{i1}(t) + jx_{i2}(t)$$

so that

$$\hat{d}(t) = w(t)^T x(t) = \sum_i (w_{i1}x_{i1} - w_{i2}x_{i2}) + j\sum(w_{i1}x_{i2} + w_{i2}x_{i1})$$

so that

$$Arg(\hat{d}(t)) = tan^{-1}[(\sum_i (w_{i1}x_{i2} + w_{i2}x_{i1})/(\sum_i w_{i1}x_{i1} - x_{i2}x_{i2})] = tan^{-1}(\hat{d}_2(t)/\hat{d}_1(t))$$

$$\frac{\partial}{\partial w_{i1}} Arg(\hat{d}(t)) =$$

11. Phase LMS: Let $\mathbf{z}(t)$ be a complex vector valued signal and $d(t)$ a complex linear combination of $\mathbf{y}(t)$ plus noise. Thus, $d(t) = \mathbf{w}_0^T \mathbf{z}(t) + n(t)$. The estimate of the desired signal at time $t$ is given by $\hat{d}(t) = \mathbf{w}(t)^T \mathbf{z}(t)$. We are bothered only about phase information and so we wish to minimize $(Arg(d(t)) - Arg(\hat{d}(t)))^2$. The adaptive algorithm for doing this minimization reads

$$\mathbf{w}(t + dt) = \mathbf{w}(t) - \mu.dt.\nabla_{\mathbf{w}(t)}(Arg(d(t)) - Arg(\hat{d}(t)))^2$$

Now observe that

$$Argz = Im(logz)$$

Suppose $w, z$ are complex numbers. We let $w = a + ib$ and $z = x + iy$. Then, $wz = (ax - by) + i(ay + bx)$. It follows that

$$Arg(wz) = tan^{-1}[(ay + bx)/(ax - by)]$$

and hence

$$\frac{\partial}{\partial a} Arg(wz) = ((ax - by)y - (ay + bx)x)/((ax - by)^2 + (ay + bx)^2)$$

$$\frac{\partial}{\partial b} Arg(wz) = ((ax - by)x + (ay + bx)y)/((ax - by)^2 + (ay + x)^2)$$

Suppose on the other hand, $z_1, ..., z_n, w_1, ..., w_n$ are complex numbers and we wish to find

$$\frac{\partial}{\partial a_i} Arg(\sum_j w_j z_j), \frac{\partial}{\partial b_i}(\sum_j w_j z_j)$$

where

$$w_j = a_j + ib_j$$

Writing

$$z_j = x_j + iy_j$$

gives

$$\sum_j w_j z_j = \sum_j [(a_j x_j - b_j y_j) + i(a_j y_j + b_j x_j)]$$

so that

$$Arg(w^T z) = Arg(\sum_j w_j z_j) = tan^{-1}[(\sum_j (a_j y_j + b_j x_j))/(\sum_j (a_j x_j - b_j y_j))]$$

Thus,

$$\frac{\partial}{\partial a_i} Arg(\sum_j w_j z_j) = ((\sum_j (a_j x_j - b_j y_j))y_i - (\sum_j (a_j y_j + b_j x_j))x_i)/|\sum_j w_j z_j|^2$$

$$= [Re(w^T z)y_i - Im(w^T z)x_i]/|w^T z|^2$$

$$\frac{\partial}{\partial b_i} Arg(w^T z) =$$

$$= [Re(w^T z)x_i + Im(w^T z)y_i]/|w^T z|^2$$

so that

$$(\frac{\partial}{\partial a_i} + j.\frac{\partial}{\partial b_j})Arg(w^T z)$$

$$= [y_i w^T z + jx_i w^T z]/|w^T z|^2 = j\bar{z}_i.w^T z/|w^T z|^2$$

Note that the LMS update equations are

$$da_i(t) = -\mu.dt\frac{\partial}{\partial a_i(t)}(Arg(d(t)) - Arg(\hat{d}(t)))^2,$$

$$db_i(t) = -\mu.dt\frac{\partial}{\partial b_i(t)}(Arg(d(t)) - Arg(\hat{d}(t)))^2$$

or equivalently,

$$da_i(t) = 2\mu.dt\frac{\partial Arg(\hat{d}(t))}{\partial a_i(t)}(Arg(d(t) - Arg(\hat{d}(t))),$$

$$db_i(t) = 2\mu.dt.\frac{\partial Arg(\hat{d}(t))}{\partial b_i(t)}(Arg(d(t)) - Arg(\hat{d}(t)))$$

This gives us the LMS update equation on using $dw_i = da_i + jdb_i$, or $dw = da + jdb$,

$$dw(t) = 2\mu.dt(Arg(d(t)) - Arg(\hat{d}(t)))(j\bar{z}(t)w(t)^T z(t))/|w(t)^T z(t)|^2$$

or

$$dw = 2\mu.dt(Arg(d(t)) - Arg(\hat{d}(t)))(j\bar{z}(t)z(t)^T w(t))/|w(t)^T z(t)|^2$$

Now let

$$d(t) = w_0^T z(t) + \sigma.dB(t)/dt = w_i^T z(t) + n(t)$$

so that we have approximately,

$$logd(t) = log(w_0^T z) + log(1 + n(t)/w_0^T z) = log(w_0^T z) + n(t)/(w_0^T z)$$

Taking imaginary parts, we get

$$Arg(d(t)) \approx Arg(w_0^T z(t)) + n(t)Im((w_0^T z(t))^{-1})$$

and the phase LMS update equation becomes

$$dw(t) = 2\mu.dt(Arg(w_0^T z(t)) + \sigma.Im((w_0^T z(t))^{-1}).dB(t)/dt - Arg(w(t)^T d(t)))j\bar{z}(t)z(t)^T w(t)/|w(t)^T z(t)|^2$$

which can be cast in stochastic differential form

$$dw(t) = 2\mu.(dt.(Arg(w_0^T z(t)) - Arg(w(t)^T z(t))) + \sigma.Im((w_0^T z(t))^{-1}).dB(t))j\bar{z}(t)z(t)^T w(t)/|w(t)^T z(t)|^2$$

12.Define the notion of impulse respose of a linear discrete time time invariant system. Distinguish between the forms of the impulse responses of linear time varying and linear time invariant systems. Give examples illustrating this distinction.

13.Hilbert transform for discrete time signals has the transfer function $H(\omega) = -j.sgn(\omega)$ where $|\omega| \leq \pi$. Determine the impulse response of such a system. Determine the action of such a system on a sinusoidal signal $sin(\omega n + \phi)$.

14.Explain the overlap add method for computing the convolution between a finite duration sequence and an infinite duration sequence.

15. Explain how you would solve the wave equation in one dimension by a two dimensional discretization procedure. If the wave field is perturbed by white Gaussian noise so that it acquires the form

$$\psi_{tt}(t, x) - c^2\psi_{xx}(t, x) = F(t, x)$$

where $F(.,.)$ is white Gaussian noise, then by the discretization of the spatial variables, transform this equation into a system of linear stochastic differential equations. Hint:The discretized system has the form

$$\psi_{tt}(t, n) - \frac{c^2}{\Delta^2}(\psi(t, n) - 2\psi(t, n-1) + \psi(t, n-2)) = F(t, n), n = 1, 2, ..., N$$

16.Let $\mathcal{H}$ be a Hilbert space with onb $\{e_1, ..., e_n\}$. For $r \leq n$, let $\mathcal{H}^{\otimes_a n}$ denote the $r$-fold antisymmetric tensor product of $\mathcal{H}$ with itself. Let $\mathcal{H}^{\otimes r}$ denote the $r$-fold tensor product of $\mathcal{H}$ with itself. For any permutation $\sigma \in S_n$, define the operator

$$U_\sigma : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$$

by the equation

$$U_\sigma(x_1 \otimes x_2 \otimes .... \otimes x_n) = x_{\sigma 1} \otimes x_{\sigma 2} \otimes ... \otimes x_{\sigma n}$$

Define the operator

$$E : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes_a n}$$

by

$$E = \frac{1}{n!} \sum_{\sigma \in S_n} sgn(\sigma) U_\sigma$$

Show that $E$ is the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{H}^{\otimes_a n}$. Hint:First show that $E^* = E$, then show that $E^2 = E$, then complete the proof by showing that the range of $E$ is the entire $\mathcal{H}^{\otimes_a n}$. Observe that

$$< U_\sigma(x_1 \otimes ... \otimes x_n), (y_1 \otimes ... \otimes y_n) >=< x_{\sigma 1} \otimes ... \otimes x_{\sigma n}, y_1 \otimes ... \otimes y_n >$$

$$= Pi_{i=1}^n < x_{\sigma i}, y_i >= \Pi_{i=1}^n < x_i, y_{\sigma^{-1} i} >$$

$$=< x_1 \otimes ... \otimes x_n, U_{\sigma^{-1}}(y_1 \otimes ... \otimes y_n) >$$

proving that

$$U_\sigma^* = U_{\sigma^{-1}} = U_\sigma^{-1}$$

It follows that

$$E^* = \frac{1}{n!} \sum_{\sigma \in S^n} sgn(\sigma) U_{\sigma^{-1}} = E$$

Also

$$E^2 = \frac{1}{n!^2} \sum_{\sigma, \rho \in S_n} sgn(\sigma\rho) U_{\sigma\rho} = E$$

The last equation needs to be justified !. Replace the summation over the permutations $\sigma$ and $\rho$ by summation over the permutations $\sigma$ and $\tau = \sigma\rho$ and then obtain the desired result using

$$\sum_{\sigma \in S_n} 1 = n!$$

17.Baire's category theorem: Let $(X, d)$ be a complete metric space and let $A$ be a subset of $X$. If $int(Cl(A))$ is empty, we then say that $A$ is meager. Suppose $X = \bigcup_{k=1}^\infty A_k$ where $A_k$ is closed for every $k$. Then, for some $k$, we must have that $Int(A_k)$ is non-empty, ie, $A_k$ is non-meager. For suppose $Int(A_k)$ is empty for every $k$. We are assuming that $X$ is non-empty. Then $A_1 \neq X$, for if $A_1 = X$, then $Int(A_1) = X$ which is non-empty. So $A_1^c$ is a non-empty open subset of $X$. Let $B(x_1, \epsilon_1) \subset A_1$ for some $x_1 \in A_1^c$ and some $\epsilon_1 > 0$. This is possible since $A_1^c$ is non-empty and open. Then $B(x_1, \epsilon_1/2)$ cannot be contained in $A_2$ since by hypothesis, $A_2$ is non-empty. It follows that $B(x_1, \epsilon_1/2) \cap A_2^c$ is non-empty and open. Hence, there exists $x_2 \in X$ and $\epsilon_2 > 0$ such that

$$B(x_2, \epsilon_2) \subset B(x_1, \epsilon_1/2) \cap A_2^c$$

where summation over the repeated index $i$ is implied. Thus,

$$L_w L_v(f) = w^j (v^i f_{,i})_{,j} = w^j v^i_{,j} f_{,i} + w^j v^i f_{,ij}$$

Interchanging $w$ and $v$ gives us

$$L_v L_w(f) = v^j w^i_{,j} f_{,i} + v^j w^i f_{,ij}$$

Taking the difference, we get

$$(L_v L_w - L_w L_v)(f) = (v^j w^i_{,j} - w^j v^i_{,j}) f_{,i}$$

so that

$$L_v L_w - L_w L_v = L_u$$

where

$$u^i = v^j w^i_{,j} - w^j v^i_{,j}$$

summation over the repeated index $j$ being implied.

21.Modification of the LMS algorithm for estimating the amplitudes and phases of a harmonic signal. Assume that the harmonic signal has the form

$$x(t) = \sum_{i=1}^{n} A_i exp(j\omega_i t) + w(t)$$

where $w(t)$ is white noise, ie, a constant $\sigma$ times the derivative of Brownian motion. The estimated signal is given by

$$\hat{x}(t) = \sum_{i=1}^{n} B_i.exp(j\theta_i t)$$

We are assuming $A_i, B_i$ to be real numbers as also $\omega_i, \theta_i$. In fact, $B_i, \theta_i$ will be real valued functions of time. The objective is to apply the LMS algorithm to the square of the difference between the amplitudes and phases of the desired and estimated signals. That is, we define the parameter estimate evolutions by

$$dB_i(t) = -\mu.dt.\frac{\partial}{\partial B_i(t)}\{(|\hat{x}(t)|^2 - |x(t)|^2)^2 + \lambda.(Arg(\hat{x}(t)) - Arg(x(t)))^2\},$$

$$d\theta_i(t) = -\mu.dt.\frac{\partial}{\partial \theta_i(t)}\{(|\hat{x}(t)|^2 - |x(t)|^2)^2 + \lambda.(Arg(\hat{x}(t)) - Arg(x(t)))^2\}$$

According to these equations, we are splitting out the total error between the true signal and the estimated signal into an amplitude error component and a phase error component. If we wish to give more weightage to the amplitude information of the signal, then we make $\lambda$ very small while if we wish to given more weightage to the phase information of the signal, then we make $\lambda$ very large. For our sinusoidal processes, we have

$$|x(t)|^2 = \sum A_i^2 + 2\sum_{i<j} A_i A_j.cos((\omega_i - \omega_j)t),$$

$$|\hat{x}(t)|^2 = \sum B_i(t)^2 + 2\sum_{i<j} B_i(t)B_j(t).cos(\theta_i(t)t - \theta_j(t)t)$$

Then,

$$\frac{\partial}{\partial B_i}(|\hat{x}(t)|^2 - |x(t)|^2)^2) = 2(|\hat{x}(t)|^2 - |x(t)|^2)\frac{\partial}{\partial B_i}|\hat{x}(t)|^2$$

where

$$\frac{\partial}{\partial B_i}|\hat{x}(t)|^2 = 2B_i + 2\sum_{j\neq i} B_j cos((\omega_j - \theta_j)t)$$

Likewise,

$$Arg(x(t)) = tan^{-1}\frac{\sum_i A_i sin(\omega_i t)}{\sum_i A_i.cos(\omega_i t)}$$

$$Arg(\hat{x}(t)) = tan^{-1}\frac{\sum_i B_i sin(\theta_i t)}{\sum_i B_i cos(\theta_i t)}$$

$$\frac{\partial}{\partial B_i} = (\sum_j B_j cos(\theta_j t) sin(\theta_i t) - \sum_j B_j sin(\theta_j t) cos(\theta_i t))/|\hat{x}(t)|^2$$

$$= \sum_j B_j sin((\theta_i - \theta_j)t)/|\hat{x}(t)|^2$$

22. Metric space:Definition and properties of a metric (symmetry, positivity, triangle inequality), Examples of metric spaces: A finite set with the Hamming (zero one) metric. Distance between a set and a point, distance between two sets.

Completeness:Definition of a Cauchy sequence: $d(x_n, x_m) \to 0, asn, m \to \infty$. If every Cauchy sequence has a limit, we say that the metric space is complete.

Positive definite operators in a Hilbert space: $< Tx, x >\geq 0$ for all $x \in \mathcal{H}$. Then, we write $T \geq 0$. This implies that $T^* = T$, ie, $T$ is self-adjoint. If $T, S$ are self adjoint and $T - S \geq 0$, we write $T \geq S$.

Norm of an operator:

$$\| T \|= sup\{\| Tx \| / \| x \|, x \neq 0\}$$

23. Adjoint of a linear operator in a Hilbert space: $T : \mathcal{H} \to \mathcal{H}$ is a linear operator. Assume $T$ is bounded, ie, $\| T \|= sup(\| Tx \|:\| x \|\leq 1\} < \infty$. Then let

$$f_T(x, y) =< Tx, y >$$

For each $y$, $x \to f_T(x, y)$ is a bounded linear functional on the Hilbert space and hence by the Riesz representation theorem, there is a unique vector $T^*(y)$ such that

$$< Tx, y >=< x, T^*(y) >, \forall x \in \mathcal{H}$$

1.Phasor theory: If a linear circuit consists of voltage and current sources all at the same frequency $\omega$, then these sources can be written as $v_a(t) = V_a.cos(\omega t+\phi_a), a = 1, 2, ..., N$ where $\{V_a\}, \phi_a$ are real. The phasor notation for these voltages is $\mathbf{V}_a = V_a exp(i\phi_a)$. A resistance $R$ is retained as it is, a capacitance $C$ is replaced by the complex impedance $1/i\omega C$ since if $v(t) = V.cos(\omega t+\phi)$ is the voltage across it, then the current through it is given by $i(t) = C.v'(t) = -\omega.C.V.sin(\omega t + \phi) = \omega.C.V.cos(\omega t + \pi/2)$. The voltage across the capacitor has the phasor representation $V.exp(i\phi)$, while the current through the capacitor has the phasor representation $i\omega C.V.exp(i\phi)$ thereby justifying the representation of the capacitance. Likewise, an inductance $L$ has the impedance $i\omega L$. This follows from the fact that if $i(t) = I.cos(\omega t + \phi)$ is the current through it, then the voltage across it is given by $v(t) = L.i'(t) = -\omega LI.sin(\omega t + \phi)$. The current is then represented by the phasor $I.exp(i\phi)$ and the voltage by the phasor $i\omega L.I$. In this way, a linear circuit with passive time invariant elements excited by sinusoidal sources can be dealt with by purely algebraic methods based on complex algebra.

2. Phasor theory for electromagnetic wave propagation: In Maxwell's equations, the partial derivatives of the electric and magnetic fields with respect to time are replaced by multiplication with $i\omega$ assuming that the exciting charge density and current density have sinusoidal variation. Then the electric field and magnetic field at each point in space can be replaced by phasors and one can derive the Helmholtz equation for these fields:

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0$$

In case the fields are to be determined in a region where there is charge and current, the vector and scalar potential satisfy the Helmholtz equations with source terms:

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = s(\mathbf{x})$$

the solution to which is obtained as

$$\psi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}')s(\mathbf{x}')s^3x'$$

where

$$G(\mathbf{x}, \mathbf{x}') = -exp(-ik|\mathbf{x} - \mathbf{x}'|)/4\pi|\mathbf{x} - \mathbf{x}'|$$

In this fashion, an electromagnetic field with sinusoidally varying sources can be completely reduced to phasor analysis wherein only spatial variables are involved, the time variable disappears.

27. General quiz in signals and systems:

[1] Define a signal in terms of pattern of variations over a set. Show that if the set is equipped with the structure of a measure space, then one can successfully define signal energy and correlations but not power.

Note: Let $(X, \mathcal{F}, \mu)$ be a measure space. If $f : X \to \mathbb{R}$ is a non-negative measurable function, then we can define the integral $\int_X f(x)d\mu(x)$. Explain this part. Specifically, show that any nonnegative measurable function is a pointwise limit of simple nonnegative measurable functions and if $f$ is simple, its integral can be readily defined: $f(x) = \sum_{i=1}^{n} c_i\chi_{E_i}(x)$ where $E_i \in \mathcal{F}, i = 1, 2, ..., n$, then

$$\int f d\mu = \sum_{i=1}^{n} c_i\mu(E_i)$$

Then show that if $0 \leq f \leq g$ are simple measurable functions, then $\int f d\mu \leq \int g d\mu$. Hence, deduce that the integral $\int f d\mu$ of any nonnegative measurable function can be defined as $lim \int f_n d\mu$ where $f_n$ are nonnegative measurable and simple and increase pointwise to $f$. Show that the definition of the integral is independent of the approximating sequence. The energy of any measurable signal $f$ may now be defined as $E(f) = \int |f(x)|^2 d\mu(x)$. If $f, g$ are two measurable signals, their cross correlation can be defined as

$$R(f,g) = \int f(x)g(x)d\mu(x)$$

Show that the space $\mathcal{H}$ of all measurable signals $f$ defined on $X$ for which $\int |f(x)|^2 d\mu(x) < \infty$ is a Hilbert space provided that we define the inner product between $f, g \in \mathcal{H}$ as

$$< f, g >= \int f(x)g(x)d\mu(x)$$

Prove the Cauchy-Schwarz inequality

$$| < f, g > |^2 \leq < f, f >< g, g >, f, g \in \mathcal{H}$$

[2] Error correcting codes: Show that if $x = (x_1, ...., x_n)$ and $y = (y_1, ..., y_n)$ are two $n$-length binary sequences with the the distance between the two being defined by

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

and if $w$ is another $n$ length binary sequence with $d(w, 0) \leq t$ interpreted as noise, then if $d(x, y) > 2t$, we can design an error correcting scheme.

Hint: Let $x, y$ be two sequences with $d(x, y) > 2t$. The noise corrupted sequence is $z = x + w$ when $x$ is the transmitted signal and is $z = y + w$ when $y$ is the transmitted signal. If $d(z, x) \leq t$ then decide that $x$ was transmitted while if $d(z, y) \leq t$ decide that $y$ was transmitted. Then this scheme is a $t$ error correcting scheme. Note that the situation $d(z, x), d(z, y) \leq t$ cannot occur since that would imply

$$d(x,y) \leq d(z,x) + d(z,y) \leq 2t$$

which is false.

[3] Matched filter over a finite duration: Let $x(t) = s(t) + w(t), t \in [0, T]$ where $s(.)$ is the noiseless signal and $w(.)$ white Gaussian noise. Let $h(t), t \geq 0$ denote the impulse response of a causal linear time invariant filter. Show that the output at time $T$ can be expressed as

$$y(t) = \int_0^T h(\tau)x(T-\tau)d\tau = y_s(T) + y_w(T)$$

where

$$y_s(T) = \int_0^T h(\tau)s(T-\tau)d\tau, y_w(T) = \int_0^T h(\tau)w(T-\tau)d\tau$$

The signal power at the filter output at time $T$ is given by $P_s = |y_s(T)|^2$ and the noise power at the output at time $T$ by

$$P_w = \mathbb{E}[y_w(T)^2] = \int_0^T h(\tau)^2 d\tau$$

The signal to noise ratio (SNR) at the output at time $T$ is thus given by

$$SNR_o = \frac{P_s}{P_w} = \frac{(\int_0^T h(\tau)s(T-\tau)d\tau)^2}{\int_0^T h(\tau)^2 d\tau}$$

and application of the Cauchy-Schwarz inequality gives

$$SNR_o = \leq \int_0^T s(T-\tau)^2 d\tau = \int_0^T s(\tau)^2 d\tau$$

with equality iff $h(t) = K.s(T-t), t \in [0,T]$. This is the matched filter. In words, the condition for the SNR at the output to be a maximum at some time $T > 0$ is that the filter impulse response be matched to the input signal.

[4] Singularity functions occurring in quantum electrodynamics: Let $x^0 = t, x^1 = x, x^2 = y, x^3 = z$, so that the contravariant position four vector is given by $(x^\mu) = (t, x, y, z)$. The covariant position four vector is given by $(x_\mu) = (t, -x, -y, -z)$. The wave operator (denoted as $\Box$) is defined by

$$\Box = \partial^\mu \partial_\mu = \partial_t^2 - \nabla^2$$

We define the retarded Greens function $D_R(x)$ by the equation

$$\Box D_R(x) = \delta^4(x), D_R(x) = 0, x^0 < 0$$

Then it is easy to see that

$$D_R(x) = C.\theta(x)\delta(x^2)$$

where $\theta(x) = 1$ if $x^0 > 0$ and $\theta(x) = 0$ if $x^0 < 0$. To prove this, we first note that

$$\delta^4(x) = (2\pi)^{-4} \int exp(ik.x)d^4k$$

the integral being over the entire $\mathbb{R}^4$. We write

$$D_R(x) = \int A(k)exp(ik.x)d^4k$$

Substituting this into the pde gives

$$-\int k^2 A(k)exp(ik.x)d^4k = (2\pi)^{-4} \int exp(ik.x)d^4k$$

This gives

$$A(k) = -(2\pi)^{-4}/k^2 = -(2\pi)^{-4}/(k^{02} - |\mathbf{k}|^2)$$

where the subscript + in the integral sign means that the $k^0$ integral is the straight line passing below the two poles. Similarly, define

$$D_A(x) = \int_- exp(ik.x)d^4k/k^2$$

where the subscript − for the integral indicates that the integral is along a line passing just above the poles. Then a simple computation gives that

$$D_A(x) = r^{-1}\delta(t+r)$$

for $t < 0$ and zero for $t > 0$. Let now $\Gamma$ denote the contour which is a rectangle with infinite length along the $k^0$ axis and negligible breadth so that both the poles are enclosed by this contour. Then, after ignoring a proportionality constant, we have

$$D(x) = D_R(x) - D_A(x) = \int_\Gamma exp(ik.x)d^4k/k^2 = r^{-1}(\delta(t-r) - \delta(t+r))$$

We shall show that after ignoring a proportionality constant,

$$D(x) = \int exp(ik.x)\delta(k^2)\epsilon(k)d^4k$$

the integral being an ordinary real four dimensional integral. In fact, we have

$$\delta(k^2) = \frac{1}{2|\mathbf{k}|}(\delta(k^0 - |\mathbf{k}|) + \delta(k^0 + |\mathbf{k}|))$$

Define

$$\epsilon(x) = sgn(x^0)$$

Then,

$$\delta(k^2)\epsilon(k) = \frac{1}{2|\mathbf{k}|}(\delta(k^0 - |\mathbf{k}|) - \delta(k^0 + |\mathbf{k}|))$$

It follows that

$$\int exp(ik.x)\delta(k^2)\epsilon(k)d^4k = \int |\mathbf{k}|^{-1}(exp(i(|\mathbf{k}|x^0 - \mathbf{k.x})) - exp(-i(|\mathbf{k}|x^0 - \mathbf{k.x})))d^3k$$

which agrees with

$$\int_\Gamma exp(ik.x)d^4k/k^2$$

obtained using residue calculus to the two poles $\pm|\mathbf{k}|$ enclosed by the contour. We observe that

$$\Box D(x) = \Box \int_\Gamma exp(ik.x)d^4k/k^2 = \int_\Gamma exp(ik.x)d^4k = 0$$

since the integrand in the last expression on the left is an analytic function of $k^0$, or more specifically, it has no poles enclosed by $\Gamma$. Another way to infer this result is to use the other expression,

$$\Box D(x) = \Box \int exp(ik.x)\delta(k^2)\epsilon(k)d^4k = -\int exp(ik.x)k^2\delta(k^2)\epsilon(k)d^4k = 0$$

$|k(t,s)| \leq c < \infty$ for all $t,s \in [a,b]$. Let $v:[a,b] \to \mathbb{R}$ be continuous and $\mu$ a real number. We ask the question:When does the integral equation

$$x(t) - \mu \int_a^b k(t,s)x(s)ds = v(t), t \in [a,b]$$

have a unique solution for $x \in C[a,b]$ ? Define the operator $T : C[a,b] \to C[a,b]$ by

$$T(x)(t) = v(t) + \mu \int_a^b k(t,s)x(s)ds$$

Then if $x, y \in C[a,b]$, we have

$$|T(x)(t) - T(y)(t)| \leq |\mu| \int_a^b |k(t,s)||x(s) - y(s)|ds \leq c\mu(b-a)d(x,y)$$

or

$$d(T(x), T(y)) \leq \alpha.d(x,y), \alpha = c|\mu|(b-a)$$

where $d$ is the standard $L^\infty$-metric on $C[a,b]$, ie,

$$d(x,y) = sup_{t\in[a,b]}|x(t) - y(t)|$$

It follows from Banach's fixed point theorem that if $\alpha < 1$, $T$ has a unique fixed point or equivalently, the above integral equation has a unique solution. Note that the whole scheme works only because $C[a,b]$ is a Banach space, ie, a complete normed linear space. By the proof of Banach's fixed point theorem, the solution to $T(x) = x$ is given by $x = lim_{n\to\infty}T^n(y)$ where $y \in C[a,b]$ is arbitrary. Equivalently, we choose any $y \in C[a,b]$, and define $x_0 = y$,

$$x_{n+1}(t) = T(x_n)(t) = v(t) + \mu \int_a^b k(t,s)x_n(s)ds, n \geq 0$$

and then

$$x(t) = lim_{n\to\infty}x_n(t)$$

is the unique solution to the integral equation.

Statement and proof of Banach's fixed point theorem: Let $(X,d)$ be a complete metric space and $T : X \to X$ a map such that $d(T(x), T(y)) \leq \alpha.d(x,y)$ for all $x, y \in X$ where $0 < \alpha < 1$. Then, $T$ has a unique fixed point. More generally, let $T : X \to X$ be a map such that for some $N = 1,2,...$, we have $d(T^N(x), T^N(y)) \leq \alpha.d(x,y), \forall x,y \in X$ for some $\alpha \in (0,1)$. Then $T$ has a unique fixed point. For let $y \in X$ be arbitrary. Then we have seen that $x = lim_n T^{Nn}(y)$ is a fixed point of $T^N$. Now, since $T^{Nn}(x) = x$ for all $n = 1,2,...$, it follows that

$$Tx = TT^{Nn}(x) = T^{Nn}(Tx)$$

and taking limit $n \to \infty$ gives us

$$Tx = x$$

Thus, $x$ is also a fixed point of $T$. Suppose $Ty = y$ for some $y \in X$. Then, $T^N y = y$, ie $y$ is a fixed point of $T^N$. But by the uniqueness part of the Banach fixed point theorem, we then have that $y = x$. This proves that $T$ has a unique fixed point.

Volterra integral equation:

Let $X = C[a, b]$ and let

$$D = \{(t, s) : s \leq t, t, s \in [a, b]\}$$

Let $k : D \to \mathbb{R}$ be continuous and let $v \in C[a, b]$. Consider the integral equation

$$x(t) - \mu \int_a^t k(t, s)x(s)ds = v(t), t \in [a, b]$$

We want to know when this integral equation has a unique solution. Define the operator $T : C[a, b] \to C[a, b]$ by

$$T(x)(t) = x(t) - \mu \int_a^t k(t, s)x(s)ds$$

Then, for $x, y \in C[a, b]$, we have

$$|T(x)(t) - T(y)(t)| \leq |\mu|c \int_a^t |x(s) - y(s)|ds$$

where

$$c = sup\{|k(t, s)| : (t, s) \in D\}$$

Thus,

$$d(T(x), T(y)) \leq c|\mu|(b - a)d(x, y)$$

Again,

$$|T^2(x)(t) - T^2(y)(t)| \leq |\mu|c \int_a^t |T(x)(s) - T(y)(s)|ds \leq |\mu|^2 c^2 \int_a^t (t - s)|x(s) - y(s)|ds$$

$$\leq |\mu|^2 (c^2 (b - a)^2/2)d(x, y)$$

etc. In general, we get by iteration,

$$d(T^n(x), T^n(y)) \leq (|\mu|^n c^n (b - a)^n/n!)d(x, y)$$

It follows that for sufficiently large $N$, we have

$$d(T^N(x), T^N(y)) \leq \alpha.d(x, y), x, y \in C[a, b]$$

for some $\alpha \in (0, 1)$. From the above theorem, we get the result that $T$ has a unique fixed point in $C[a, b]$, no matter what the values of $\mu, c$ are. Thus, the Volterra integral equation has a unique solution.

29. Strain tensor, concept of(Landau and Lifshitz, elasticity): Let the point $\mathbf{x} = (x_1, .x_2, x_3)$ inside the elastic body be displaced to the point $\mathbf{x} + \mathbf{u}(\mathbf{x})$, in terms of components, $x_i + u_i$. The point $x_i + dx_i$

Linearize this system about a quiescent trajectory. Then we end up with a first order state variable system:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t)$$

where

$$\mathbf{x}(t) = [\delta q(t)^T, \delta p(t)^T]^T,$$

$$a_{ij}(t) = \frac{\partial^2 H(q(t),p(t))}{\partial p_i \partial q_j}, 1 \le i,j \le n,$$

$$a_{i,n+j}(t) = \frac{\partial^2 H(q(t),p(t))}{\partial p_i \partial q_j}, 1 \le i,j \le n,$$

$$a_{n+i,j}(t) = -\frac{\partial^2 H(q(t),p(t))}{\partial q_i \partial q_j}, 1 \le i,j \le n,$$

$$a_{n+i,n+j}(t) = -\frac{\partial^2 H(q(t),p(t))}{\partial q_i \partial p_j}, 1 \le i,j \le n$$

State variable equations with time varying matrix coefficients can be solved using the Dyson series, or equivalently the chronological product:

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0),$$

$$\Phi(t) = I + \sum_{k=1}^{\infty} \int_{0<t_k<t_{k-1}<...<t_2<t_1<t} A(t_1)A(t_2)...A(t_k)dt_k dt_{k-1}...dt_2 dt_1$$

Note that $\Phi(t)$ satisfies

$$\Phi'(t) = A(t)\Phi(t), t \ge 0, \Phi(0) = I$$

31. Chebyshev polynomials as used in antenna arrays

32. Define the three notions of convergence of a sequence of linear operators on a normed linear space.

33. Convolution and its generalization: Let two measurable functions $x, h$ on $\mathbb{R}$ be given. Assume that $h$ is integrable while $x$ is bounded, ie, $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ and $sup_{t\in\mathbb{R}}|x(t)| = M < \infty$. Then for each $t \in \mathbb{R}$, the signal $z_t : \mathbb{R} \to \mathbb{R}$ defined by $z_t(\tau) = h(\tau)x(t-\tau)$ is integrable. In fact, we have

$$\int_{-\infty}^{\infty} |z_t(\tau)|d\tau \le M \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

It follows that the signal $y : \mathbb{R} \to \mathbb{R}$ defined by

$$y(t) = \int_{-\infty}^{\infty} z_t(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

is well defined and bounded by

$$K = M \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

The signal $y(.)$ however need not be integrable. If however $x$ is also integrable (which is a stronger condition than being bounded), then $y(.)$ will be integrable. In fact, we have by Fubini's theorem

$$\int |y(t)|dt \leq \int |h(t-\tau)||x(\tau)|d\tau dt = \int_{-\infty}^{\infty} |h(t)|dt \int_{-\infty}^{\infty} |x(\tau)|d\tau$$

In this case (ie, the case in which both $x, h$ are integrable), the Fourier transforms of all the signals $x, h, y$ are well defined and we have

$$Y(j\omega) = H(j\omega).X(j\omega)$$

where

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)exp(-i\omega t)dt$$

34. Problem: Construct examples of signals $x, y$ such that $x \in L^1(\mathbb{R}) - L^2(\mathbb{R})$ while $y \in L^2(\mathbb{R}) - L^1(\mathbb{R})$.

Ans: Let $x(t) = t^{-1/2}.exp(-t), t > 0$ and $x(t) = 0$ if $t \leq 0$. Thios signal is integrable but not square integrable. Let $y(t) = t^{-1}$ if $t \geq 1$ and $y(t) = 0$ if $t < 1$. This signal is square integrable but not integrable.

35. Problem: Let $(X, \| . \|)$ be a normed linear space. Choose vectors $x, y_1, ..., y_n \in X$ and define a function $f : \mathbb{C}^n \to \mathbb{R}$ by the equation

$$f(\alpha_1, ..., \alpha_n) = \| x - \sum_{i=1}^{n} \alpha_i y_i \|$$

Show that $f$ is convex, ie, for any $\lambda \in [0, 1]$, we have

$$f(\lambda.\alpha + (1-\lambda).\beta) \geq \lambda.f(\alpha) + (1-\lambda)f(\beta), \alpha, \beta \in \mathbb{C}^n$$

Soln:

$$f(\lambda.\alpha + (1-\lambda).\beta) = \| x - \sum_{i=1}^{n} (\lambda.\alpha_i + (1-\lambda)\beta_i)y_i \|$$

$$= \| \lambda(x - \sum \alpha_i y_i) + (1-\lambda)(x - \sum \beta_i y_i) \|$$

$$\leq \lambda. \| x - \sum \alpha_i y_i \| + (1-\lambda) \| x - \sum \beta_i y_i \|$$

$$= \lambda.f(\alpha) + (1-\lambda)f(\beta)$$

proving the claim.

36. Problem: Let $(X, \| . \|)$ be a normed linear space. Let

$$B = \{x \in X : \| x \| \leq 1\}$$

the closed unit ball in $X$. Show that $X$ is finite dimensional iff $B$ is compact as a metric space.

Soln: Suppose $X$ is infinite dimensional. Choose any vector $e_1 \in X$ such that $\| e_1 \|= 1$. Then $sp(e_1) \neq X$ implies by the Riesz theorem that there is an $e_2 \in X$ such that $\| e_2 \|= 1$ with $d(e_2, sp(e_1)) > 1/2$. Again since $sp(e_1, e_2) \neq X$, by Riesz theorem, there is an $e_3 \in X$ with $\| e_3 \|= 1$ such that $d(e_3, sp(e_1, e_2)) > 1/2$. Continuing this process, it follows that there is an infinite sequence $\{e_1, e_2, ...\} \subset X$ such that $\| e_i \|= 1$ for all $i$ such that $d(e_{i+1}, sp(e_1, ..., e_i)) > 1/2$ for every $i = 1, 2, ....$ It follows in particular that for any $i \neq j$, $\| e_i - e_j \|> 1/2$ and hence no subsequence of $\{e_i : i = 1, 2, ...\}$ can converge in $X$. It follows that $B$ is not compact. Conversely, suppose $X$ is finite dimensional. Let $x_i, i = 1, 2, ...$ be any infinite sequence in $B$. Choose a basis $\{e_1, ..., e_N\}$ for $X$. We can write $x_i = \sum_{j=1}^N a(i, j)x_j$ for some complex scalars $a(i, j)$. It follows from the equivalence of all norms on a finite dimensional normed linear space that there exist constants $c, c' > 0$ such that

$$c.\sum_{j=1}^{N} |a(i,j)| \leq\| x_i \|\leq c' \sum_{j=1}^{N} |a(i,j)|$$

for all $i = 1, 2, ....$ In particular, $\{(a(i, 1), a(i, 2), ..., a(i, N)) : i = 1, 2, ...\}$ is a bounded sequence in $\mathbb{R}^N$ and hence by the Bolzano Weierstrass property has a convergent subsequence, say $(a(n_i, 1), a(n_i, 2), ..., a(n_i, N)), i = 1, 2, ....$ Let $(a(1), ..., a(N))$ be the limit of this subsequence. Then, define

$$x = \sum_{i=1}^{N} a(i)x_i$$

We have

$$\| x - x_{n_m} \|\leq \sum_{i=1}^{N} |a(i) - a(n_m, i)| \to 0, m \to \infty, m \to \infty$$

Since $\| x_i \|\leq 1$, it follows then that $\| x \|\leq 1$ and $x_{n_m} \to x$. This proves that $B$ is compact. The proof is complete.

37. Problem: Let $(X, d)$ be a compact metric space and $f : X \to \mathbb{R}$ a continuous function. Show that $f$ attains its supremum at some point $x_0 \in X$.

Ans: Let $M = sup(f(x) : x \in X)$. Then by definition of supremum, there is a sequence $x_n \in X, n = 1, 2, ...$ such that $f(x_n) \to M$. Since $X$ is compact, there is a convergent subsequence $(x_{n_m} : m = 1, 2, ...)$ of $(x_n)$, say $x_{n_m} \to x_0, m \to \infty$. Then by continuity of $f$, we get

$$M = limf(x_n) = limf(x_{n_m}) = f(limx_{n_m}) = f(x_0)$$

38. Problem:Let $X$ be a normed linear space and let $x \in X$. Let $Y$ be a subspace of $X$ and define $B = \{y \in Y :\| y \|\leq 2 \| x \|\}$. Let $\delta = inf(\| x - y \|: y \in Y)$. Since $0 \in Y$, it follows that $\| x \|\geq \delta$. If $y \notin B$, then $\| y \|> 2 \| x \|$ and then

$$\| x - y \|\geq\| y \| - \| x \|>\| x \|\geq \delta$$

It follows that

$$\delta = d(x, B) = inf(\| x - y \|: y \in B)$$

3. Linear superposition: Given vectors $v_1, ..., v_k$ in a vector space $V$ and scalars $c_1, ..., c_n$ in the underlying field, we define the linear superposition

$$v = \sum_{i=1}^{n} c_i v_i$$

If the $v_i's$ are linearly independent, then this combination can be the zero vector when and only when all the $c_i's$ are zero. If further, all vectors $v \in V$ are covered by such a superposition by varying the $c_i's$, then the $v_i's$ are said to form a basis for the vector space $V$. $n$ is then the dimension of the vector space. It can be shown that every basis has the same number of elements and hence the dimension of the vector space is uniquely determined.

4. Linear transformations on a vector space:Let $V, W$ be vector spaces over the same field $\mathbb{F}$. A mapping $T : V \to W$ is called a linear transformation if for all $x, y \in V$ and all scalars $c_x, c_y \in \mathbb{F}$, one has

$$T(c_x x + c_y y) = c_x T(x) + c_y T(y)$$

41.

1.Let $X$ be a normed linear space and $T : X \to X$ a compact linear operator. Then for all $\lambda \neq 0$, we have that $\mathcal{R}(T - \lambda.I)$ is closed.

2.Let $X$ be a normed linear space and $T : X \to X$ a compact linear operator, ie, if $B$ is the open unit ball in $X$, then $ClT(B)$ is compact as a metric space. Show that compactness of $T$ implies boundedness of $T$.

Ans:Suppose $B$ is unbounded. Then there exists a sequence $x_n, n = 1, 2, ...$ such that $x_n \in B$ but $T(x_n), n = 1, 2, ...$ has the property:$\| Tx_n \| > n, n = 1, 2, ....$ It follows that $\{Tx_n\}$ cannot have any convergent subsequence, for if $Tx_n \to y, n \to \infty$, then for all sufficiently large $n$, we must have

$$\| Tx_n - y \| \leq 1$$

and hence

$$\| Tx_n \| \leq 1 + \| y \|$$

which is a contradiction. This means that $T$ cannot be compact and we are done.

3.Let $X, Y$ be the normed spaces and $T : X \to Y$ be compact, ie, if $B$ is the unit open ball in $X$, then $ClT(B)$ is compact in $Y$. Then, show that if $M$ is any bounded subset of $X$, $ClT(M)$ is compact in $Y$. For let $M \subset X$ be such that for all $x \in M$ we have $\| x \| < K < \infty$ where $K$ is fixed. Then, $K^{-1}M \subset B$ and hence,

$$K^{-1}ClT(M) = ClT(K^{-1}M) \subset ClT(B)$$

By definition of compactness of $T$, $ClT(B)$ is compact and hence $K^{-1}ClT(M)$ is compact (A closed subset of a compact metric space is compact). It follows that $ClT(M)$ is compact. We are done and this furnishes us with an alternative definition of compactness: If $X, Y$ are normed linear spaces and

$T : X \to Y$ is a linear map, then $T$ is said to be compact if $ClT(M)$ is compact for every bounded set $M \subset X$.

4. Let $X_1, X_2, X_3, X_4$ be normed linear spaces and $T_1 : X_1 \to X_2, T_2 : X_2 \to X_3, T_3 : X_3 \to X_4$ be bounded linear operators. If $T_2$ is compact, then show that $T_3T_2T_1$ is compact. For let $x_n, n = 1, 2, ...$ be any sequence in $X_1$ such that $\| x_n \| < 1$ for all $n$. Then, $\{T_1(x_n)\}$ is bounded since $T_1$ is bounded. It follows from the definition of compactness that $\{T_2T_1(x_n)\}$ has a convergent subsequence, say $T_2T_1(x_{n_k})$. Since $T_3$ is bounded, it is also continuous and hence $T_3T_2T_1(x_{n_k})$ is a convergent sequence in $X_3$. It follows that $T_3T_2T_1$ is compact and we are done.

5. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator and $T^*$ the adjoint of $T$. Then $T$ is compact iff $T^*T$ is compact. For let $T$ be compact and $\{x_n\}$ a bounded sequence. Then, $Tx_n$ has a convergent subsequence say $Tx_{n_k}$. $T^*$ is bounded since for all $x, y \in \mathcal{H}$, we have

$$| < T^*x, y > | = | < x, Ty > | \leq \| T \| \| x \| \| y \|$$

and hence

$$\| T^*x \|^2 \leq \| T \| \| T^*x \| \| x \|$$

or equivalently,

$$\| T^* \| \leq \| T \|$$

Hence, $T^*Tx_{n_k}$ is convergent proving compactness of $T^*T$. Conversely suppose $T^*T$ is compact. Let $x_n$ be a bounded sequence and let $T^*Tx_{n_k} \to x$. For simplicity of notation, set $y_k = x_{n_k}$. Then

$$\| Ty_n - Ty_m \|^2 = < Ty_n, Ty_n > + < Ty_m, Ty_m > - < Ty_n, Ty_m > - < Ty_m, Ty_n >$$

Now,

$$| < Ty_n, Ty_n > - < x, y_n > | = | < T^*Ty_n - x, y_n > | \leq \| T^*Ty_n - x \|$$

Likewise,

$$| < Ty_m, Ty_m > - < x, y_m > | = | < T^*Ty_m - x, y_m > | \leq \| T^*Ty_m - x \|$$

$$| < Ty_n, Ty_m > - < x, y_m > | = | < T^*Ty_n - x, y_m > | \leq \| T^*Ty_n - x \|$$

$$| < Ty_m, Ty_n > - < x, y_n > | = | < T^*Ty_m - x, y_n > | \leq \| T^*Ty_m - x \|$$

so combining all these we get

$$\| Ty_n - Ty_m \| \leq 2 \| T^*Ty_n - x \| + 2 \| T^*Ty_m - x \|$$

proving that $\{Ty_n\}$ is Cauchy in $\mathcal{H}$ and hence convergent.

6. Notion of spectrum of an operator: Let $X$ be a normed linear space and $T : X \to X$ a bounded linear operator. For any $\lambda \in \mathbb{C}$, we define $T_\lambda = T - \lambda.I$. Consider three properties: (1) $T_\lambda$ is not one-one, (2) $R(T_\lambda)$ is not dense in $X$, (3) $T_\lambda^{-1}$ is not bounded. Suppose (1) holds, then we say that $\lambda$ belongs to the point spectrum of $T$. Suppose (1) does not hold but (2) holds. Then we say that $\lambda$ belongs to the residual spectrum of $T$. Finally, suppose (1) and (2) do not hold but (3) holds. Then, we say that $\lambda$ belongs to the continuous spectrum of $T$. We denote the point spectrum by $\sigma_p(T)$, the continuous spectrum by $\sigma_c(T)$ and the residual spectrum by $\sigma_r(T)$. The spectrum of $T$ is the set

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

Closed graph theorem: Suppose $X, Y$ are Banach spaces and $T : \mathcal{D}(T) \to Y$ is a closed linear operator. If $\mathcal{D}(T)$ is closed in $X$, then $T$ is bounded.

Proof: $X \times Y$ under the above norm and usual addition and scalar multiplication, is a Banach space. For suppose $(x_n, y_n)$ is Cauchy in $X \times Y$. Then

$$\| x_n - x_m \| + \| y_n - y_m \| \to 0, n, m \to \infty$$

Hence, $x_n$ is Cauchy in $X$ and $y_n$ Cauchy in $Y$. Hence, $x_n \to x, y_n \to y$ for some $(x, y) \in X \times Y$. It follows that

$$\| (x_n, y_n) - (x, y) \| = \| x_n - x \| + \| y_n - y \| \to 0$$

proving completeness of $X \times Y$. since $Gr(T)$ is closed in $X \times Y$ and $\mathcal{D}(T)$ is closed in $X$, it follows that $Gr(T)$ and $\mathcal{D}(T)$ are Banach spaces. Consider the map

$$P : Gr(T) \to \mathcal{D}(T)$$

defined by

$$(x, Tx) \to x$$

This map is linear. Further, $P$ is bounded because

$$\| Px \| = \| x \| \leq \| x \| + \| Tx \| = \| (x, Tx) \|$$

Further, $P$ is bijective with the inverse map given by

$$P^{-1}(x) = (x, Tx)$$

The bounded inverse theorem (corollary of the open mapping theorem) then implies that $P^{-1}$ is bounded. Thus,

$$\| Tx \| \leq \| x \| + \| Tx \| = \| P^{-1}x \| \leq \| P^{-1} \| . \| x \|$$

proving boundedness of $T$.

Theorem: Let $X, Y$ be normed spaces and $T : \mathcal{D}(T) \to Y$ a linear operator. Then, $T$ is closed iff $x_n \to x$ and $Tx_n \to y$ imply $x \in \mathcal{D}(T)$ and $y = Tx$.

Proof:First assume that $T$ is closed. Let $x_n \to x$ and $Tx_n \to y$. Then, $(x_n, Tx_n) \to (x, y)$. Since $Gr(T)$ is closed and $(x_n, Tx_n) \in Gr(T)$, it follows that $(x, y) \in Gr(T)$, ie, $x \in \mathcal{D}(T)$ and $y = Tx$ and we are through with the only if part. Conversely, suppose $x_n \to x, Tx_n \to y$ imply $x \in \mathcal{D}(T)$ and $y = Tx$. We must show that $Gr(T)$ is closed. So suppose $z_n \in Gr(T)$ and $z_n \to z$. Then, $z_n = (x_n, Tx_n)$ for some $x_n \in \mathcal{D}(T)$. Write $z = (x, y)$. Then we have by hypothesis,

$$0 = lim \| z_n - z \| = \| x_n - x \| + \| Tx_n - y \|$$

and hence

$$x_n \to x, Tx_n \to y$$

It follows by hypothesis that $x \in \mathcal{D}(T)$ and $y = Tx$, ie, $(x, y) \in Gr(T)$ completing the proof of the if part.

42.Rajeev's Ph.d problem: Consider the wave equation with random forcing. The motion corresponds to a string stretched between the points $x = 0$ and $x = L$. The displacement at time $t$ at the point $x$, $u(t, x)$ satisfies the one dimensional wave equation with random forcing $F(t, x)$:

$$u_{tt}(t,x) - u_{xx}(t,x) = F(t,x)$$

We expand $u, F$ as a half wave Fourier series:

$$u(t,x) = \sum_{n=1}^{\infty} c_n(t).sin(n\pi x/L), F(t,x) = \sum_{n=1}^{\infty} f_n(t).sin(n\pi x/L)$$

Then the wave equation gives

$$c_n''(t) + (n\pi/L)^2 c_n(t) = f_n(t), n = 1, 2, ...$$

The energy of the string at time $t$ can be expressed as

$$E(t) = \frac{\lambda}{2}\int_0^L u_x(t,x)^2 dx + \frac{1}{2}\int_0^L u_t(t,x)^2 dx$$

where $\lambda$ is a constant related to the Young's modulus of the string. We express this energy in terms of the Fourier coefficients:

$$u_x = \sum c_n (n\pi/L) cos(n\pi x/L)$$

$$\int_0^L u_x^2 dx = \frac{L}{2}\sum (n\pi/L)^2 c_n^2 = \sum_n c_n^2 (n^2\pi^2/2L)$$

$$u_t = \sum_n c_n'.sin(n\pi x/L)$$

$$\int_0^L u_t^2 dx = \frac{L}{2}\sum c_n'^2$$

so that

$$E(t) = \sum_n c_n^2 (n^2\pi^2/4L) + c_n'^2 L/4$$

We shall use the simplified notation

$$E(t) = A\sum_n n^2 c_n(t)^2 + B\sum_n c_n(t)'^2$$

where

$$A = \pi^2/4L, B = L/4$$

Assume that the Fourier series has been truncated to $N$ coefficients. We assume that $f_n(t) = \sigma_n dB_n(t)/dt$, where $B_n's$ are standard Brownian motion processes independent of each other. Then the equations of motion are

$$dc_n(t) = v_n(t)dt, dv_n(t) = \sigma_n dB_n(t) - (n\pi/L)^2 c_n(t)dt$$

Thus, the solution to the above state variable equations is given by

$$(c_n(t), v_n(t))^T = \int_0^t exp((t-\tau)A_n)(0,\sigma_n)^T dB_n(\tau)$$

$$= \sigma_n \int_0^t (\omega_n^{-1} sin(\omega_n(t-\tau)), cos(\omega_n(t-\tau)))dB_n(\tau)$$

or equivalently,

$$c_n(t) = \sigma_n \omega_n^{-1} \int_0^t sin(\omega_n(t-\tau))dB_n(\tau),$$

$$v_n(t) = \sigma_n \int_0^t cos(\omega_n(t-\tau))dB_n(\tau)$$

The expression for $v_n$ may also have been obtained directly from the formula

$$v_n(t) = c_n'(t) = \sigma_n \int_0^t h_n'(t-\tau)dB_n(\tau)$$

We can write

$$u_t(t,x) = \sum_n c_n'(t) sin(n\pi x/L) = \sum_n \sigma_n.sin(n\pi x/L) \int_0^t cos(\omega_n(t-\tau))dB_n(\tau)$$

$$= \sum_{n=1}^N \int_0^t F_n(t-\tau, x)dB_n(\tau)$$

where

$$F_n(t,x) = \sigma_n.cos(\omega_n t) sin(\omega_n x)$$

Note that

$$K_n(t,x) = \sigma_n.\omega_n^{-1}.sin(\omega_n t) sin(\omega_n x)$$

We are now in a position to calculate (1) space-time correlations in the displacement and velocity field of the vibrating string, mean energy and energy fluctuations of the string.

43. Application of Laurent Schwarz and Sobolev's theory of distributions to signal theory. $\Omega$ is an open subset of $\mathbb{R}^n$ and $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions on $\Omega$ having compact support. More precisely, for any compact subset $K$ of $\Omega$ $\mathcal{D}_K$ is the space of infinitely differentiable functions on $K$ and $\mathcal{D}(\Omega)$ is the union of all the $\mathcal{D}_K's$. The support of any function $f$ on a topological space is the closure of the set $\{x : f(x) \neq 0\}$. Choose compact sets $K_i, i = 1, 2, ...$ such that $K_i \subset int(K_{i+1})$ for all $i$. For $f \in C^\infty(\Omega)$ and $N = 1, 2, ...,$ define

$$p_N(f) = max(|D^\alpha f(x)| : x \in K_N, |\alpha| \leq N\}$$

These seminorms define a metrizable locally convex topology on $C^\infty(\Omega)$. We can show that $\{p_N : N = 1, 2, ...\}$ is a separating family of seminorms on $C^\infty(\Omega)$. Indeed, suppose $f \neq 0$. Then there exists $x \in \Omega$

such that $f(x) \neq 0$. Choose $N$ such that $x \in K_N$ and then $p_N(f) > |f(x)| > 0$ proving the claim. If $f, g \in \Omega$ and $f \neq g$, then the argument is applied to $f - g$ resulting in an $N$ such that $p_N(f - g) > 0$.

Result: Suppose $X$ is a vector space and $\mathcal{P}$ is a separating family of seminorms. For each $p \in \mathcal{P}$, define the set

$$V(p, n) = \{x : p(x) < 1/n\}$$

Let $\mathcal{B}$ be the collection of all finite intersections of sets of the form $V(p, n)$. Say that a set $A \subset X$ is open iff $A$ is a union of translates of members of $\mathcal{B}$. Then we get a topology for $X$. $\mathcal{B}$ is a local base for this topology. Each element of $\mathcal{B}$ is convex and balanced. To prove convexity, suppose $x, y \in V(p_1, n_1) \cap ... \cap V(p_k, n_k)$. Then, $p_j(x) < 1/n_j, p_j(y) < 1/n_j$, so that if $a \in [0, 1]$, then

$$p_j(ax + (1 - a)y) \leq ap_j(x) + (1 - a)p_j(y) < 1/n_j$$

proving that

$$ax + (1 - a)y \in V(p_1, n_1) \cap ... \cap V(p_j, n_j)$$

This proves convexity of the elements of $\mathcal{B}$. Suppose

$$x \in \bigcap_{j=1}^{k} V(p_j, n_j)$$

and $|a| \leq 1$, then

$$p_j(ax) = |a|p_j(x) < |a|/n_j \leq 1/n_j$$

proving that

$$ax \in \bigcap_{j=1}^{k} V(p_j, n_j)$$

Using this result applied to $C^\infty(\Omega)$, this becomes a locally convex topological vector space. Further this space is metrizable since in the above result, we may define

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} p_i(x - y)/(1 + p_i(x - y))$$

Then $d$ is a metric. The fact that $d(x, y) > 0$ if $x \neq y$ follows from the fact that $\{p_i\}$ is a separating family of seminorms. If $K$ is a compact subset of $\Omega$, then $\mathcal{D}_K$, the set of all $f \in C^\infty(\Omega)$ with support $K$ is closed in $C^\infty(\Omega)$. This follows from the fact that for any $x \in \Omega$, the map $f \to f(x)$ from $C^\infty(\Omega)$ into $\mathbb{C}$ is continuous. Indeed, suppose $f_n \to f$ in $C^\infty(\Omega)$. Let $x \in \Omega$ be arbitrary and choose $N$ sufficiently large so that $x \in K_N$. Then since $f_n \to f$, it follows that $p_N(f_n - f) \to 0$ and since

$$p_N(f_n - f) = max(|D^\alpha(f_n - f)(u)| : u \in K_N, |\alpha| \leq N) \geq |f_n(x) - f(x)|$$

it follows that

$$f_n(x) \to f(x), n \to \infty$$

46. Consider two kernels $K, H$ on $\mathbb{R}^n$. Show that if $f$ is defined on $\mathbb{R}^n$, then

$$K * (H * f) = H * (K * f)$$

what is the system theoretic interpretation of this identity.

47. Show that the system of Maxwell equations with charge density and current density as input and the electric and magnetic fields as output define a linear shift invariant multiinput-multioutput system for four dimensional signals. Determine the impulse response and transfer function of this system. Hint:Start with the retarded potential formula for the electric potential $V(t, \mathbf{x})$ and magnetic vector potential $\mathbf{A}(t, \mathbf{x})$ in terms of the charge and current density:

$$V(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(t - |\mathbf{x} - \mathbf{y}|/c, \mathbf{y}) d^3y / |\mathbf{x} - \mathbf{y}|$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(t - |\mathbf{x} - \mathbf{y}|/c, \mathbf{y}) d^4y / |\mathbf{x} - \mathbf{y}|$$

Show using these formulae that

$$V(t, \mathbf{x}) = H(t, \mathbf{x}) * \rho(t, \mathbf{x}), \mathbf{A}(t, \mathbf{x}) = \mu_0\epsilon_0 H(t, \mathbf{x}) * \mathbf{J}(t, \mathbf{x})$$

where

$$H(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \delta(t - |\mathbf{x}|/c)/|\mathbf{x}|$$

Show that if $\hat{V}, \hat{\mathbf{A}}$ are respectively the four dimensional Fourier transforms of $V, \mathbf{A}$, then

$$\hat{V}(k) = \hat{H}(k)\hat{\rho}(k)$$

with

$$\hat{H}(k) = \int H(t, \mathbf{x}) exp(-i(k_0 t - \mathbf{k}.\mathbf{x})) dt d^3x, k = (k_0, \mathbf{k})$$

Note that

$$\hat{V}(k) = \int V(t, \mathbf{x}) exp(-i(k_0 t - \mathbf{k}.\mathbf{x})) dt d^3x, \hat{\rho}(k) = \int \rho(t, \mathbf{x}) exp(-i(k_0 t - \mathbf{k}.\mathbf{x})) dt d^3x$$

Determine the explicit formula for $\hat{H}(k)$. This is easier to do if we start directly from the wave equation satisfied by $V, \mathbf{A}$:

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\rho(t, \mathbf{x})/\epsilon_0$$

Taking Fourier transforms, we get taking $c = 1$,

$$\hat{V}(k)(-\mathbf{k}^2 + k_0^2) = -\hat{\rho}(k)/\epsilon_0$$

so that

$$H(k) = -\frac{1}{\epsilon_0(k_0^2 - \mathbf{k}^2)}$$

We use the shorthand notation

$$k.k = k^2 = k_0^2 - \mathbf{k}^2$$

so that

$$H(k) = -\frac{1}{\epsilon_0 k^2}$$

Explain how you would determine $h(t, \mathbf{x})$ using $H(k)$ by choosing the contour for $k_0$ appropriately.

48. DSP implementation of the creation and annihilation operators for bosons.

49. Minkowski functional on a topological vector space (construction of seminorms)

Let $X$ be a vector space and $A \subset X$ an absorbing convex subset. By absorbing, we mean that given any $x \in X$, we have $x \in tA$ for some $t = t(x) > 0$. If $X$ is a topological vector space, then we know that every nhood of zero is absorbing. In fact, we know that if $V$ is a nhood of zero and $r_n$ is a sequence of positive reals increasing to infinity, then $X = \bigcup_n r_n V$. It is easy to see that $A$ must contain the zero vector. For by definition of absorbing, $0 \in tA$ for some $t > 0$. it follows that $0 = ty$ for some $y \in A$. Since $t > 0$, it follows that $y = 0$ and hence $0 \in A$.

Then define

$$\mu_A(x) = inf(t > 0 : t^{-1}x \in A\} = \inf(t > 0 : x \in tA)$$

Then, $\mu_A : X \to [0, \infty)$ is a well defined map. First, we want to show that

$$\mu_A(x+y) \leq \mu_A(x) + \mu_A(y), x, y \in X$$

For $x \in X$, define

$$H_A(x) = \{t > 0 : t^{-1}x \in A\} = \{t > 0 : x \in tA\}$$

Obviously

$$\mu_A(x) = inf(t : t \in H_A(x))$$

Let $t \in H_A(x)$ and $s > t$. Since $0 \in A$ and $A$ is convex, it follows that $s \in H_A(x)$. Here is the proof. $t \in H_A(x)$ implies $t^{-1}x \in A$. Since $0 \leq t/s < 1$ and $A$ is convex, we get

$$s^{-1}x = (t/s)(t^{-1}x) + (1 - t/s)0 \in A$$

and hence $s \in H_A(x)$. It follows that $H_A(x) = [\mu_A(x), \infty)$. Let $\mu_A(x) < s, \mu_A(y) < t$ and $u = s + t$. Then, $s^{-1}x \in A, t^{-1}y \in A$ and since $A$ is convex,

$$u^{-1}(x+y) = (s/u)s^{-1}x + (t/u)t^{-1}y \in A$$

so that

$$\mu_A(x+y) \leq u$$

Setting $s = \mu_A(x) + \delta, t = \mu_A(y) + \delta$ with $\delta > 0$, it follows then that

$$\mu_A(x+y) \leq \mu_A(x) + \mu_A(y) + 2\delta$$

Further, scalar multiplication $(c,a) \to c.a$ from $\mathbb{C} \times A \to A$ satisfies $c(a_1 a_2) = (ca_1)a_2 = a_1(ca_2)$ where $c \in \mathbb{C}, a_1, a_2 \in A$. An example of a non-commutative algebra is the space of all $2 \times 2$ complex matrices with addition and multiplication being specified by matrix addition and matrix multiplication. A Banach algebra is an algebra with a norm such that under this norm $A$ becomes a complex normed linear space and in addition, the norm function satisfies

$$\| a_1 a_2 \| \leq \| a_1 \| . \| a_2 \|, a_1, a_2 \in A$$

An example of a commutative Banach algebra is the space of all complex valued functions on a set $X$ with norm defined by

$$\| f \| = sup(|f(x)| : x \in X), f \in A$$

where

$$A = \{f : X \to \mathbb{C}\}$$

We verify that

$$\| f + g \| =$$

$$sup(|f(x) + g(x)| : x \in X) \leq sup(|f(x)| + |g(x)| : x \in X) \leq sup(|f(x)| : x \in X) + sup(|g(x)| : x \in X)$$

$$= \| f \| + \| g \|$$

We have

$$|f(x)| \leq \| f \|, |g(x)| \leq \| g \|, x \in X$$

so that

$$|f(x)g(x)| = |f(x)||g(x)| \leq \| f \| \| g \|$$

and taking sup over all $x$ gives

$$\| fg \| \leq \| f \| \| g \|$$

51. Wavelet based system identification with advantage as lesser data storage.

Consider the problem of identification of parameters of a second order Volterra system using wavelets. The input-output equation for the system is given by

$$y(t) \approx \int_{-\infty}^{\infty} h(u,\theta)x(t-u)du + \int_{-\infty}^{\infty} g(u,v,\theta)x(t-u)x(t-v)dudv$$

Here, $\theta = (\theta_1, ..., \theta_p)$ is the parameter vector to be identified. For each value of $\theta$ in a finite set say $\Omega = \{\theta_i, i = 1, 2, ..., p\}$, we calculate the wavelet coefficients $c(n,k|\theta)$ and $d(n,k,m,r|\theta)$ of respectively of the first order kernel $h(u,\theta)$ and the second order kernel $g(u,v,\theta)$ ,ie, if

$$\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k), n, k \in \mathbb{Z}$$

are the wavelet functions, then

$$h(u,\theta) = \sum_{n,k} c(n,k|\theta)\psi_{n,k}(u), g(u,v,\theta) = \sum d(n,k,m,r|\theta)\psi_{n,k}(u)\psi_{m,r}(v)$$

Example: Let $X = \mathbb{R}^n$ with its usual topology, Then, let

$$B(\delta) = (-\delta, \delta)^n$$

with $\delta > 0$. It is easy to see that $\{B(\delta) : \delta > 0\}$ is a local base for $X$. Another example of a local base for $\mathbb{R}^n$ is the collection of all sets of the form

$$S(\delta) = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 < \delta\}$$

Suppose $X$ is a normed vector space. Then, a local base for the topology is the set of all open spheres with zero as centre.

[c] If $X$ is a topological vector space such that there exists a local base $\mathcal{B}$, such that every element of $\mathcal{B}$ is convex, then $X$ is said to be a locally convex topological vector space. A set $E \subset X$ is convex if for every $x, y \in E$ and $0 < a < 1$, we have $ax + (1-a)y \in E$. For example, let $X$ be a normed vector space and let $\mathcal{B}$ be the local base consisting of all open spheres with origin as the centre. Then, every element of $\mathcal{B}$ is convex since $x, y \in S(\delta)$ and $0 < p < 1$ imply

$$\| px + (1-p)y \| \leq p \| x \| + (1-p) \| y \| < p\delta + (1-p)\delta = \delta$$

Thus, every normed space is a locally convex topological vector space.

[d] $X$ is called a locally compact space if 0 has a nhood whose closure is compact. For example, suppose $X$ is a finite dimensional normed linear space. Then it is one of the important theorems of functional analysis that the closed unit ball is compact in $X$ . Hence, every finite dimensional normed linear space is locally compact.

[e] Let $X$ be a topological vector space and let $K, C$ be disjoint subsets of $X$ with $K$ compact and $C$ closed. Then there is a nhood $V$ of 0 such that

$$(K + V) \cap (C + V) = \phi$$

Choose $x \in K$. Since $C^c$ is open and $x \in C^c$, we can choose a symmetric nhood $V_x$ of zero such that

$$V_x + V_x + V_x \subset C^c - x$$

or

$$x + V_x + V_x + V_x \subset C^c$$

It follows that $(x + V_x + V_x) \cap (C + V_x) = \phi$ (since $-V_x = V_x$). $\{x + V_x : x \in K\}$ is an open covering for $K$, hence there is a finite subcovering $\{x_i + V_{x_i} : i = 1, 2, ..., n\}$. Put $V = \bigcap_{i=1}^n V_{x_i}$. Then,

$$K + V \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \subset \bigcup_{i} (x_i + V_{x_i} + V_{x_i})$$

Since

$$(x_i + V_{x_i} + V_{x_i}) \cap (C + V) = \phi$$

it follows that

$$(K+V)\cap(C+V)=\phi$$

and we are done.

Remark:If $W$ is an open nhood of 0, then since vector addition is continuous, there exists a nhood $U$ of 0 such that $U+U\subset W$. Then define $V=U\cap(-U)$. Then, $V$ is a symmetric nhood of zero and $V+V\subset W$. By replacing $W$ with $V$ in this argument, there exists a symmetric nhood $S$ of zero such that $S+S+S+S\subset W$. Let now $x\in X$ be arbitrary and let $W$ be a nhood of $x$. Then $W-x$ is a nhood of zero and hence there exists a symmetric nhood $U$ of zero such that $U+U+U+U\subset W-x$. This implies in particular that $x+U+U+U\subset W$, a fact that has been made use of in the above proof.

Topology: Let $X$ be a topological space and $E\subset X$. $int(E)$ is the set of all $x\in E$ for which there exists an open nhood $U$ such that $x\in U\subset X$. In other words,

$$int(E)=\bigcup\{U:U open, U\subset E\}$$

A subset $E$ of $X$ is said to be nowhere dense if $int(\bar{E})=\phi$. In that case, we may show that $\bar{E}^c$ is dense in $X$. For any subset $E$, $int(\bar{E})\subset\bar{E}$ implies $\bar{E}^c\subset int(\bar{E})^c$. it follows that

$$Cl(\bar{E}^c)\subset int(\bar{E})^c$$

since $int(\bar{E})^c$ is closed. Conversely, suppose $x\notin Cl(\bar{E}^c)$. Then, since $Cl(\bar{E}^c)$ is closed, its complement is open and hence there exists an open set $U$ such that $x\in U\subset Cl(\bar{E}^c)^c$. Thus,

$$U\cap Cl(\bar{E}^c)=\phi$$

Hence,

$$U\cap\bar{E}^c=\phi$$

and hence

$$U\subset\bar{E}$$

Since $U$ is open, it follows that

$$U\subset int(\bar{E})$$

and hence

$$x\notin int(\bar{E})^c$$

Thus we have proved that for any subset $E$ of $X$, we have

$$Cl(\bar{E}^c)=int(\bar{E})^c$$

It follows that if $int(\bar{E})=\phi$, then $Cl(\bar{E}^c)=X$, or equivalently that $\bar{E}^c$ is dense in $X$.

Let $A$ be a commutative Banach algebra and $J$ a proper closed ideal in $A$. Define the quotient map $\pi:A\rightarrow A/J$ so that $\pi(x)=x+J, x\in A$. Define

$$\|\pi(x)\|=inf\{\|x+y\|:y\in J\}$$

Then $A/J$ equipped with this norm is a Banach space.

Proof: First we must show that the norm is well defined. Let $\pi(x) = \pi(y)$. Then, $z = y - x \in J$. Since then $z + J = J$, it easily follows that

$$inf(\| x + u \|: u \in J) = inf(\| y + z + u \|: u \in J) = inf(\| y + u \|: u \in J)$$

showing that $\| \pi(x) \|$ is well defined for all $x \in A$. Now, let $x, y \in A$. Let $\delta > 0$ be arbitrary. Choose $u, v \in J$ so that $\| x + u \| \leq \| \pi(x) \| + \delta$ and $\| y + v \| \leq \pi(y) + \delta$. Then $u + v \in J$ and

$$\| \pi(x) + \pi(y) \| = \| \pi(x + y) \| \leq \| x + y + u + v \| \leq \| x + u \| + \| y + v \| \leq \| \pi(x) \| + \| \pi(y) \| + 2\delta$$

and letting $\delta \downarrow 0$ gives the triangle inequality:

$$\| \pi(x) + \pi(y) \| \leq \| \pi(x) \| + \| \pi(y) \|$$

Now for the Completeness property. Let $x_n, n = 1, 2, ...$ be a sequence in $A$ such that $\pi(x_n), n = 1, 2, ...$ is Cauchy in $A/J$. This means that $\| \pi(x_n) - \pi(x_m) \| \to 0, n, m \to \infty$. It follows that there is a subsequence $n_i$ of $\mathbb{N}$ such that $\| \pi(x_{n_{i+1}}) - \pi(x_{n_i}) \| < 1/2^i$. In particular, $\| \pi(x_{n_2}) - \pi(x_{n_1}) \| < 1/2$. So we can choose a $u_1 \in J$ so that

$$\| x_{n_2} - x_{n_1} + u_2 \| < 1/2$$

Since $\pi(x_{n_2} + u_2) = \pi(x_{n_2})$, we can choose $u_3 \in J$ so that

$$\| x_{n_3} - x_{n_2} - u_2 + u_3 \| < 1/2^2$$

and continuing this process, we get a sequence $u_i, i = 1, 2, ...$ in $J$ such that

$$\| x_{n_{i+1}} + u_{i+1} - x_{n_i} - u_i \| < 1/2^i, i = 1, 2, ...$$

In other words, $x_{n_i} + u_i, i = 1, 2, ...$ is Cauchy in $A$. It follows that there exists a $y \in A$ such that $x_{n_i} + u_i \to y$ as $i \to \infty$. Since

$$\| \pi(x_{n_i}) - \pi(y) \| \leq \| x_{n_i} + u_i - y \|$$

it follows that $\pi(x_{n_i}) \to \pi(y)$ in the normed space $A/J$. Since $\pi(x_n)$ is Cauchy, it follows then that $\pi(x_n) \to \pi(y)$ proving that $A/J$ is a Banach space.

53. Let $X$ be a topological vector space and $Y$ a locally compact subspace of $X$. Then, $Y$ is closed in $X$.

1.Define a bounded linear functional on a normed linear space $X$. Show that $\mathcal{M}$ is a closed subspace of $X$ and $f$ is a linear functional on $M$ that satisfies $|f(x)| \leq \| x \|$ for all $x \in M$, then there exists a linear functional $g$ on $X$ such that $g = f$ on $M$ and $|g(x)| \leq \| x \|$ for all $x \in X$. Hint:Use the Hahn-Banach theorem. Deduce that if $x \in X$, then there exists an $f \in X^*$ ($X^*$ is the class of all bounded linear functionals on $X$), such that $\| f \| = \| x \|$

2. Define a convex subset of a vector space and give examples. Specifically show that if $f_1, ..., f_k$ are linear functionals on a vector space $X$ and $c_1, ..., c_k$ are arbitrary scalars, then the set

$$\{x \in X : f_i(x) \leq c_i, i = 1, 2, ..., k\}$$

is the error signal.

55. Invariant functions: Let $X$ be a set on which a group $G$ acts and let $\mathcal{F}(X)$ denote the set of all functions $f : X \to \mathbb{C}$. $G$ acts on $\mathcal{F}(X)$ in the natural way:

$$(g.f)(x) = f(g^{-1}x)$$

A function $f \in \mathcal{F}(X)$ is said to be $G$-invariant if $g.f = f$. Note that if $f_1, f_2 \in \mathcal{F}(X)$ and $c_1, c_2 \in \mathbb{C}$, then

$$(g.(c_1 f_1 + c_2 f_2))(x) = c_1 f_1(g^{-1}x) + c_2 f_2(g^{-1}x) = (c_1 g_1.f + c_2 g_2.f)(x)$$

so that each $g \in G$ acts linearly on the algebra $\mathcal{F}(X)$. Also each $g \in G$ preserves the multiplication operation in $\mathcal{F}(X)$ as may be seen from

$$(g.(f_1 f_2))(x) = (f_1 f_2)(g^{-1}x) = f_1(g^{-1}x) f_2(g^{-1}x) = ((g.f_1).(g.f_2))(x)$$

The set of $G$-invariant functions on $X$ forms an algebra as can be seen from $g \in G$, $g.f_1 = f_1, g.f_2 = f_2$ implies $g.(c_1 f_1 + c_2 f_2) = c_1 g.f_1 + c_2 g.f_2 = c_1 f_1 + c_2 f_2$ and $g.(f_1 f_2) = (g.f_1)(g.f_2) = f_1 f_2$.

56. Bilinear forms:Let $V$ be a vector space over $\mathbb{R}$ of dimension $n < \infty$ and let $f : V \times V \to \mathbb{R}$ be a symmetric non-degenerate bilinear form, ie, $f$ is linear in both of its arguments, $f(u, v) = f(v, u)$ for all $u, v \in V$ and $f(u, v) = 0$ for all $v \in V$ iff $u = 0$. Choose a basis $B = \{e_1, ..., e_n\}$ for $V$ and consider the matrix

$$A = ((f(e_i, e_j))$$

$A$ is a real Hermitian matrix and hence, by the spectral theorem, it can be represented as

$$A = UDU^T$$

where $U$ is a real orthogonal matrix and $D$ is a real diagonal matrix. Let $x, y \in V$ and set $x = \sum x_i e_i, u = \sum y_i e_i$ so that $x_i, y_i \in \mathbb{R}$. Then,

$$f(x, y) = \sum_{i,j} x_i y_j f(e_i, e_j) = x^T A y = [x]_B^T U D U^T [y]_B$$

There exists a basis $B'$ of $V$, such that $[x]_{B'} = U^T [x]_B$ for all $x \in B$. To see this, suppose $B' = \{f_1, ..., f_n\}$ so that $x = \sum x'_i f_i$ with $[x]_{B'} = (x'_i)$, then

$$\sum x_j e_j = x = \sum x'_i f_i = \sum (U^T [x]_B)_i f_i = \sum u_{ji} x_j f_i$$

so that the new basis is given by

$$\sum_i u_{ji} f_i = e_j$$

or

$$f_i = \sum_j u_{ji} e_j$$

using orthogonality of $U$. Thus,

$$f(x,y) = [x]_{B'}^T D[y]_{B'}$$

which means that

$$f(f_i, f_j) = d_i \delta_{ij}$$

By applying a permutation to the basis elements $(f_i)$, we may assume without loss of generality that $d_i > 0$ for $t = 1, 2, ..., r$, and $< 0$ for $i = r+1, ..., n$. Note that since we are assuming that $f$ is non-degenerate, no $d_i$ equals zero. Then letting $g_i = |d_i|^{-1/2} f_i$, it follows that $((f(g_i, g_j)))$ has the diagonal form

$$C = \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}$$

This is the canonical form of a symmetric, non-degenerate real bilinear form. Now we shall derive the canonical form for the antisymmetric bilinear form. Let $V$ be a real vector space and $f : V \times V \to \mathbb{R}$ be an antisymmetric, non-degenerate bilinear form. By antisymmetric, we mean that $f(u,v) = -f(v,u)$ for all $u, v \in V$. Then choose a basis $B = (e_i)$ as before and define $A = ((f(e_i, e_j))$. $A$ is then a real antisymmetric matrix and $iA$ is a Hermitian matrix. We can diagonalize $iA$ relative to an orthonormal basis for $\mathbb{C}^n$. Thus, we choose vectors $\xi_1, ..., \xi_n$ in $\mathbb{C}^n$ so that

$$iA\xi_k = c_k \xi_k, k = 1, 2, ..., n$$

$c_k' s$ are all real and $\xi_k$ are complex vectors. If The eigenvalues $c_k$ appear in pairs as $(c, -c)$ for taking the complex conjugate gives us $-iA\bar{\xi}_k = \bar{c}_k \bar{\xi}_k$, None of the eigenvalues is zero since the form is non-degenerate. Writing $\xi_k = u_k + iv_k$ where $u_k, v_k$ are real, we get

$$iA(u_k + iv_k) = c_k(u_k + iv_k)$$

and equating real and imaginary parts, we get

$$Au_k = c_k v_k, Av_k = -c_k u_k$$

Thus, $sp\{u_k, v_k\}$ is an invariant subspace for $V$ and the restriction of $A$ relative to this space is

$$\begin{pmatrix} 0 & -c_k \\ c_k & 0 \end{pmatrix}$$

By appropriately permuting the elements $u_k, v_k$, and applying a scaling, we may obtain a real basis for $V$ relative to which the form has the matrix

$$\begin{pmatrix} ) & I_r \\ -I_r & 0 \end{pmatrix}$$

where $r = 2n$. This means that a non-degenerate bilinear skew symmetric form on a real vector space exists iff the space has even dimension.

57. Let $G$ be a group and let $H, K$ be subgroups of $G$. Consider the $G$ action on the set $(G/H) \times (G/K)$. This action is defined by the equation

$$g(g_1 H, g_2 K) = (gg_1 H, gg_2 K), g, g_1, g_2 \in G$$

If $x(t)$ is the received signal, the likelihood ratio for performing the decision is given by

$$\Lambda(x) = exp(-\int_0^T (x(t) - s_1(t))^2/N_0)/exp(-\int_0^T (x(t) - s_0(t))^2/N_0)$$

If $\Lambda$ is greater than 1, we decide in favour of $H_1$ while if $\Lambda$ is smaller than one, we decide in favour of $H_0$. Here, we are assuming the two hypotheses to be equally probable apriori. The likelihood ratio test can be simplified by defining

$$l(x) = \int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt$$

If $l(x) > 0$, decide $H_1$ while if $l(x) < 0$, decide $H_0$. Define two impulse response functions $h_i(t) = s_i(T-t), i = 1, 0$. Then

$$\int_0^T x(t)s_i(t)dt = \int_0^T h_i(\tau)x(T-\tau)d\tau = \int_0^T h_i(T-\tau)x(\tau)d\tau, i = 1, 2$$

So the decision process can be viewed as the construction of two filters having impulse responses $h_i, i = 1, 0$. We pass the received signal over the duration $[0, T]$ through each of these filters and select the maximum. To implement this using MATLAB, we do the following. Generate two discrete time signals $s_i[n], n = 0, 1, 2, ..., N-1$. For example, we can take $s_i$ to be one and minus one, or one and zero. We also generate white Gaussian noise $w[n], n = 0, 1, ..., N-1$ using the randn command followed by an amplitude scaling. Then select $i = 0, 1$ at random and generate $x[n] = s_i[n] + w[n], n = 0, 1, ..., N-1$. Using this $x$ generate the numbers $\sum_{n=0}^{N-1} x[n]s_i[n], i = 1, 0$ and choose the maximum. Repeat this process $K$ times for the same $s_i's$ but different realizations of the noise $w[.]$ and count the proportion of times that an error has occurred. Plot this error propagability as a function of the signal to noise ratio $\sum s_i[n]^2/\mathbb{E}w[n]^2$. Here we are assuming that $s_i's$ have been amplitude scaled appropriately so that

$$\sum s_1[n]^2 = \sum s_0[n]^2$$

59. Implement an adaptive equalizer based on LMS algorithm and study the effect of step size on the MSE.

Choose $p << N$ real numbers (filter tap weights) $h(0), h(1), ..., h(p)$. Generate an input sequence $x(n), n = 0, 1, 2, ..., N-1$. Generate white Gaussian noise $w[n], n = 0, 1, ..., N-1$ and generate the output process

$$y[n] = \sum_{k=0}^{p} h(k)x(n-k) + w(n), n = p, p+1, ..., N$$

Choose an initial guess for the tap weights $g_0[k], k = 0, 1, 2, ..., p$ and generate the tap weights in accord with the LMS algorithm:

$$g_{n+1}[k] = g_n[k] - \mu\frac{\partial}{\partial g_n[k]}(y[n] - \sum_{m=0}^{p} g_n[k]x[n-k])^2$$

or

$$g_{n+1}[k]=g_n[k]+2\mu e[n]x[n-k], k=0,1,...,p$$

where

$$e[n]=y[n]-\sum_{m=0}^{p} g_n[k]x[n-k]$$

is the error at time $n$. $\mu$ is the adaptation weight. The aim is to show that this algorithm for sufficiently small $\mu$ guarantees convergence of the weights $g_n[k]$ to $h[k]$ and also guarantees good tracking in case the $h's$ vary slowly with time. This gives an implementation of an adaptive system identifier. For an adaptive equalizer, we pass the input through an fir filter $\{h(n)\}$ and then letting $y(n)$ denote the output, we apply the above algorithm taking as input $y(n)$ and output $x(n)$. The resulting adaptive filter $\{g_n(k)\}$ is then expected to converge to an inverse of the filter $\{h(k)\}$. We may take $h(n)$ as an IIR filter of the form

$$H(z)=\frac{1}{1-az^{-1}}$$

The process $y(n)$ is thus generated by the algorithm

$$z(n)+az(n-1)=x(n), y(n)=z(n)+w(n)$$

The adaptive filter output is

$$\hat{x}(n)=\sum_{k=0}^{p} g_n(k)y(n-k)$$

so that

$$g_{n+1}(k)=g_n(k)+2\mu e(n)y(n-k), e(n)=x(n)-\sum_{k=0}^{p} g_n(k)y(n-k)$$

If the noise is small, we then expect

$$G_n(z)\approx 1-az^{-1}$$

for large $n$. Check this.

60. Monte carlo simulation of a QPSK communication system. Generate a sequence of ones and zeroes. Pair this sequence so that we get a string of elements of the form $(ij), i, j=0,1$. To transmit a (00) we use the signal $cos(\omega_c n)$, to transmit (01), we transmit $cos(\omega_c n+2\pi/4)$, to transmit (10), we transmit $cos(\omega_c n+\pi/2)$ and finally to transmit (11), we transmit $cos(\omega_c n+3\pi/4)$. We add white Gaussian noise $w(n)$ to the transmitted signal with a variance of 1/10. Denote the resulting received signal by $x(n), n=0,1,2,...,N-1$. Take $\omega_c=2\pi/100$ and $N=100$, or more generally, $\omega_c=2\pi/N$ so that the transmitted signal consists of one complete cycle of the carrier signal. To decode at the receiver end, we use the correlation method. Specifically compute

$$((2/N)\sum_{n=0}^{N-1} x(n)cos(\omega_c n), (2/N)\sum_{n=0}^{N-1} x(n)sin(\omega_c n))$$

Compute the distance of this pair from the four points $(1,0)$ , $(cos(2\pi/4), -sin(2\pi/4))$, $(cos(\pi/2)\ -sin(\pi/2))$ and $(cos(3\pi/4), -sin(3\pi/4))$ and if the minimum distance is to the first pair, then decide (00), if it is to

62. Commutative Banach algebras:Let $A$ be a commutative Banach algebra and $\Delta$ the set of all complex homomorphisms of $A$. If $M$ is a maximal ideal of $A$, then $M = kerh$ for some homomorphism $h$ of $A$ and conversely if $h$ is a homomorphism of $A$, then $Kerh$ is a maximal ideal of $A$.

Lemma: If $A$ is a Banach algebra in which every nonzero element is invertible, then $A$ is isometrically isomorphic to the complex field.

For choose any two distinct complex numbers $\lambda_i, i = 1, 2$. Then at most one of the elements $\lambda_i e - x, i = 1, 2$ is zero and hence at least one of them is invertible. Since the spectrum $\sigma(x)$ is non-empty, it follows that $\sigma(x)$ has exactly one point say $\lambda(x)$ for each $x \in A$. $\lambda(x)e - x$ is not invertible and hence it is zero. It is easy to see that $\lambda$ is a linear map from $A$ into $\mathbb{C}$ and that

$$|\lambda(x)| = \| \lambda(x)e \| = \| x \|$$

proving isometry.

Note:We have used an important result that the spectrum of any element in a Banach algebra is nonempty. For suppose $x \in A$ where $A$ is a Banach algebra. Define $g : \mathbb{C} \to A$ by $g(\lambda) = \lambda.e - x$. $g$ is a continuous function. However the set $G(A)$ of invertible elements in $A$ is an open set (Let $y$ be invertible. Then $z = y - (y - z)$ gives

$$z^{-1} = (y - (y - z))^{-1} = (1 - y^{-1}(y - z))^{-1} y^{-1}$$

exists if $\| y^{-1}(y - z) \| < 1$ which is in turn true if $\| y - z \| < \| y^{-1} \|^{-1}$. Note that $\| y \| \| y^{-1} \| \geq 1$. Thus, we have shown that $z^{-1}$ exists for all $z$ inside an open ball of sufficiently small radius with $y$ as centre.). It follows that $g^{-1}(G(A))$ is an open set. But $g^{-1}(G(A)) = \sigma(x)^c$, ie, the resolvent set of $x$. The resolvent set is open and hence the spectrum is closed. We now show that the spectrum is bounded. For let $x$ be an element of the Banach algebra and let $\lambda$ be a complex number with $|\lambda| > \| x \|$. Then, $e - \lambda^{-1}x$ is invertible and hence $\lambda \in \sigma(x)^c$. It follows that if $\lambda \in \sigma(x)$, then $|\lambda| \leq \| x \|$. This proves that the spectrum of $x$ is bounded by $\| x \|$. Since then $\sigma(x)$ is a closed and bounded subset of $\mathbb{C}$, it is compact. Define $f : \sigma(x)^c \to A$ by

$$f(\lambda) = (\lambda.e - x)^{-1}$$

We now make use of the inequality if $x \in G(A)$ and $h \in A$ is such that $\| h \| < \frac{1}{2} \| x^{-1} \|^{-1}$, then $x + h \in G(A)$ (this we have already proved above) and

$$L = \| (x + h)^{-1} - x^{-1} + x^{-1}hx^{-1} \| \leq 2 \| x^{-1} \|^3 \| h \|^2$$

Indeed,

$$(x + h)^{-1} = (x(1 + x^{-1}h))^{-1} = (1 + x^{-1}h)^{-1}x^{-1} = (x^{-1} - x^{-1}hx^{-1} + \sum_{n=2}^{\infty}(-1)^n(x^{-1}h)^n x^{-1}$$

giving

$$L \leq \sum_{n=2}^{\infty} \| x^{-1} \|^{n+1} \| h \|^n$$

$$= \| h \|^2 \| x^{-1} \|^3 / (1 - \| x^{-1} \| . \| h \|)$$

$$\leq 2 \parallel x^{-1} \parallel^3 . \parallel h \parallel^2$$

It follows that if $|\mu - \lambda|$ is sufficiently small, then

$$\parallel f(\mu) - f(\lambda) + (\mu - \lambda) f(\lambda)^2 \parallel \leq 2 f(\lambda)^3 |\mu - \lambda|^2$$

From which we deduce that $f$ is strongly differentiable at $\lambda$ and is hence holomorphic on $\sigma(x)^c$. If $\Gamma$ is a circle in $\mathbb{C}$ with origin as center and radius $r > \parallel x \parallel$, then we have the result that for all $\lambda \in \Gamma$, the series

$$f(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} x^n$$

is uniformly convergent and hence term by term integration gives

$$x^n = (2\pi i)^{-1} \int_\Gamma \lambda^n f(\lambda) d\lambda, n = 0, 1, 2, ...$$

Suppose $\sigma(x)$ is empty. Then, $f$ is holomorphic in the entire complex plane $\mathbb{C}$ and hence the above integrals are zero for all $n$ which contradict the fact that $x^0 = e$. This proves that $\sigma(x)$ is non-empty.

63. Parseval's theorem: Let $G$ be a locally compact Abelian group, for example $\mathbb{R}^n$ and for $g \in G$, suppose $\chi(g)$ is a complex number of unit magnitude such that $\chi(g + g') = \chi(g)\chi(g')$ for all, $g, g' \in G$, in other words, $\chi : G \rightarrow \mathbb{T}$ where $\mathbb{T}$ is the unit torus, is a homomorphism. Then $\chi$ is called a character of $G$. For example, if $G = \mathbb{R}^n$, then for any $(t_1, ..., t_n) \in \mathbb{R}^n$, we may define

$$\chi_{t_1,...,t_n}(x_1, ..., x_n) = exp(i \sum_{a=1}^{n} t_a x_a)$$

or equivalently,

$$\chi_{\mathbf{t}}(\mathbf{x}) = exp(i < \mathbf{t}, \mathbf{x} >)$$

We easily verify that $|\chi_{\mathbf{t}}(\mathbf{x})| = 1$ and

$$\chi_{\mathbf{t}}(\mathbf{x} + \mathbf{y}) = \chi_{\mathbf{t}}(\mathbf{x}) + \chi_{\mathbf{t}}(\mathbf{y}), \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

The Fourier transform theorem on $\mathbb{R}^n$ tells us that any square integrable function on $\mathbb{R}^n$ can be expanded as a superposition of the characters, ie,

$$f(\mathbf{x}) = \int \hat{f}(\mathbf{t}) \chi_{\mathbf{t}}(\mathbf{x}) d\mathbf{t}$$

where

$$\hat{f}(\mathbf{t}) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\mathbf{x}) \bar{\chi}_{\mathbf{t}}(\mathbf{x}) d\mathbf{x}$$

The Parseval theorem then gives for two functions $f, g \in L^2(\mathbb{R}^n)$,

$$< f, g >= (2\pi)^n < \hat{f}, \hat{g} >$$

Computing the statistics of the output of systems described by differential and difference equations with Gaussian input , white or colored, (c) Simulating nonlinear filters for trajectory tracking problems.

(3). Developing software for simulating quantum mechanical systems. These include: (a) Solving Schrodinger's equation for various kinds of potential numerically, (b) Applying independent and time dependent perturbation theory for computing the energy levels of many electron atoms and transition probabilities for atoms excited by the radiation field, (c) Numerically evaluating the Feynman path integral for computing the propagation for various kinds of quantum states e.g. harmonic oscillator and hydrogen atom (Coulamb potential), (d) Numerical evaluation of the amplitude for various kinds of scattering processes based on Feynman diagrams.

**Requirements:** Ph.D. students and M. Tech. students and 5 computers with sitting space and facilities.

**Lab Development:** A lab adjoining the DSP lab is to be developed for carrying out M. Tech. projects.

**Requirements:** 10 computer tables and 10 chairs, 1 printer table, 1 staff table and chair, 5 computers and a printer cum scanner cum copier.

65.**Proposal for music laboratory**

Main features of the laboratory

1.Installation of musical instruments like flute, Tabla, Veena etc.

2. Development of Signal processing toolboxes like Fourier and Wavelet techniques for the analysis of the music.

For example: let $x(t)$ be the music coming from an instrument. Suppose that $x(t)$ consists of the discrete frequencies $\omega_k, k = 1, 2, ..., n$. Then,

$$x(t) = \sum_{k=1}^{n} A_k cos(\omega_k t + \phi_k)$$

and its Fourier transform is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)exp(-i\omega t) = \pi \sum_{k=1}^{n} [exp(i\phi_k)\delta(\omega - \omega_k) + exp(-i\phi_k)\delta(\omega + \omega_k)]$$

The frequencies $\{\omega_k\}$ can be estimated by simply forming the periodogram of $x(t)$ which is a finite data set based approximation of the Fourier transform and then locating its peaks. The problem of estimating the phases $\phi_k$ is harder. Spectral analysis suppresses the phase information. Bispectral analysis can then be used to estimate the phases. The periodogram of the signal based on the data set $x(t), t \in [0, T]$ is defined by

$$X_T(\omega) = \int_{-T}^{T} x(t)exp(-i\omega t)dt$$

$$= \frac{1}{2} \sum_{k=1}^{n} exp(i\phi_k) \int_{-T}^{T} [exp(i(\omega_k - \omega)t)dt + exp(-i\phi_k) \int_{-T}^{T} exp(-i(\omega_k + \omega)t)]dt$$

$$= \sum_{k=1}^{n} exp(i\phi_k)\frac{sin((\omega-\omega_k)T)}{(\omega-\omega_k)} + exp(-i\phi_k)\frac{sin((\omega+\omega_k)T)}{(\omega+\omega_k)}$$

$X_T(\omega)$ shows smeared peaks at the points $\{\omega_k\}$ and can be used to obtain good estimates for the frequencies $\{\omega_k\}$. For doing local spectral analysis, we may use the STFT (short time Fourier transform). To this end, let $h(t)$ be a window function. The STFT of the signal $x(.)$ with respect to this window is defined by

$$S_h(t,\omega) = \int_{-\infty}^{\infty} h(\tau - t)x(\tau)exp(-i\omega\tau)d\tau$$

For a given value of $t$, the peaks of this function with respect to the frequency variable $\omega$ give the dominant frequencies present in the vicinity of the time $t$. Let

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)exp(i\omega t)d\omega$$

Then,

$$S_h(t,\omega) = (2\pi)^{-1}\int H(\theta)x(\tau)exp(i\theta(\tau - t))exp(-i\omega\tau)d\tau d\theta$$

$$= \int_{-\infty}^{\infty} H(\theta)X(\omega-\theta)exp(-i\theta t)d\theta$$

3.Bispectral analysis: Suppose $x(t), t \in \mathbb{R}$ is a signal. Assume that this signal is passed through a linear time invariant filter having impulse response $h(t), t \in \mathbb{R}$. The output is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

The third order moment function of the input signal with calculations based on time averages are given by

$$C_x(u,v) = \int_{-\infty}^{\infty} x(t)x(t+u)x(t+v)dt$$

Its bivariate Fourier transform equals

$$B_x(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_x(u,v)exp(-j(\omega_1 u + \omega_2 v))dudv$$

$$= X(\omega_1)X(\omega_2)\bar{X}(\omega_1+\omega_2)$$

Suppose $\phi_x(\omega) = ArgX(\omega), \phi_y(\omega) = ArgY(\omega)$ and $\phi_h(\omega) = ArgH(\omega)$. Then we have

$$ArgB_x(\omega_1,\omega_2) = \phi_x(\omega_1) + \phi_x(\omega_2) - \phi_x(\omega_1+\omega_2)$$

This formula can be used to estimate the signal phase spectrum from bispectral measurements. In musical notes the relative phase difference between two harmonics plays a fundamental role. For example, suppose that there are just two frequencies.

We can analyze the mean and fluctuations of the process $\{y(t)\}$ using the Ito differential rule:

$$\frac{d}{dt}\mathbb{E}y(t) = \mathbb{E}[\nabla\phi(\mathbf{x}(t))^T\mu(\mathbf{x}(t))] + \frac{1}{2}Tr[\mathbb{E}\sigma(\mathbf{x}(t))\sigma(\mathbf{x}(t))^T\nabla\nabla^T\phi(\mathbf{x}(t))]$$

The mean of the process $y(.)$ is a good estimate of the music itself and fluctuations must be minimized for they correspond to noisy disturbances. To reduce fluctuations, we must use nonlinear filtering theory. This involves taking measurements of some other functions of the displacement process $\mathbf{x}(.)$ and solving Kushners' filter for the conditional probability density of the displacement process given the measurements. Another important kind of disturbance that enters into musical tones generated by stringed instruments is spikes. Such noise can be modeled using the Poisson process or the compound Poisson process in which the time difference between two spikes has the exponential distribution. If $N(t), t \geq 0$ is a Poisson process with arrival times $\tau_1 < \tau_2 < ...$ and the spikes have random amplitudes represented by independent random variables $X_1, X_3, ...$, then the spiky noise can be represented by the process

$$Z(t) = \sum_{i=1}^{N(t)} X_i$$

Such a process satisfies the generalized Ito rule:

$$df(Z(t)) = [f(Z(t) + X_{N(t)+1}) - f(Z(t))].dN(t)$$

6. Study of music produced by a flute. The governing equations are obtained by considering pressure waves inside a tube. These are longitudinal waves. Assume that the length of the tube is $L$. Consider the infinitesimal element $[x, x+dx]$ of air inside the pipe at time $t = 0$. At time $t$, the point $x$ gets displaced by a small amount $\xi(t,x)$, so that the column of air now occupies the length $[x+\xi, x+dx+d\xi] = [x+\xi, x+dx(1+\frac{\partial\xi}{\partial x})]$. The density of air in this new region $\rho$ is is obtained by applying the mass conservation equation

$$\rho(t,x) = \rho_0 dx/(dx+d\xi) \approx \rho_0(1 - \frac{\partial\xi}{\partial x})$$

The air inside the tube is assumed to be adiabatic, so that the pressure is given by

$$p = p_0(\rho/\rho_0)^\gamma \approx p_0(1 - \gamma\frac{\partial\xi}{\partial x})$$

The equation of motion of this element of air is given by

$$\rho_0 dx\frac{\partial^2\xi}{\partial t^2} = -dx.\frac{\partial p}{\partial x}$$

or

$$\xi_{tt} = c^2\xi_{xx}, c^2 = \gamma p_0/\rho_0$$

A flute is a long pipe of air with holes along its length. Suppose we take a typical example in which the flute extends between the points $x = 0$ and $x = L$ and there is a single open hole at $x = L/2$, ie, midway. The blow is applied at $x = 0$ and we assume that this blow produces a pressure signal $p(t, 0)$. The end $x = L$ is open and hence $p(t, L) = 0$. Also $p(t, L/2) = 0$ since the hole at $x = L/2$ is open. The problem is then to obtain a complete solution to the wave equation for the pressure with thse boundary conditions. This problem can be extended to include a flute with more than one hole.

66. Numerical integration: Consider the Banach space $X = C[0, 1]$ and for $x \in X$, define

$$f(x) = \int_0^1 x(t)dt$$

$f$ is a bounded linear functional on $X$, ie, $f \in X^*$. Choose points $0 = t_0 < t_1 < ... < t_n = 1$ and $\alpha(k), k = 0, 1, ..., n$ and define

$$g(x) = \sum_{k=0}^{n} \alpha(k)x(t_k)$$

Then $g \in X^*$. We want to approximate $f$ by $g$. This is achieved by choosing the coefficients $\{\alpha(k)\}$ such that $g(x) = f(x)$ for all polynomials $x$ having degree $\leq n$. This means that

$$f(t^m) = g(t^m), m = 0, 1, ..., n$$

or

$$\frac{1}{m+1} = \sum_{k=0}^{n} \alpha(k)t_k^m, m = 0, 1, ..., n$$

This is a system of $n + 1$ linear equations for the $n + 1$ unknowns $\alpha(k), k = 0, 1, ..., n$ and can be solved readily by MATLAB.

(Taken from Kreyszig, Introductory Functional Analysis).

67. Let $\mathcal{H}$ be an inner product space and let $T$ be a Hermitian operator in $\mathcal{H}$. Then

$$\| T \| = sup(\| Tx \| : \| x \| \leq 1) = sup(| < Tx, x > | : \| x \| = 1)$$

Note that $< Tx, x >$ is real so the modulus sign on the inner product on the right can be removed.

Proof: Let

$$K = sup(| < Tx, x > | : \| x \| = 1)$$

Then the Schwarz inequality implies

$$K \leq \| T \|$$

Choose $z$ such that $\| z \| = 1$ but $Tz \neq 0$ and define $v = \| Tz \|^{1/2} z$ and $w = \| Tz \|^{-1/2} Tz$. Then

$$\| v \| = \| w \| = \| Tz \|^{1/2}$$

Define

$$y_1 = v + w, y_2 = v - w$$

Then,

$$< Ty_1, y_1 > - < Ty_2, y_2 > = < Tv, v > + < Tv, w > + < Tw, v > + < Tw, w >$$

authors then prove another theorem which states that the error in the Gibbs vector alone also converges to zero. The paper is well written and may be published after incorporating the suggestions made.

69. **Wavelet based estimation of Volterra kernels**

Sudipta's thesis work: Assume that the input output equation in continuous time is given by

$$y(t) = \int h_1(u)x(t-u)du + \int h_2(u,v)x(t-u)x(t-v)dudv$$

We wish to estimate the first and second order Volterra kernels $h_1(.)$ and $h_2(.,.)$ from input-output data. Assume that the kernels $h_1$ and $h_2$ are respectively of duration $[0,T]$ and $[0,T]\times[0,T]$ and that the maximum frequency of $h_1$ is $f_1$ and the maximum frequency of $h_2$ along both the two directions is $f_2$. We expand the kernels in terms of a wavelet basis. Specifically, let $\psi$ be the mother wavelet function that is concentrated over the band $[a,b]$. Then set $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$ with $n,k \in \mathbb{Z}$. The wavelet function $\psi_{n,k}$ is concentrated over all $t$ for which $2^n t - k \in [a,b]$ or equivalently, over all $t$ such that $t \in [(k+a)/2^n, (k+b)/2^n]$. The duration of this wavelet function is given by $2^{-n}(b-a)$ and this must be set to $1/f_1$. Thus, the maximum value of the resolution $n_1$ for the first order component of the impulse response is given by $2^{-n_1}(b-a) = 1/f_1$, or equivalently,

$$n_1 = log_2(f_1(b-a))$$

The minimum value of the resolution $n$ is $n_2$ and may be chosen so that $\psi_{nk}$ is focussed over an interval of length $T$. Thus, $(b-a)/2^{n_2} = T$, or equivalently,

$$n_2 = log_2((b-a)/T)$$

For a given value of the resolution $n$, the translation variable $k$ must vary in such a way so that $\psi_{n,k}$ covers the entire range $[0,T]$. This means that if $k_1(n) \le k \le k_2(n)$, then we must have

$$(k_1+a)/2^n = 0, (k_2+b)/2^n = T$$

or

$$k_1 = -a, k_2 = 2^n T - b$$

Thus, we must seek an expansion of $h_1(.)$ in the form

$$h_1(t) \approx \sum_{n_1 \le n \le n_2, k_1(n) \le k \le k_2(n)} c(n,k)\psi_{n,k}(t)$$

We denote this range of the ordered doublet $(n,k)$ by $D$. In other words,

$$h_1(t) \approx \sum_{(n,k)\in D} c(n,k)\psi_{n,k}(t)$$

Likewise, we choose a domain $E$ for the resolution and translation indices for expanding the second order Volterra kernel:

$$h_2(u,v) \approx \sum_{(n,k,p,q)\in E} d(n,k,p,q)\psi_{n,k}(u)\psi_{p,q}(v)$$

Specifically, $n, p$ each varies over $[log_2((b-a)/T), log_2(f_2(b-a))]$ and for a given $n, p$, $k, q$ respectively vary over $[-a, k_2(n)]$ and $[-a, k_2(p)]$. This domain, we denote by $E$. The output data can then be represented as

$$y(t) = \sum_{(n,k)\in D} c(n,k) \int x(t-u)\psi_{n,k}(u)du + \sum_{(n,k,p,q)\in E} d(n,k,p,q)\psi_{n,k}(u)\psi_{p,q}(v)x(t-u)x(t-v)dudv$$

Assume that $y(t)$ has a maximum frequency $f_3$ and a duration $T'$, so that in accord with the above method, we can choose a maximum and minimum resolution index $n_3, n_4$ and a corresponding translation variable range. Specifically,

$$n_4 = log_2((b-a)/T), n_3 = log_2(f_3(b-a))$$

and for a given resolution index $n$, the translation index $k$ varies in the range $[-a, 2^n T' - b]$. We denote this range of indices by $F$. Thus, the wavelet coefficients of $\{y(t)\}$ are given by

$$f(r,s) = \int y(t)\psi_{r,s}(t)dt = \sum_{(n,k)\in D} A(r,s|n,k)c(n,k) + \sum_{(n,k,p,q)\in E} B(r,s|n,k,p,q)d(n,k,p,q), (r,s) \in F$$

where

$$A(r,s|n,k) = \int x(t-u)\psi_{r,s}(t)\psi_{n,k}(u)dtdu, B(r,s|n,k,p,q) = \int x(t-u)x(t-v)\psi_{n,k}(u)\psi_{p,q}(v)\psi_{r,s}(t)dudvdt$$

This system of equations can be solved for the wavelet coefficients $c, d$ of the two Volterra kernels provided the number of points in the set $F$ is larger than the sum of the number of points in the sets $D$ and $E$. For any finite set $A$, let $\mu(A)$ denote the number of elements in $A$. Then the condition for solvability is that

$$\mu(F) \geq \mu(D) + \mu(E)$$

We order the elements of $D, E, F$ and construct matrices $\mathbf{A}, \mathbf{B}$ respectively of sizes $\mu(D) \times \mu(D)$ and $\mu(E) \times \mu(E)$. Then the above set of linear equations can be expressed in vectorial form as

$$\mathbf{f} = \mathbf{Ac} + \mathbf{Bd} = [\mathbf{A}|\mathbf{B}] \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

Suppose that the kernels $h_1$ and $h_2$ are known except for some parameters $\theta$. We write $h_1(t,\theta)$ and $h_2(u,v,\theta)$. The problem is to estimate these parameters. Suppose $\mu(F) >> \mu(D) + \mu(E)$. Then, we can use a least squares matching procedure combined with the gradient algorithm to estimate $\theta$. This is achieved by defining the energy function

$$E(\theta) = \| \mathbf{f} - \mathbf{Ac}(\theta) - \mathbf{Bd}(\theta) \|^2$$

where $\mathbf{c}(\theta)$ and $\mathbf{d}(\theta)$ are respectively obtained by arranging $c(n,k,\theta)$ and $d(n,k,p,q,\theta)$ as vectors with

$$c(n,k,\theta) = \int h_1(u,\theta)\psi_{n,k}(u)du, d(n,k,p,q,\theta) = \int h_2(u,v,\theta)\psi_{n,k}(u)\psi_{p,q}(v)dudv$$

Equivalently,

$$E(\theta) = \sum_{(r,s)\in F} (f(r,s) - \sum_{(n,k)\in D} A(r,s|n,k)c(n,k,\theta) - \sum_{(n,k,p,q)\in E} B(r,s|n,k,p,q)d(n,k,p,q,\theta))^2$$

To minimize this function, we may use the gradient algorithm:

$$\theta[k+1] = \theta[k] - \mu\nabla_\theta E(\theta[k]), k = 0,1,2,...$$

Note that the optimal equations are

$$\nabla E(\theta) = 0$$

which gives

$$\sum_{(r,s)\in F}(f(r,s) - \sum_{(n,k)\in D} A(r,s|n,k)c(n,k,\theta) - \sum B(r,s|n,k,p,q)d(n,k,p,q,\theta))$$

$$\times(\sum_{(n',k')\in D} A(r,s|n',k')\nabla_\theta c(n',k',\theta) - \sum_{(n',k',p',q')\in E} B(r,s|n',k',p',q')\nabla_\theta d(n',k',p',q',\theta)) = 0$$

This expression is complicated and not amenable to closed form solutions.

Comparison with a direct least squares based identification: If we were to use directly the least squares method for determining the filter kernels $h_1, h_2$, we would have to replace the integrals by sums over samples and likewise the integral of the squared error by the sum of the squared errors of the samples and this approximation could lead to serious errors. The wavelet based method inherently accounts for the resolution and localization of the kernels and hence the sum of error squares over the wavelet coefficients reflects better the exact error energy of the signals. In other words, in the direct least squares approach, we are replacing $\int_0^T e^2(t)dt$ by $\sum_{n=0}^{N-1} e^2(n\Delta)$ while in the wavelet method we are replacing it by $\sum_{k=1}^{p} |< e, \phi_k)|^2$ where $\phi_k, k = 1,2,...,p$ is a truncated orthonormal basis for the $L^2$ space and

$$< e, \phi_k >= \int_0^T e(t)\phi_k(t)dt$$

When the basis functions $\phi_k$ are chosen appropriately (wavelet basis in accord with resolution and duration), then the latter can give even for small values of $p$ a very good approximation to $\int e^2 dt$ that what the former gives with even large values of $N$. Thus, the wavelet method is superior in performance.

70.**Compact operators**: Let $\mathcal{H}$ be a Hilbert space and let $T_n, n = 1,2,...$ be a sequence of compact operators on $\mathcal{H}$ and let $T$ be a bounded linear operator in $\mathcal{H}$ such that

$$\| T_n - T \| \to 0, n \to \infty$$

Then, $T$ is compact. For let $(v_m : m = 1,2,...)$ be a sequence in $\mathcal{H}$ with $\| v_m \| = 1$ for all $m$. We want to show that $Tv_m, m = 1,2,...$ has a convergent subsequence. Choose a subsequence $(s_1(m))$ of $(m)$ such that $T_1 v_{s_1(m)}$ converges. Then choose a subsequence $(s_2(m))$ of $(s_1(m))$ such that $T_2 v_{s_2(m)}$ converges. In general, $(s_i(m))$ is a subsequence of $s_{i-1}(m)$ with the property $T_i v_{s_i(m)}$ converges. Define $w(m) = s_m(m)$. We shall show that $Tv_{w(m)}$ is Cauchy. Let $n \geq m$. Then, $w(n) = s_n(n) = s_m(k)$ for some $k \geq n$. We have

$$\| T(v_{w(n)}) - T(v_{w(m)}) \| \leq$$

$$\leq \| T(v_{w(n)}) - T_m(v_{w(n)}) \| + \| T_m(v_{w(n)}) - T_m(v_{w(m)}) \| + \| T_m(v_{w(m)}) - T(v_{w(m)}) \|$$

$$\leq 2 \| T - T_m \| + \| T_m(v_{w(n)}) - T_m(v_{w(m)}) \|$$

This coefficient is clearly

$$\binom{-N}{n}(-1)^n = \binom{N+n-1}{n}$$

Verify that this dimension is the same as $Tr(S)$.

2.Quantum Markov process (MATLAB simulation)(Simulation of a star unital homomorphism): Let $\mathcal{H}_0$ be a Hilbert space and $\mathcal{H}$ another Hilbert space. Let $\theta : \mathcal{B}(H_0) \to \mathcal{B}(H_0) \otimes \mathcal{B}(H)$ be an algebra homomorphism that preserves the star operation and $\theta(1) = 1$. Choose an orthonormal basis $(e_0, ..., e_{d-1})$ for $\mathcal{H}$ and define $\theta_{ij} : \mathcal{B}(H_0) \to \mathcal{B}(H_0)$ by the equation

$$\theta_{ij}(X) = \mathbb{E}_{|e_j><e_i|}(\theta(X))$$

This notation means that for any $u, v \in \mathcal{B}(H_0)$, we have

$$< u, \theta_{ij}(X)v >=< u \otimes e_i, \theta(X)(v \otimes e_j) >, i, j = 0, 1, ..., d-1$$

Then we have

$$< u, \theta_{ij}(1)v >=< u, \mathbb{E}_{|e_j><e_i|}(\theta(1))v >=< u \otimes e_i, \theta(1)(v \otimes e_j) >$$

$$=< u \otimes e_i, v \otimes e_j >=< u, v > \delta_{ij}$$

so that

$$\theta_{ij} = \delta_{ij}.I$$

Also

$$< u, \theta_{ij}(X^*)v >=< u \otimes e_i, \theta(X^*)(e_j \otimes v) >$$

$$=< u \otimes e_i, \theta(X)^*(e_j \otimes v) >=< \theta(X)(u \otimes e_i), e_j \otimes v >$$

$$=< e_j \otimes v, \theta(X)(u \otimes e_i) >^*=< v, \theta_{ji}(X)u >^*=< \theta_{ji}(X)u, v >=< u, \theta_{ji}(X)^*v >$$

so that

$$\theta_{ij}(X^*) = \theta_{ji}(X)^*$$

Finally, let $\{u_r\}$ be an orthonormal basis for $H_0$. Then,

$$< u, \theta_{ij}(XY)v >=< u \otimes e_i, \theta(XY)(v \otimes e_j) >$$

$$=< u \otimes e_i, \theta(X)\theta(Y)(v \otimes e_j) >= \sum_{r,s} < u \otimes e_i, \theta(X)(u_r \otimes e_s) >< (u_r \otimes e_s), \theta(Y)(v \otimes e_j) >$$

$$= \sum_{r,s} < u, \theta_{is}(X)u_r >< u_r, \theta_{sj}(Y)v >= \sum_s < u, \theta_{is}(X)\theta_{sj}(Y)v >$$

$$=< u, \sum_s \theta_{is}(X)\theta_{sj}(Y)v >$$

or equivalently,

$$\theta_{ij}(XY) = \sum_s \theta_{is}(X)\theta_{sj}(Y)$$

Example:Let $(S,\mu)$ be a measure space and let $\phi_j : S \to S, j = 0,1,2,...,d-1$ be measurable maps and let $p_j : S \to [0,1], j = 0,1,...,d-1$ be measurable functions such that $p_j(x) \geq 0$ and $\sum_{j=0}^{d-1} p_j(x) = 1$. Let $H_0 = L^2(S,\mu)$ and $H = \mathbb{C}^d$. Define the map $T : \mathcal{B}(H_0) \to \mathcal{B}(H_0)$ so that

$$Tf(x) = \sum_{j=0}^{d-1} p_j(x) f(\phi_j(x))$$

The physical interpretation of $T$ is as follows. Let $X_n, n = 0,1,2,...$ be a Markov chain with state space $S$ such that conditioned on the event $X_n = x$, we have $X_{n+1} = \phi_j(x)$ with probability $p_j(x)$. Thus,

$$Pr(X_{n+1} = \phi_j(x) | X_n = x) = p_j(x), j = 0,1,...,d-1, x \in S$$

Then for any $f \in L^2(\mu)$, we have

$$\mathbb{E}[f(X_{n+1})|X_n = x] = Tf(x)$$

Define $\mathcal{B}(H_0)$ consists of complex valued functions on $H_0$, $H_0 \otimes H$ consists of $\mathbb{C}^d$ valued functions on $S$. $\mathcal{B}(H_0) \otimes \mathcal{B}(H)$ consists of $d \times d$ complex matrix valued functions on $S$ Define $\theta : \mathcal{B}(H_0) \to \mathcal{B}(H_0) \otimes \mathcal{B}(H)$ by

$$\theta(f) = U.diag[fo\phi_0, ..., fo\phi_{d-1}]U^*$$

where $U$ is a unitary $d \times d$ matrix valued function on $S$. Then, $\theta$ is a star unital homomorphism. Indeed, we have for two functions $f, g$ on $S$,

$$\theta(f)\theta(g) = \theta(fg)$$

We have

$$\theta_{ij}(f) = \sum_r u_{ir}\bar{u}_{jr} fo\phi_r$$

We can arrange the matrix $U$ so that

$$|u_{0r}(x)|^2 = p_r(x)$$

In that case, we get the equation

$$\theta_{00}(f) = T(f)$$

the transition generator of the Markov chain. This idea provides the means for realizing a classical Markov process via quantum Markov processes, ie, via, star unital homomorphisms.

Note: Let $\mathcal{H}_n, n = 0,1,2,...$ be a sequence of Hilbert spaces. For each $n = 0,1,2,...,$ define

$$\mathcal{H}_{n]} = \mathcal{H}_0 \otimes ... \otimes \mathcal{H}_n, n = 0,1,2,...$$

and also define

$$\mathcal{H}_{[n} = \mathcal{H}_n \otimes \mathcal{H}_{n+1} \otimes ...$$

with respect to a stabilizing sequence $\phi_n \in H_n, n = 0,1,2,....$ This means that every vector in $\mathcal{H}_{[n}$ has the form $u \otimes \phi_k \otimes \phi_{k+1} \otimes ...$ for some $k > n$ and

$$u \in \mathcal{H}_n \otimes ... \otimes H_k$$

Let $\mathcal{B}_{n]}$ denote the Von-Neumann algebra of all bounded operators in $\mathcal{H}_{n]}$ and $\mathcal{B}_{[n}$ the Von-Neumann algebra of all bounded operators in $\mathcal{H}_{[n}$. Note that we can identify the elements of $\mathcal{B}_{n]}$ with elements of the form $B \otimes I_{[n+1}$ where $B$ is a bounded operator in $\mathcal{H}_{n]}$. Let $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1 ...$ with respect to the given stabilizing sequence. This means that

$$\mathcal{H} = \mathcal{H}_{n]} \otimes \mathcal{H}_{[n+1}, n = 0, 1, 2, ...$$

Let $\mathcal{B} = \mathcal{B}(H)$, ie, the algebra of all bounded operators in $H$. For $B \in \mathcal{B}$, we then define $\mathcal{E}_{n]}B \in \mathcal{B}_{n]}$ by the formula that if $u, v \in \mathcal{H}_{n]}$, then

$$< u, \mathbb{E}_{n]}(B)v >=< u \otimes \phi_{[n+1}, B(v \otimes \phi_{[n+1}) >$$

where

$$\phi_{[n+1} = \phi_{n+1} \otimes \phi_{n+2} \otimes ...$$

Remark: $\mathcal{H}_{n]}, n = 0, 1, 2, ...$ in the quantum theory plays the role of a filtration in discrete time while the Hermitian operators in $\mathcal{B}_{n]}$ play the role of random variables measurable with respect to $\mathcal{F}_n$, the filtration at time $n$.

72. Problem:(Ph.D thesis problem of Dr.Rajbir Yaduvanshi working under guidance of Dr.Dhananjay and H.Parthasarathy) Implementation of Largrangian and Hamiltonian systems on DSP processors:Assume that the Lagrangian has a kinetic energy term quadratic in the velocities, ie,

$$T(q_1, ..., q_n, q_1', ..., q_n') = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q_1, ..., q_n) q_i' q_j'$$

The Lagrangian for the system of particles moving under the potential energy $U(q_1, ..., q_n)$ is then given by

$$\mathcal{L}(q_1, ..., q_n, q_1', ..., q_n') = T(q, q') - U(q)$$

and the Euler-Lagrange equations are

$$\frac{d}{dt} \frac{\partial T(q, q')}{\partial q_i'} = \frac{\partial T(q, q')}{\partial q_i} - \frac{\partial U(q)}{\partial q_i}$$

We compute

$$\frac{\partial T}{\partial q_i'} = \sum_j a_{ij}(q) q_j'$$

and

$$\frac{d}{dt} \frac{\partial T}{\partial q_i'} = \sum_{j,k} a_{ij,k}(q) q_k' q_j' + \sum_j a_{ij}(q) q_j''$$

$$\frac{\partial T}{\partial q_i} = \sum_{k,m} a_{km,i}(q) q_k' q_m'$$

$$= 2\mu.dt.(\nabla_{W(t)} Arg(\hat{d}(t))).(Arg(d(t)) - Arg(\hat{d}(t)))$$

$$Arg(d(t)) = Imlog(d(t)) = Imlog(W^T Z(t) + n(t)) \approx Im[log(W^T Z(t)) + n(t)/(W^T Z(t))]$$

$$= Arg(W^T Z(t)) + Im(n(t)/(W^T Z(t)))$$

Write

$$Z(t) = Z_1(t) + iZ_2(t), W = W_1 + iW_2$$

so that

$$W^T Z(t) = W_1^T Z_1(t) - W_2^T Z_2(t) + i(W_1^T Z_2(t) + W_2^T Z_1(t))$$

It follows that

$$Arg(W^T Z(t)) = tan^{-1}\frac{(W_1^T Z_2(t) - W_2^T Z_1(t))}{(W_1^T Z_1(t) - W_2^T Z_2(t))}$$

Also writing

$$n(t) = n_1(t) + in_2(t)$$

we have

$$\frac{n(t)}{W^T Z(t)} = \frac{n_1 + in_2}{A + iB}$$

where

$$A = Re(W^T Z(t)) = W_1^T Z_1 - W_2^T Z_2, B = W_1^T Z_2 + W_2^T Z_1 = Im(W^T Z(t))$$

giving

$$Im(\frac{n(t)}{W^T Z(t)}) = \frac{An_1 - Bn_2}{A^2 + B^2}$$

$$= \frac{(W_1^T Z_1 - W_2^T Z_2)n_1 - (W_1^T Z_2 + W_2^T Z_1)n_2}{|W^T Z|^2}$$

Likewise, we have

$$Arg(W^T(t)Z(t)) = Imlog(W^T(t)Z(t)) = Imlog(W_1(t)^T Z_1(t) - W_2(t)^T Z_2(t)$$

$$+i(W_1(t)^T Z_2(t) + W_2(t)^T Z_1(t)))$$

so that

$$\nabla_{W(t)} Arg(W(t)^T Z(t)) = \nabla_{W_1(t)} Arg(W(t)^T Z(t)) + i\nabla_{W_2(t)} Arg(W(t)^T Z(t))$$

where

$$\nabla_{W_1(t)} Arg(W(t)^T Z(t)) = Im((Z_1(t) + iZ_2(t))/W(t)^T Z(t)) = Im(Z(t)/W(t)^T Z(t))$$

$$\nabla_{W_2(t)} Arg(W(t)^T Z(t)) = Im\frac{(-Z_2(t) + iZ_1(t))}{W(t)^T Z(t)} = Re(Z(t)/W(t)^T Z(t))$$

This gives us

$$\nabla_{W(t)} Arg(W(t)^T Z(t)) = Im(Z(t)/W(t)^T Z(t)) + i.Re(Z(t)/W(t)^T Z(t)) = i.(Z(t)/W(t)^T Z(t))^*$$

Thus,

$$dW(t) = 2\mu.dt.i(Z(t)/W(t)^T Z(t))^*(Arg(W^T Z(t)) - Arg(W(t)^T Z(t)) + (An_1(t) - Bn_2(t))/|W^T Z|^2)$$

or in full expanded form,

$$dW(t) = 2\mu.dt.i(Z(t)/W(t)^T Z(t))^*(Arg(W^T Z(t)) - Arg(W(t)^T Z(t))$$

$$+|W^Z(t)|^{-2}(Re(W^T Z(t))n_1(t) - Im(W^T Z(t))n_2(t)))$$

74.Concluding remarks for a paper by Tarun and Myself: The paper starts by developing a differential equation model for the motion of a plane under the influence of noise. If there were no noise, the plane would execute motion with uniform velocity. The presence of noise causes the plane to turn by small random amounts. Equations for the evolution of mean and covariance of the position and velocity variables are derived using approximation techniques. These equations are valid in between observations which are taken at discrete time instants. The evolution of the conditional mean and covariance just after an observation has been made are obtained. In filtering theory terminology, this method of estimating the trajectory of plane is called the continuous-discrete filter. The observations are made on the range and bearing of the plane. The methods used in this paper can be applied to any mechanical problem, for example estimating the trajectory of the Foucault pendulum used for determining the angular velocity of the earth's rotation. Any mechanical system is generally described by Newton's equations of motion in the presence of a potential field and frictional forces. In addition, to incorporate uncertainties in the model like random impacts on the moving object, we include white noise terms resulting thereby in a system of stochastic differential equations describing the motion. Measurements may be taken on any set variables to which we have access, for example, if we take a movie photograph of the object from one angle, we are then measuring a projection of the trajectory (position and velocity) on a plane, for example, one component of the particle displacement. Or we may be able to measure the energy of the object which is a nonlinear function of its position and displacement. The point is that measurements are usually discrete and then estimating a continuous trajectory from discrete measurements becomes important. Another example where the same techniques can be used is in non-linear circuit theory. For example, suppose we are given an LCR circuit with $R$ a nonlinear function of the current with some unknown parameters. We write $R(i, \theta)$. We can excite this circuit with white Gaussian noise and write down the state variable equations for the current and its derivative. We may then extend the state by writing a sde for the unknown parameter $\theta$ with noise incorporated and from noisy measurements on the current through the circuit taken at discrete time instants, use the continuous discrete filter to estimate the parameter. The general philosophy is to write down the dynamical equations for the states of a system having unknown parameters, and then write down an equation for the parameter variation. In case the parameters are constant, the dynamical equation for the parameter variation will involve only noise and the parameters can be estimated by taking discrete measurements on some functions of the system state using the continuous discrete filter.

75. Product topology, subspace topology: Let $(X_i, \tau_i), i = 1, 2$ be topological spaces and let $\mathcal{C}$ be the collection of all subsets of $X_1 \times X_2$ having the form $E_1 \times E_2$ where $E_i \in \tau_i, i = 1, 2$. Then, let $\tau_1 \times \tau_2$ be the topology on $X_1 \times X_2$ generated by $\mathcal{C}$. $(X_1 \times X_2, \tau_1 \times \tau_2)$ is then a topological space called the product of the two topological spaces $(X_i, \tau_i), i = 1, 2$. Let $(X_i, \tau_i), i = 1, 2$ be topological spaces and let $f : X_1 \to X_2$ be a map such that $f^{-1}(\mathcal{E})) \subset \tau_1$ where $\mathcal{E}$ is any collection of subsets of $X_2$ that generates

$\tau_2$. Then $f$ is continuous. To see this, let

$$\mathcal{D} = \{E \subset X_2 : f^{-1}(E) \in X_1\}$$

Then $\mathcal{D}$ is easily verified to be a topology on $X_2$ and $\mathcal{E} \subset \mathcal{D}$ by hypothesis. Hence, $\mathcal{D}$ contains $\tau_1$ proving that $f$ is continuous.

76. Sudipta's Ph.D. problem: This problem involves estimating using wavelets the parameters of a simple pendulum. The equation of motion of the pendulum is given by

$$\theta''(t) = -a.sin(\theta(t))$$

where $\theta(t)$ is the deflection angle of the pendulum at time $t$ and $a = g/L$ with $g$ as the gravitational constant and $L$ the length of the pendulum. The problem is to identify the parameter $a$ by giving a noisy input. The approximate equation of motion of the pendulum (for small deflections) is given by

$$\theta''(t) = -a\theta(t) + a\theta^3(t)/6 + w(t)$$

The problem is to calculate the wavelet coefficients of $\theta(t)$ using a second order Volterra approximation to the motion. To this end we write

$$\theta(t) = \theta_0(t) + \theta_1(t)$$

where $\theta_0$ is the deflection obtained if only the linear component of the pendulum is taken into consideration and $\theta_1(t)$ is the correction produced by the nonlinear component $\theta^3$ in the motion. We have

$$\theta_0''(t) = -a\theta_0(t) + w(t),$$

$$\theta_1''(t) = -a\theta_1(t) + a\theta_0^3(t)$$

The first gives on Laplace transforming,

$$(s^2 + a)\hat{\theta}_0(s) = W(s)$$

so that

$$\theta_0(t) = a^{-1/2}\int_0^t sin((t-u)\sqrt{a})w(u)du$$

The second gives on Laplace transforming,

$$(s^2 + a)\hat{\theta}_1(s) = aL(\theta_0^3)(s)$$

so that

$$\theta_1(t) = a^{1/2}\int_0^t sin((t-u)\sqrt{a})\theta_0^3(u)du$$

$$= a^{-1}\int_{0<u_1,u_2,u_3<u<t} sin((t-u)\sqrt{a})sin((u-u_1)\sqrt{a})sin((u-u_2)\sqrt{a})sin((u-u_3)\sqrt{a})w(u_1)w(u_2)w(u_3)$$

$$du_1du_2du_3du$$

4.Calculate and store the wavelet coefficients of the theoretically expected result $\theta(t)$ using the above third order Volterra formula. Do this calculation for various choices of the parameter $a$.

5.Match the wavelet coefficients of the simulated output with the theoretical expressions for different $a's$ and determine the optimum choice of $a$.

77. Justification for the procurement of items for the DSP laboratory.

1.High speed scanner: Currently, a lot of literature exists in the form of unpublished reprints and books that are out of print in the field of "Group Representations in Signal and Image Processing". The authors who have contributed much to the field of group representations are mainly eminent mathematicians like I.Schur, Burnside, Frobenius, H.Weyl, Harish-Chandra, E.Wigner and more recently, Barry Simon, Zelobenko, Sternberg, Procesi and V.S.Varadarajan. The manuscripts written by these authors contain chiefly mathematical ideas like the construction of irreducible representations of semi-simple Lie algebras that are not accessible to the engineering student owing to the absence of such books in the institute library. For this purpose, the scanner will be a useful tool. We can borrow the books and reprints from well known mathematical institutions like the Indian Statistical Institute, The Tata Institute of Fundamental Research, scan these and store the manuscript on the PC. The engineering student will then be exposed to the mathematical ideas (like Harish-Chandra's use of maximal ideals in the construction of both finite and infinite dimensional irreducible representations of semisimple Lie groups) and they can directly apply it to image processing problems. The current literature available in image processing deals only with the rotation group $SU(2), SO(3)$ and the Lorentz group which can be regarded as $SL(2,\mathbb{C})$. For other semisimple Lie groups, one must take resort to the theory developed by Harish-Chandra and the antique papers of this famous mathematician will be available directly to the student if a scanner is purchased.

2.Lap-top computer: Mechanics is the most ancient of all the physical sciences. Ever since Sir Isaac Newton laid down the fundamental laws of mechanics in the form of the three laws, mechanics has taken rapid advances. After Newton formulated his laws of mechanics and the inverse square law of gravitation and used these two laws and calculus (which he called fluxions) to arrive at Kepler's laws of planetary motion, there have been several developments. After Newton, Joseph Louis Lagrange derived the laws of mechanics from a variational principle according to which the trajectories of particles between two time instants can be obtained by minimizing a cost functional of the path. Lagrange was able to formulate equations of motion for more complex systems like the simple pendulum double pendulum and other systems without having to introduce auxiliary forces like normal reaction, tension in a rod etc. His method involved merely the introduction of generalized coordinates to describe the motion of the system and then write down an expression for the difference between the kinetic and potential energies of the system in terms of these generalized coordinates and by minimizing the integral of this difference over a path, he arrived at the equations of motion. After Lagrange, came Hamilton who taught us that by applying the Legendre transformation to the Lagrangian system, one can describe the equations of motion in terms of first order coupled non-linear differential equations for the positions and momenta of the system. Recently, there has been a surge of interest in the formulation of the equations of mechanics for generalized systems using the tool of differentiable manifolds. The manifold can be any topological space with a differentiable structure. This has led to unexpected discoveries by mathematicicans such as Stephen Smale, Rene Thom, V.I.Arnold etc. In short, mechanics has become a big industry for obtaining and hence the transfer function of the ensemble of layers is given by

$$H(\omega) = \frac{X(\omega)}{S(\omega)} = \sum_i h(i) exp(-j\omega\tau(i))$$

If we assume that the signals have been discretized, then the corresponding model for the received signal can be expressed as

$$x(n) = \sum_{kh=1}^{p} (k)s(n-k)$$

Assuming that $s(n)$ is white non-Gaussian noise, we can recover the magnitude transfer function

$$|\sum_n h(n) exp(-j\omega n)|^2$$

from a spectral analysis of $\{x(n)\}$, ie by estimating the autocorrelation function of $x(.)$ followed by an FFT. The phase of the transfer function can be recovered by performing a bispectral analysis of $x(n)$, ie, estimate the third order moment sequence followed by a bivariate FFT. The spectrum of the received signal has the form $\sigma^2|H(\omega)|^2$ while the bispectrum has the form $\gamma.H(\omega_1)H(\omega_2)\bar{H}(\omega_1+\omega_2)$. The phase can be recovered from the bispectrum by a simple recursive algorithm. On can generalize this technique by sending an electromagnetic signal distributed over a spatial domain. Assume that the transmitted signal on the surface is $E(x,y,t)$. Let $\tau(i,x,y)$ be the delay suffered by the signal at a layer below the point $(x,y)$ and $h(i,x,y)$ the corresponding attenuation. Then, the received space-time signal has the form

$$F(t,x,y) = \sum_i h(i,x,y)E(x,y,t-\tau(i,x,y))$$

Signal processing can be performed on such signals to receive information about the layer structure distributed under a region of space. The above equation decouples the received signal at different points $(x,y)$. More generally, we can have coupling.

79. Vireshwar's M.Tech disseration synopsis: Suppose $x(n,m)$ is an image obtained after discretization using pixels. Assume that it is blurred by noise. Then the blurred image can be represented as $y(n,m) = x(n,m)+w(n,m)$ where $x$ is the unblurred image, $w$ is the blurring noise and $y$ is the blurred image. The problem of removing the blurring noise can be achieved by passing this two dimensional signal through an FIR filter having impulse response $h(n,m)$. The output is

$$\hat{x}(n,m) = \sum_{i,j} h(i,j)y(n-i,m-j)$$

with $h(i,j)$ chosen so that

$$\mathbb{E}(x(n,m) - \sum_{i,j} h(i,j)y(n-i,m-j))^2$$

is a minimum. The optimization can be carried out by setting to zero the derivative of the above energy with respect to the filter coefficients $\{h(i,j)\}$. The resulting optimal normal equations are

$$\mathbb{E}(x(n,m)y(n-i,m-j)) = \sum_{p,q} h(p,q)\mathbb{E}(y(n-p,m-q)y(n-i,m-j))$$

These equations are solved to obtain the optimal filter. In practice, we will be given $N$ copies of original images $x_i(n,m)$ and their blurred versions $y_i(n,m)$. Then the impulse response of the deblurring filter will be determined by solving the system of linear equations

$$\sum_i x_i(n,m)y_i(n-r,m-s) = \sum_{p,q} h(p,q)\sum_i y_i(n-p,m-q)y_i(n-r,m-s)$$

This method of removing blurs can be extended to situations when there is a distorting linear transformation. In this case, the original image $x(n,m)$ and its distorted cum blurred version $y(n,m)$ are related via an equation of the form

$$y(n,m) = g(n,m) * x(n,m) + w(n,m)$$

where $*$ stands for convolution. In the ideal case of known statistics, the optimal Wiener filter $h(n,m)$ that takes as input $y(n,m)$ and outputs an optimal estimate $\hat{x}(n,m)$ of the unblurred image can be designed easily. In our work, we shall explore the applicability of second order Volterra filters for image deblurring. Such a deblurring filter will have the form

$$\hat{x}(n,m) = \sum h_1(p,q)x(n-p,m-q) + \sum h_2(p_1,q_1,p_2,q_2)x(n-p_1,m-q_1)x(n-p_2,m-q_2)$$

with the kernels $h_1, h_2$ designed so that

$$\mathbb{E}(x(n,m) - \hat{x}(n,m))^2$$

is a minimum. In practice, the expectation will be replaced by a sum over prototype unblurred and blurred images. The effectiveness of the second order Volterra filters in edge smoothing will also be investigated. We shall also explore the applicability of recursive Volterra filters in image modeling. Specifically, given an image $y(n,m)$, we shall try to determine parameters $h_1(k)$ and $h_2(k,m)$ so that if

$$\epsilon(n,m) = y(n,m) + \sum_{p,q} h_1(p,q)y(n-p,m-q) + \sum_{p_1,q_1,p_2,q_2} h_2(p_1,q_1,p_2,q_2)y(n-p_1,m-q_1)y(n-p_2)y(m-q_2)$$

then $\sum_{n,m}\epsilon^2(n,m)$ is a minimum. We shall then explore the possibility of using such a model to recursively generate the given image. Such a model will provide and efficient and accurate parametrization of the given image.

80. Replies to referees comments for Tarun's paper:Under what conditions can be model the inductance as $L + w(t)$ where $L$ is a constant and $w(t)$ is a correlated random process. First note that by application of Ampere's law, the magnetic flux through the inductance is given by $N\Phi = N^2AI/d$ where $d$ is the length of the inductance, $I$ the current flowing through the coil and $N$ the number of turns. $A$ is the cross sectional area of the inductance. The inductance is therefore $L = N\Phi/I = N^2A/d$. While actually computing the inductance, the parameters $N, A, d$ are subject to random measurement errors $\delta N, \delta A, \delta d$. It follows that the first order error in the inductance computation is given by

$$\delta L = (2NA/d)\delta N + (N^2/d)\delta A - (N^2A/d^2)\delta d$$

This expression gives us a method for computing the statistics of the random fluctuation in the inductance. However, this fluctuation is a static measurement error, it is not a random process. The random process

# Chapter 3

# Quantum Mechanics

1.Path integral for the harmonic oscillator; A discrete Fourier transform based approach. The action for the particle in traveling from $(x(0), 0)$ to $(x(T), T)$ is given by

$$\mathcal{A}(x) = \int_0^T (Mx'(t)^2/2 - M\omega^2 x(t)^2/2)dt$$

we write

$$x(t) = x_{cl}(t) + \delta x(t)$$

where $x_{cl}(t)$ is the action for the classical trajectory. We have

$$x''_{cl}(t) = -\omega^2 x_{cl}(t)$$

whose solution is given by

$$x_{cl}(t) = A.cos(\omega t) + B.sin(\omega t)$$

The boundary conditions are

$$x_{cl}(0) = x(0) = A, x_{cl}(T) = A.cos(\omega T) + B.sin(\omega T)$$

so that

$$A = x(0), B = (x(T) - x(0).cos(\omega T))/sin(\omega T)$$

The classical action for motion between the two end points can now be evaluated:

$$Mx^{'2}/2 - M\omega^2 x^2/2 = (M/2)(-A\omega.sin(\omega t) + B\omega.cos(\omega t))^2 - (M\omega^2/2)(A.cos(\omega t) + B.sin(\omega t))^2$$

$$= -(MA^2\omega^2/2)cos(2\omega t) + (MB^2\omega^2/2)cos(2\omega t) - MAB\omega^2 sin(2\omega t)$$

giving

$$\mathcal{A}_{cl}(x(0), x(T), T) = (M(A^2 - B^2)\omega/4)(1 - sin(2\omega T)) + (MAB\omega/2)(cos(2\omega T) - 1)$$