Chapter 2

MATHEMATICAL MODELS OF PHYSICAL SYSTEMS

2.1. INTRODUCTION

For the analysis and design of control systems, we need to formulate a mathematical description of the system. The process of obtaining the desired mathematical description of the system is known as “modeling”. The basic models of dynamic physical systems are differential equations obtained by application of the appropriate laws of nature. These equations may be linear or nonlinear depending on the phenomena being modeled.

The differential equations are inconvenient for the analysis and design manipulations and so the use of Laplace Transformation which converts the differential equations into algebraic equations is made use of. The algebraic equations may be put in transfer function form, and the system modeled graphically as a transfer function block diagram. Alternatively, a signal flow graph may be used.

This chapter is concerned with differential equations, transfer functions, block diagrams, signal flow graphs, etc., of different physical systems namely, mechanical, electrical, hydraulic, pneumatic and thermal systems. Analysis of a dynamic system requires the ability to predict its performance. This ability and the precision of the results depend on how well the characteristics of each component can be expressed mathematically.

One of the most important tasks in the analysis and design of control systems is mathematical modeling of the systems. The two most common methods are the transfer function approach and the state equation approach. The transfer function method is valid only for linear time-invariant systems, whereas the state equations are first-order to use transfer functions and linear state equations the system must first be linearized, or its range of operation must be confined to a linear range.

Although the analysis and design of linear control systems have been well developed, their counterparts for nonlinear systems are usually quite complex. Therefore, the control systems engineer often has the task of determining not only how to accurately describe a system mathematically, but also, more important, how to make proper assumptions and approximations, whenever necessary, so that the system may be adequately characterized by a linear mathematical model.

Table 2.1. Some Laplace transform pairs

$x(t)$	$x(s)$
Unit impulse $\delta(t)$	1
Unit step $u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-\lambda t}$	$\frac{1}{s+\lambda}$
$t^n e^{-\lambda t}$	$\frac{n!}{(s+\lambda)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-\lambda t} \sin \omega t$	$\frac{\omega}{(s+\lambda)^2+\omega^2}$
$e^{-\lambda t} \cos \omega t$	$\frac{s+\lambda}{(s+\lambda)^2+\omega^2}$

We can thus conclude that

$$\mathcal{L}\{e^{\lambda t}\} = \frac{1}{S-\lambda} \tag{2.3}$$

The result of Example 2.1 is useful in many ways. First, the Laplace transform of the exponential function is of interest in its own right. Second, since the exponential is a generating function of many other interesting functions, we can use Eq. (2.3) to provide the Laplace transform of these other functions. To start, note that a step function $u(t)$ is defined as

$$\begin{aligned} u(t) &= 1 \quad t \geq 0 \\ &= 0 \quad t < 0 \end{aligned}$$

Note that

$$u(\mathrm{t}) = \mathrm{e}^{0t} \tag{2.4}$$

$$= - \ldots - sx^{(n-2)}(0) - x^{n-1}(0) \tag{2.11}$$

where $x(0)$, $\dot{x}(0)$, $\ddot{x}(0)$, ... , x^{n-2}, and x^{n-1} denote the values of

$x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, ... , $(d^{\,n-2}/dt^{\,n-2})\, x(t)$, $(d^{\,n-1}/dt^{\,n-1})\, x(t)$ evaluated at $t = 0$.

Note that if all initial values of $x(t)$ and its derivatives are zero, the Laplace transform of the nth derivative of $x(t)$ is $s^nX(s)$ and the differentiation operation is equivalent in the Laplace domain to the s operator.

We can arrive at the *integration theorem* through use of Eq. (2.10) as follows. Let $y(t) = \dot{x}(t)$; thus

$$x(t) = \int y(t)dt$$

As a result of Eq. (2.10), we have

$$\mathscr{L}\,\{y(t)\} = s\,\mathscr{L}\left\{\int y(t)dt\right\} - \int y(t)dt\Big|_{t=0}$$

Thus,
$$\mathscr{L}\left\{\int y(t)dt\right\} = \frac{Y(s)}{s} + \frac{y^{-1}(0)}{s} \tag{2.12}$$

where $y^{-1}(0) = y(t)\; dt$ evaluated at $t = 0$.

The shift theorem deals with the Laplace transorm of $x(t - \tau)$, where τ is a delay given that $x(t)$ is defined and $x(t) = 0$, $t < 0$. Applying the fundamental expression (2.1), we have

$$X(s) = \int_0^\infty x(\sigma)e^{-\sigma s}d\sigma$$

$$= \int_0^\infty x(t-\tau)e^{-s(t-\tau)}dt$$

$$= e^{\tau s}\int_0^\infty x(t-\tau)e^{-st}dt$$

Thus,

$$\mathscr{L}\,\{x(t - \tau)\} = \mathrm{e}^{-\tau s}X(s) \tag{2.13}$$

The *final-value theorem* can be obtained from the differentiation theorem (2.10) by taking the limits as $s \to 0$, with the result

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{2.14}$$

The *initial-value theorem* is stated as

$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s) \tag{2.15}$$

We will consider now an important case involving the *convolution integral* defined for the two functions $h(t)$ and $x(t)$ by the relation

$$y(t) = h(t) * x(t) \tag{2.16}$$

or
$$y(t) = \int_0^1 h(t-\tau)x(\tau)d\tau \tag{2.17}$$

The asterisk denotes the convolution operation. The Laplace transform of $y(t)$ is obtained using Eq. (2.1) as

$$Y(s) = \int_0^\infty y(t)e^{-st}dt \tag{2.18}$$

Let us note that both $h(t)$ and $x(t)$ are assumed to be zero for $t < 0$. We can thus conclude that $h(t - \tau)$ is zero for $\tau > t$. As a result, we can write

$$y(t) = \int_0^\infty h(t-\tau)x(\tau)d\tau \tag{2.19}$$

As a result, Eq. (2.18) can be written as

$$Y(s) = \int_0^\infty \left[\int_0^\infty h(t-\tau)x(\tau)d\tau\right]e^{st}dt$$

Interchanging the order of integration, we get

$$Y(\mathrm{s}) = \int_0^\infty \int_0^\infty e^{-st}h(t-\tau)x(\tau)dtd\tau$$

$$= \int_0^\infty x(\tau)\left[\int_0^\infty e^{-st}h(t-\tau)dt\right]d\tau$$

Consider the integral with respect to t and let $\sigma = t - \tau$; thus

$$\int_0^\infty e^{-st}h(t-\tau)dt = \int_{\sigma=-t}^\infty e^{-s(\sigma+\tau)}h(\sigma)d\sigma$$

$$= e^{-\tau s}\int_0^\infty e^{-s\sigma}h(\sigma)d\sigma$$

$$= H\ (s)\ \mathrm{e}^{-\tau s}$$

Thus, we have

$$Y(s) = \int_0^\infty e^{-\tau s}x(\tau)H(s)d\tau$$

Since $H(s)$ is independent of τ we get

$$Y(s) = H(s)\int_0^\infty e^{-s\tau}x(\tau)d\tau$$

As a result,

$$Y(s) = H(s)\ X(s) \tag{2.20}$$

Our conclusion is that the Laplace transform of the convolution of $h(t)$ and $x(t)$ is the product of the Laplace transforms $H(s)$ and $X(s)$.

The Inverse Laplace Transform

Consider the simple differential equation

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = u(t)$$

Application of the Laplace transform to both sides, assuming zero initial conditions, gives us

$$(a_2s^2 + a_1s + a_0)\ X(s) = U(s)$$

This is an algebraic equation which can be written as

$$X(s) = \frac{U(s)}{a_2 s^2 + a_1 s + a_0}$$

Suppose now that the input function $u(t)$ is a unit step; thus

$$U(s) = \frac{1}{s}$$

As a result, the Laplace transform of $x(t)$ is given by

$$X(s) = \frac{1}{s(a_2 s^2 + a_1 s + a_0)}$$

Finding the function $x(t)$ whose transform is as given above is symbolized by the inverse transform operator; $\mathscr{L}^{-1}$ thus

$$s(t) = \mathscr{L}^{-1}\{X(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s(a_2 s^2 + a_1 s + a_0)}\right\}$$

A formal definition of the *inverse Laplace transform* is given by

$$x(t) = \mathscr{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{a-j\infty}^{a+\infty} X(s)e^{st}\,ds \qquad (2.21)$$

where a is a real constant. It is quite possible (although somewhat involved) to obtain the inverse Laplace transform by performing the integration indicated in Eq. (2.21). A much more effective way is commonly employed in control systems engineering, which relies on performing a partial fraction expansion that results in an expression of $X(s)$ as the sum of the functions $X_1(s)$, $X_2(s)$,..., $X_n(s)$.

$$X(s) = X_1(s) + X_2(s) + \ldots + X_n(s) \qquad (2.22)$$

The functions $X_1(s)$, $X_2(s)$,... can then be looked up in a table of Laplace transform pairs and hence we can obtain the corresponding inverses $X_1(t)$, $X_2(t)$,...,$X_n(t)$. The final result is thus

$$X(t) = X_1(t) + X_2(t) + \ldots + X_n(t) \qquad (2.23)$$

Partial Fraction Expansion

In most control systems engineering applications it is desired to find the inverse Laplace transform of a function F(s) expressed as the ratio of two functions N(s) and D(s) according to

$$F(s) = \frac{N(s)}{D(s)} \qquad (2.24)$$

The numerator and denominator functions are commonly obtained as polynomials in s of the form

$$N(s) = \sum_{j=0}^{m} b_j s^j \qquad (2.25)$$

$$D(s) = \sum_{i=0} a_i s^i \tag{2.26}$$

The denominator is an nth-degree polynomial, while the numerator is of mth degree, with $n > m$. For example, a typical function $F(s)$ is

$$F(s) = \frac{4+s}{2+3s+s^2}$$

Thus

$$N(s) = 4 + s$$
$$D(s) = 2 + 3s + s^2$$

In order to carry out the partial fraction expansion procedure it is necessary to obtain $N(s)$ and $D(s)$ in the following factored form:

$$N(s) = (s - z_1)(s - z_2) \ldots (s - z_m) \tag{2.27}$$
$$D(s) = (s - p_1)(s - p_2) \ldots (s - p_n) \tag{2.28}$$

In Eq. (2.27), the z_1, z_2, ..., z_m are called the zeros of the function $F(s)$, while in Eq. (2.28), the p_1, p_2, ..., p_n are called the poles of the function $F(s)$. Obtaining the poles and zeros from definitions of Eqs. (2.25) and (2.26) may or may not be straightforward. In the preceding example it is easy to see that

$$z_1 = -4$$
$$p_1 = -1$$
$$p_2 = -2$$

This follows since $D(s)$ is a second-order polynomial which can be easily factored.

For higher-order polynomials it is often necessary to resort to an iterative procedure to obtain the necessary factors. Consider, for example, the expression

$$D(s) = s^4 + 9s^3 + 26s^2 + 24s$$

It is easy to see that s is a common factor.

$$D(s) = s\, D_1(s)$$

with

$$D_1(s) = s^3 + 9s^2 + 26s + 24$$

Now $D_1(s)$ is a third-order polynomial and we have to find its roots.

Althouth there is a well-defined procedure for doing just that, we illustrate the use of an iterative method such as *Newton's method.* We start with an estimate $s^{(0)}$ if the solution, and obtain $s^{(1)}$ using

$$s^{(1)} = s^{(0)} - \frac{D_1(s^{(0)})}{D'_1(s^{(0)})}$$

where $D'_1(s)$ is the derivative of $D_1(s)$. In our example we have

$$D'_1(s) = 3s^2 + 18s + 26$$

Thus take $s^{(0)} = -1$ to obtain

$$s^{(1)} = -1 - \frac{-1+9-26+24}{3-18+26}$$
$$= -1.55$$

Put

$$s = -5; \text{ thus } A_2 = \frac{-1}{18}$$

Put

$$s = -11; \text{ thus } A_3 = \frac{1}{54}$$

As a result,

$$F(s) = \frac{1}{54}\left(\frac{2}{s+2} - \frac{3}{s+5} + \frac{1}{s+11}\right)$$

The inverse Laplace transform is thus

$$f(t) = \frac{1}{54}(2e^{-2t} - 3e^{-5t} + e^{-11t})$$

Consider now the case when a pole is repeated in the $F(s)$ of Eq. (2.29). Assume that $p_1 = p_2 = ... = p_{n1}$ in Eq. (2.29), so that we have

$$F(s) = \frac{N(s)}{(s-p_1)^{n1}(s-p_{n1+1})...(s-p_n)} \tag{2.37}$$

Clearly, we need an alternative expression to Eq. (2.30). The required expression is

$$F(s) = \frac{A_1}{s-p_1} + \frac{A_2}{(s-p_1)2} + ...$$

$$= \frac{A_{n1}}{(s-p_1)^{n1}} + \frac{A_{n1+1}}{s-p_{n1+1}} + ... + \frac{A_n}{s-p_n} \tag{2.38}$$

To obtain the coefficients A_1, A_2, ..., A_n we adopt the procedure outlined previously with a slight modification, as shown in the next two examples.

Example 2.4. Consider the function $f(t)$ defined by its Laplace transform

$$F(s) = \frac{1}{s^2(s+3)}$$

Find $f(t)$ using partial fraction expansion.

Solution: We write

$$F(s) = \frac{A_1}{s} + \frac{A_2}{s2} + \frac{A_3}{s+3}$$

Thus,

$$A_{1s}(s + 3) + A_2(s + 3) + A_3s^2 = 1$$

Put

$$s = 0; \text{ thus } A_2 = \frac{1}{3}$$

The inverse Laplace tranform is thus given by

$$f(t) = \frac{1}{9}(1 - e^{-3t} - 3te^{-3t})$$

There are situations where the presence of complex-conjugate poles makes it impossible to find a real root using Newton's method. The following example involves such a situation and proposes a method for treatment.

Example 2.6. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^4 + 6s^3 + 17s^2 + 24s + 18}$$

Solution: The denominator function is given by

$$D(s) = s^4 + 6s^3 + 17s^2 + 24s + 18$$

We attempt to factor $D(s)$ into the product of two second-order polynomials

$$D(s) = D_1(s)D_2(s)$$

where,

$$D_1(s) = s^2 + a_1 s + b_1$$
$$D_2(s) = s^2 + a_2 s + b_2$$

Performing the multiplications and equating coeffcients of equal powers, we conclude that

$$a_1 + a_2 = 6$$
$$b_1 + b_2 + a_1 a_2 = 17$$
$$a_1 b_2 + a_2 b_1 = 24$$
$$b_1 b_2 = 18$$

The equations above can be combined to yield one equation in b_1 given by

$$b^6_1 - 17b^5_1 + 126b^4_1 - 612b^3_1 + 2268b^2_1 - 5508b_1 + 5832 = 0$$

An iterative solution yields

$$b_1 = 3$$

Thus,

$$b_2 = \frac{18}{b_1} = 6$$

Hence, $\quad 6a_1 + 3a_2 = 24$

Also, $\quad a_1 + a_2 = 6$

As a result, we find that

$$a_1 = 2$$
$$a_2 = 4$$

We can thus conclude that

$$D(s) = (s^2 + 2s + 3)(s^2 + 4s + 6)$$

Instead of performing the partial fraction expansion in terms of simple poles, we do it in terms of the second-order terms as

$$F(s) = \frac{A_1 + B_1 s}{s^2 + 2s + 3} + \frac{A_2 + B_2 s}{s^2 + 4s + 6}$$

Since

$$F(s) = \frac{1}{(s^2+2s+3)(s^2+4s+6)}$$

we thus have

$$(A_1 + B_1 s)(s^2 + 4s + 6) + (A_2 + B_2 s)(s^2 + 2s + 3) = 1$$

Expanding, we obtain

$$(B_1 + B_2)s^3 + (A_1 + A_2 + 4B_1 + 2B_2)s^2 + (4A_1 + 6B_1 + 2A_2 + 3B_2)\mathrm{s} + (6A_1 + 3A_2) = 1$$

Equating the coefficients of equal power in s on both sides, we get

$$\begin{aligned} B_1 + B_2 &= 0 \\ A_1 + A_2 + 4B_1 + 2B_2 &= 0 \\ 4A_1 + 2A_2 + 6B_1 + 3B_2 &= 0 \\ 6A_1 + 3A_2 &= 1 \end{aligned}$$

Solving the equations above, we obtain

$$A_1 = -\frac{1}{9}$$

$$A_2 = \frac{5}{9}$$

$$B_1 = -\frac{2}{9}$$

$$B_2 = \frac{2}{9}$$

Thus, the partial fraction of $F(s)$ is

$$F(s) = \frac{1}{9}\left[\frac{5+2s}{s^2+4s+6} - \frac{1+2s}{s^2+2s+3}\right]$$

To get the inverse Laplace transform, we write F(s) as

$$\mathscr{L}^{-1}F(s) = \mathscr{L}^{-1}\frac{1}{9}\left[\frac{1}{(s+2)^2+2} + \frac{2(s+2)}{(s+2)^2+2} + \frac{1}{(s+1)^2+2} - \frac{2(s+1)}{(s+1)^2+2}\right]$$

Thus, the inverse Laplace transform is obtained as

$$f(t) = \frac{1}{9}\left(\frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t + 2e^{-2t}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2} - 2e^{-t}\cos\sqrt{2}t\right)$$

2.3. MECHANICAL SYSTEM ELEMENTS

Most feedback control systems contain mechanical as well as electrical components. From a mathematical viewpoint, the descriptions of electrical and mechanical elements are analogous. In fact, we can show that given an electrical device, there is usually an analogous mechanical counterpart, and vice versa.

The analogy, of course, is a mathematical one; that is, two systems are analogous to each other if they are described mathematically by similar equations.

The motion of mechanical elements can be described in various dimensions as translational, rotational, or a combination of both. The equations governing the motions of mechanical systems are often directly or indirectly formulated from Newton's law of motion.

2.3.1. Translational motion

Translational motion takes place along a straight line and the variables involved in describing a straight-line motion are displacement, velocity and acceleration. Newton's law of motion governs the linear motion. According to this law, the product of mass and acceleration is equal to the algebraic sum of forces acting on it.

Newton's law of motion states that *the algebraic sum of forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction.* The law can be expressed as

$$\Sigma\text{forces} = Ma \tag{2.39}$$

where M denotes the mass and a is the acceleration in the direction considered.

2.3.2. Mass

The function of mass in linear motion is to strore kinetic energy. Mass cannot store potential energy. Suppose a force is applied to mass M as shown in Fig. 2.1, the mass starts moving in x direction as shown. For the time being, we will assume other forces such as friction, etc. to be zero. Hence, according to Newton's law,

$$M\frac{d^2x}{dt^2} = f(t) \tag{2.40}$$

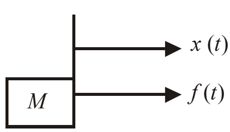


Fig. 2.1

2.3.3. Linear spring

A spring can store potential energy. In a system, there may be a spring where some components such as elastic string, cable, etc. may work as a spring. Strictly speaking, a spring is a non-linear element. However, we can assume it to be linear for small deformations.

Let us assume a spring has negligible mass and connected to a rigid support as shown in Fig. 2.2.

system and vice versa. It is always advantageous to obtain **electrical analogous** of the given mechanical system as we are well familiar with the methods of analysing electrical network than mechanical systems.

There are two methods of obtaining electrical analogous networks, namely,

1. Force-voltage Analogy, i.e. Direct Analogy. (Table 2.3)
2. Force-current Analogy, i.e. Inverse Analogy. (Table 2.4)

Mechanical Systems

Consider simple mechanical system as shown in the Fig. 2.7.

Due to the applied force, mass M will displace by an amount $x(t)$ in the direction of the force $F(t)$ as shown in the Fig. 2.7.

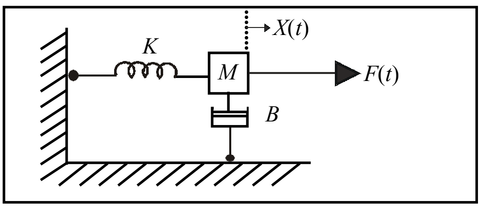


Fig. 2.7

According to Newton's law of motion, applied force will cause displacement $x(t)$ in spring, acceleration to mass M against frictional force having constant B.

$$F(t) = Ma + Bv + K\,x(t) \tag{2.45}$$

where, a = acceleration, v = velocity

$$F(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) \tag{2.46}$$

Taking Laplace, $$F(s) = Ms^2\,X(s) + Bs\,X(s) + K\,X\,(s) \tag{2.47}$$

This is equilibrium equation for the given system.

Now we will try to derive analogous electrical network.

Force-voltage Analogy (Loop Analysis)

In this method, to the force in mechanical system, voltage is assumed to be analogous one. Accordingly we will try to derive other analogous terms. Consider electric network as shown in the Fig. 2.8. The equation according to Kirchhoff's law can be written as,

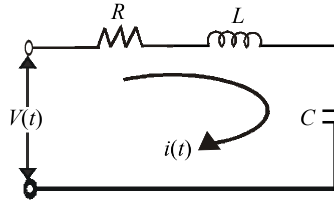


Fig. 2.8

The equation according to Kirchhoff's current law for above system is,

$$I = I_L + I_R + I_C \quad (2.53)$$

Let node voltage be V,

$$\therefore \quad I = \frac{1}{L}\int V dt + \frac{V}{R} + C\frac{dV}{dt} \quad (2.54)$$

Taking Laplace,

$$I(s) = \frac{V(s)}{sL} + \frac{V(s)}{R} + sCV(s) \quad (2.55)$$

But to get this equation in the similar form as that of $F(s)$ we will use,

$$V(t) = \frac{d\phi}{dt} \quad \text{where } \phi = \text{flux} \quad (2.56)$$

$$V(s) = s\,\phi\,(s) \quad \text{i.e. } \phi\,(s) = \frac{V(s)}{s} \quad (2.57)$$

Substituting in equation for $I(s)$

$$I(s) = Cs^2\phi(s) + \frac{1}{R}s\phi(s) + \frac{1}{L}\phi(s) \quad (2.58)$$

Comparing equations for $F(s)$ and $I(s)$ it is clear that,

(*i*) Capacitor 'C' is analogous to mass M.

(*ii*) Reciprocal of resistance $\frac{1}{R}$ is analogous to frictional constant B.

(*iii*) Reciprocal of inductance $\frac{1}{L}$ is analogous to spring constant K.

Table 2.6. Force-Current Analogy

Translational	Rotational	Electrical
F Force M Mass B Friction K Spring x Displacement x Velocity $= \frac{dx}{dt}$	T J B K θ $0 = \frac{d\theta}{dt} = \omega$	Current I C $1/R$ $1/L$ ϕ Voltage 'e' $= \frac{d\phi}{dt}$

Note: The elements which are in series in F—V analogy, get connected in parallel in F—I analogous network and which are in parallel in F—V analogy, get connected in series in F—I analogous network.

Stepwise Procedure to Solve Problems on Analogous Systems

1 : Identify all the displacements due to the applied force. The elements spring and friction between two moving surfaces cause change in displacement.

Substituting value of $\theta_1(s)$ from equation (2) in equation (1), we get

$$\left[\frac{J_1 s^2 + K)(J_2 s^2 + fs + K)}{K} - \frac{K}{1}\right]\theta(s) = T(s)$$

$$\therefore \qquad \text{Transfer function} = \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)(J_2 s^2 + fs + K) - K^2}$$

$$= \frac{K}{J_1 J_2 s^4 + J_1 f s^3 + (KJ_1 + KJ_2)s^2 + Kfs} \quad \textbf{Ans.}$$

Example 2.8. Obtain the differential equations describing the complete dynamics of the mechanical system shown in Fig. 2.12.

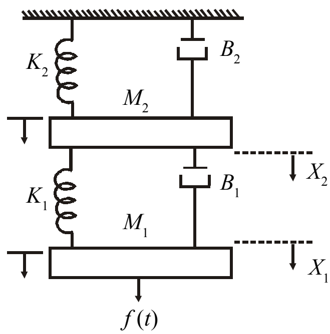


Fig. 2.12

Solution: The mechanical network diagram is shown in Fig. 2.13.

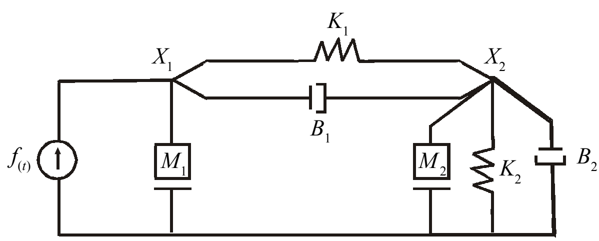


Fig. 2.13

Node 'x_1'

$$f_{(t)} = M_1\ddot{x}_1 + K_1(x_1 - x_2) + B_1(\dot{x}_1 - \dot{x}_2)$$

or $$F(s) = (M_1 s^2 + B_1 s + K_1)\, X_1(t) - (B_1 s + K_2)\, X_2(s)$$

Node 'x_2'

$$K_1(x_1 - x_2) + B_1(\dot{x}_1 - \dot{x}_2) = M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2 x^2$$

or $$[M_2 s^2 + (B_1 + B_2)s + (K_1 + K_2)]\, X_2(s) - [B_1 s + K_1]\, X_{1(s)} = 0$$

The electrical analog based on force-voltage analogy is shown in Fig. 2.14.

The electrical analog circuit is drawn with the help of electrical analog equations which are obtained from nodal equations in Laplace domain. The electrical analog equations are

$$E(s) = \left(L_1 s^2 + R_1 s + \frac{1}{C_1}\right)Q_1(s) - \left(R_1 s + \frac{1}{C_1}\right)Q_2(s)$$

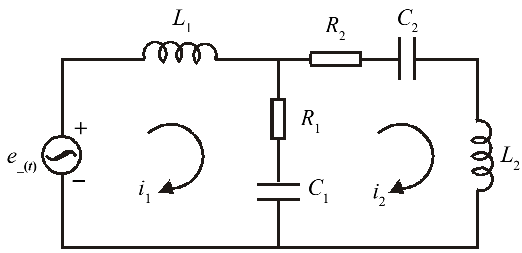


Fig. 2.14

and

$$\left(L_2 s^2 + \frac{R_2}{C_2}s + R_1 s + \frac{1}{C_1}\right)Q_2(s) - \left(R_1 s + \frac{1}{C_1}\right)Q_2(s) = 0$$

where Q = charge.

Converting the above equations into differential equation form

$$e_{(1)} = L_1\frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + R_1 i_1 - R_1 i_2 - \frac{1}{C_1}\int i_2 dt$$

and

$$L_2\frac{di_2}{dt} + R_2 i_2 + R_1 i_2 + \frac{1}{C_1}\int i_2 dt + \frac{1}{C_1}\int i_2 dt - R_1 i_1 - \frac{1}{C_1}\int i_1 dt = 0$$

Note: Above equations are also obtained, if we apply Kirchhoff's laws to the electrical network shown in Fig. 2.14.

Example 2.9. Obtain the nodal equations for the system shown in Fig. 2.15 and draw its analogous electrical network.

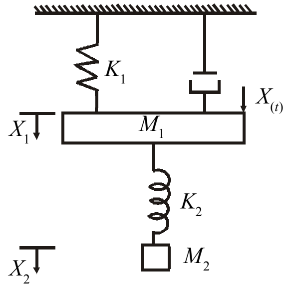


Fig. 2.15

or

$$L_2 \frac{d^2 q_2}{dt} + \frac{q_2}{C_2} - \frac{q_1}{C_1} = 0$$

$$\left(s^2 L_1 + \frac{1}{C_2} \right) Q_2(s) - \frac{Q_1(s)}{C_1} = 0$$

Note: These equations are similar to the nodal equations.

Example 2.10. Obtain the nodal equations for the system shown in Fig. 2.18 and draw its electrical analog based on force-current analogy.

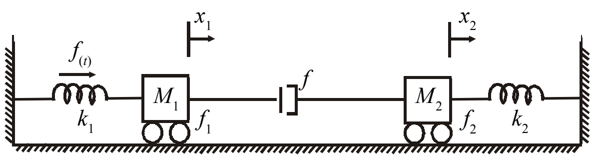


Fig. 2.18

Solution: The mechanical network is shown in Fig. 2.19.

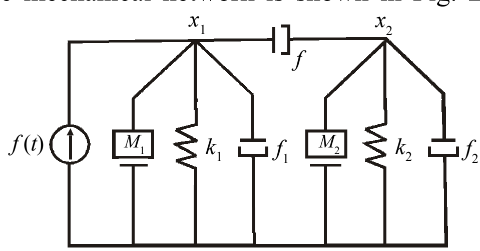


Fig. 2.19

Node 'x_1'

$$M_1\ddot{x}_1 + f_1\dot{x}_1 + f\left(\dot{x}_1 - \dot{x}_2\right) + K_1 x_1 = f_{(t)}$$

Node 'x_2'

or

$$M_2\ddot{x}_2 + f_2\dot{x}_2 + K_2 x_2 = f\left(\dot{x}_1 - \dot{x}_2\right)$$

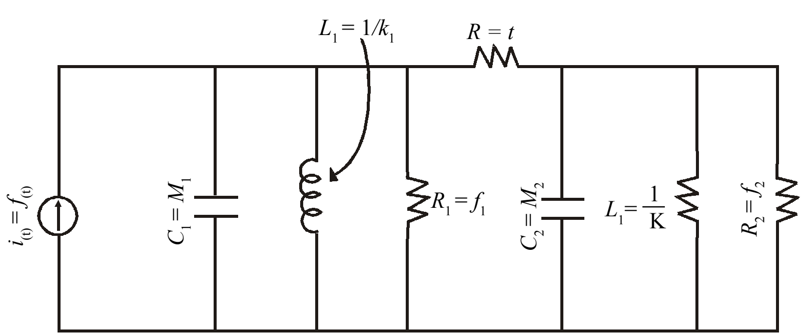


Fig. 2.20

Electrical analog circuit based on force-current analogy is shown by Fig. 2.20.

Example 2.11. Obtain transfer function for the system shown in Fig. 2.21.

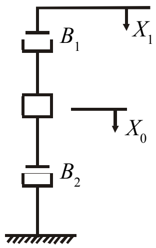


Fig. 2.21

Solution: Writing equations for the given system based on Newton's law, we get

$$B(\dot{x}_1 - \dot{x}_0) = M\dot{x}_0 + B_2\dot{x}_{(0)}$$

$$B_1 s\ X_1(s) = (Ms^2 + B_2 s + B_1 s)\ X_0(s)$$

$$\frac{X_0(s)}{X_1(s)} = \frac{B_1 s}{M_1 s^2 + B_2 s + B_1 s} = \frac{B_1}{M_1 s + B_1 + B_2} \quad \textbf{Ans.}$$

Example 2.12. Obtain transfer function for the system shown in Fig. 2.22.

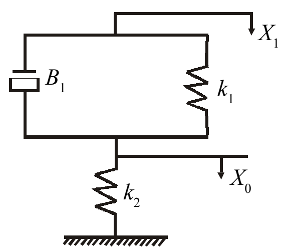


Fig. 2.22

Solution: Writing Newton's law equations

$$K_1\ (x_1 - x_0) + B_1(\dot{x}_1 - \dot{x}_0) - K_2\ x_0 = 0$$

$$[B_1 + K_1]\ X_1(s) - (B_1 + K_1 + K_2)\ X_0(s) = 0$$

$$\frac{X_0(s)}{X_1(s)} = \frac{B_1 + K_1}{B_1 + K_1 + K_2} \quad \textbf{Ans.}$$

2.6. TRANSFER FUNCTION

It has been shown already that the input and output of a linear system in general, is related by a linear or a set of linear differential equations. Such relationships are capable of completely describing the system behaviour in the presence of a particular input excitation and known initial conditions.

differential equation of Eq. (2.63) is seldom used in its original form for the analysis and design of control systems.

To obtain the transfer function of the linear system that is represented by Eq. (2.63), we simply take the Laplace transform on both sides of the equation, and assume *zero initial conditions.* The result is

$$(s^n + a_n s^{n-1} + \ldots + a_2 s + a_1)C(s) = (b_{m+1}s^m + b_m s^{m-1} + \ldots + b_2 s + b_1)R(s) \qquad (2.64)$$

The transfer function between $r(t)$ and $c(t)$ is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_{m+1}s^m + b_m s^{m-1} + \ldots + b_2 s + b_1}{s^n + a_n s^{n-1} + \ldots + a_2 s + a_1} \qquad (2.65)$$

We can summarize the properties of the transfer function as follows:

1. Transfer function is defined only for a linear time-invariant system. It is meaningless for nonlinear systems.
2. The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response. Alternately, the transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input.
3. When defining the transfer function, all initial conditions of the system are set to zero.
4. The transfer function is independent of the input of the system.
5. Transfer function is expressed only as a function of the complex variable s. It is not a function of the real variable, time, or any other variable that is used as the independent variable.

Transfer Function (Multivariable Systems)

The definition of transfer function is easily extended to a system with a multiple number of inputs and outputs. A system of this type is often referred to as the *multivariable system.* In a multivariable system, a differential equation of the form of Eq. (2.63) may be used to describe the relationship between a pair of input and output variables. When dealing with the relationship between one input and one output, it is assumed that all other inputs are set to zero. Since the principle of superposition is valid for linear systems, the total effect on any output variable due to all the inputs acting simultaneously is obtained by adding up the outputs due to each input acting alone.

A number of examples is appropriate to illustrate the concept of transfer function.

Example 2.13. Consider the *RC* integrating configured network shown in Figure 2.23. The current at the output terminals in zero, and we can write the input voltage as

$$V_i(s) = R + \frac{1}{Cs} I(s)$$

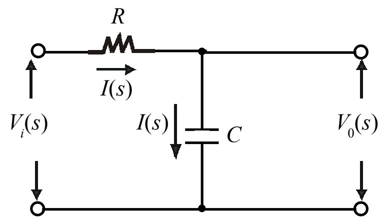


Fig. 2.23. *RC* Integrating Configuration Network

The output voltage is given by

$$V_0(s) = \frac{1}{Cs} I(s)$$

The transfer function is thus obtained as

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

Example 2.14. For the differentiating configured *RC* network shown in Figure 2.24, we can write the transfer function as

$$\frac{V_0(s)}{V_i(s)} = \frac{R}{R + 1/Cs} = \frac{RCs}{1 + RCs}$$

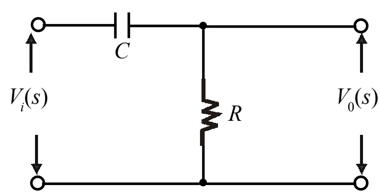


Fig. 2.24. *RC* Differentiating Configuration Network

Example 2.15. For the spring-dashpot system shown in Figure 2.25, we can write a force balance equation as

$$B\dot{x}_0 + Kx_0 = Kx_i$$

Employing the Laplace transform, we thus have

$$(Bs + K)\, X_0(s) = KX_i(\mathrm{s})$$

As a result, the transfer function is given by

$$\frac{X_0(s)}{X_i(s)} = \frac{K}{Bs + K}$$

or

$$\frac{X_0(s)}{X_i(s)} = \frac{1}{1 + (B/K)s}$$

Note the similarity of this transfer function and that of Example 2.13.

Thus, the transfer function is given by

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

This is clearly similar to the transfer function of Example 2.18.

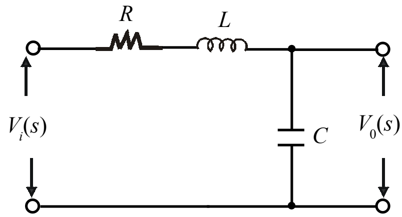


Fig. 2.29 *RLC* Network

Example 2.20. For the lead-lag RC network shown in Figure 2.30, we can write the following impedance functions:

$$Z_1 = \frac{R_1/C_1s}{R_1 + 1/C_1s}$$

$$= \frac{R_1}{1 + R_1C_1s}$$

Let

$$\tau_a = R_1C_1$$

Thus,

$$Z_1 = \frac{R_1}{1 + \tau_a s}$$

Also,

$$Z_2 = R_2 + \frac{1}{C_2s}$$

$$= \frac{1 + R_2C_2s}{C_2s}$$

Let

$$\tau_b = R_2C_2$$

Thus,

$$Z_2 = \frac{1 + \tau_b s}{C_2s}$$

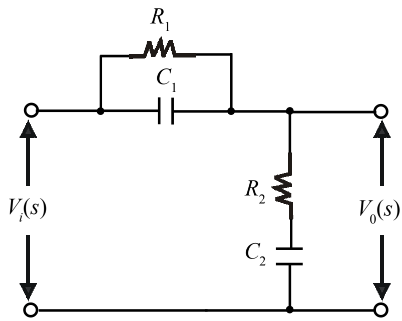


Fig. 2.30. Lead-Lag *RC* Network

The transfer function is thus given by

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

This is written in terms of the network elements as

$$\frac{V_0(s)}{V_i(s)} = \frac{(1+\tau_a s)(1+\tau_b s)}{(1+\tau_a s)(1+\tau_b s) + R_1 C_2 s}$$

Let $\tau_{ab} = R_1 C_2$; thus

$$\frac{V_0(s)}{V_i(s)} = \frac{(1+\tau_a s)(1+\tau_b s)}{\tau_a \tau_b s^2 + (\tau_a + \tau_b + \tau_{ab})s + 1}$$

We can rewrite this as

$$\frac{V_0(s)}{V_i(s)} = \frac{1+\tau_a s \; 1+\tau_b s}{1+\tau_1 s \; 1+\tau_2 s}$$

where,

$$\tau_1 \tau_2 = \tau_a \tau_b$$
$$\tau_a + \tau_b + \tau_{ab} = \tau_1 + \tau_2$$

The examples above help to illustrate the concept of a transfer function.

2.7. BLOCK DIAGRAM ALGEBRA

Introduction

If a given system is complicated, it is very difficult to analyse it as a whole, with the help of transfer function approach, we can find transfer function of each and every element of the complicated system. And by showing connection between the elements, complete system can be splitted into different blocks and can be analysed conveniently. This is the basic concept of block diagram representation.

Basically block diagram is a pictorial representation of the given system. It is very simple way of representing the given complicated practical system. In block diagram, the interconnection of system components to form a system can

A pictorial representation of the relationships between system variables is offered by the block diagram. In a block diagram, three ingredients are commonly present.

1. Functional block. This is a symbol representing the transfer between the input $U(s)$ to an element and the output $X(s)$ of the element. The block contains the transfer function $G(s)$, as shown in Figure 2.31. The arrow directed into the block

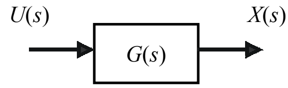


Fig. 2.31. Functional Block

represents the input $U(s)$, while that directed out of the block represents the output $X(s)$. The block shown represents the algebraic relationship

$$X(s) = G(s)U(s) \tag{2.66}$$

2. Summing point. This is a symbol denoted by a circle, the output of which is the algebraic sum of the signals entering into it. A minus sign close to an input signal arrow denotes that this signal is reversed in sign in the output expression. Figure 2.17 shows the relationship

$$E(s) = R(s) - C(s) \tag{2.67}$$

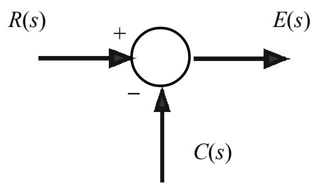


Fig. 2.32. Summing Point

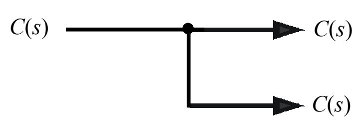


Fig. 2.33. Takeoff Point

3. Takeoff point. A takeoff point on a branch in a block diagram signifies that the same variable is being utilized elsewhere, as shown in Figure 2.33.

A fundamental block diagram configuration is the single-loop feedback system shown in Figure 2.34a. The output variable $C(s)$ is modified by the feedback element with transfer function $H(s)$ to produce the signal $B(s)$:

$$B(s) = C(s)\ H(s) \tag{2.68}$$

The signal $B(s)$ is compared to a reference signal $R(s)$ to produce the error.

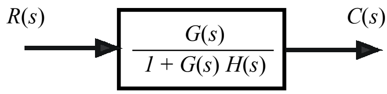


Fig. 2.35. Reduced form of Fig. 2.34

Equation (2.71) is valid for negative feedback system. Hence, for a positive feedback system we have

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)} \tag{2.72}$$

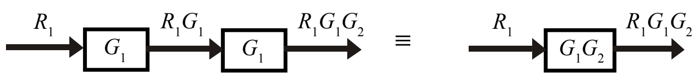
In general, for a positive/negative feedback systems, the control ratio is given by

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)} \tag{2.73}$$

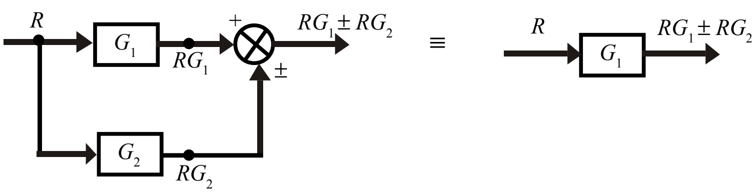
as the case may be.

Let us now discuss about the block diagram Reduction Techniques useful in the analysis of complex control systems.

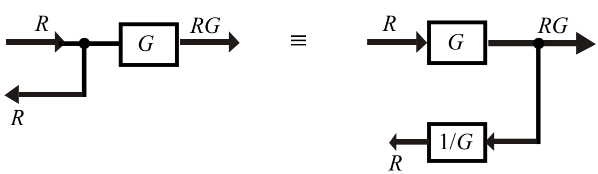
Rule (1): Combining blocks in cascade



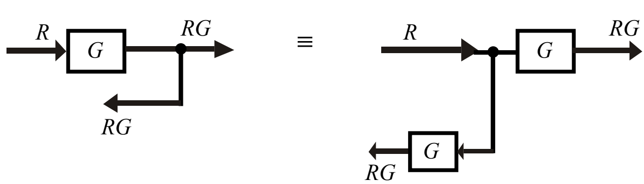
Rule (2): Combining blocks in Parallel



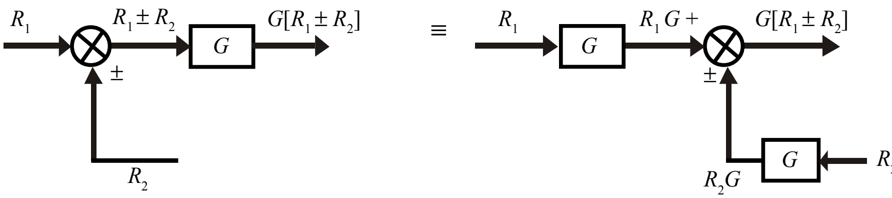
Rule (3): Moving a pick-off point after a block



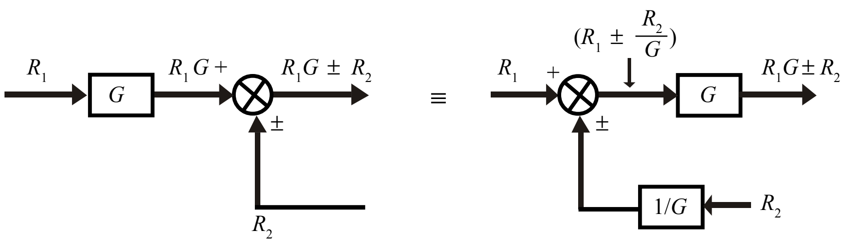
Rule (4): Moving a take off point ahead of a block



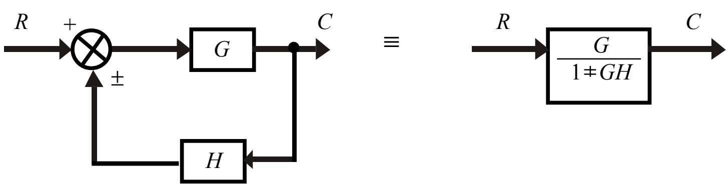
Rule (5): Moving a summing point after a block



Rule (6): Moving a summing point ahead of a block



Rule (7): Eliminating a feedback loop



Using the rules given above, let us reduce the block diagrams to arrive at the transfer functions of the system.

Example 2.21. Simplify the block diagram shown in Fig. 2.36 and obtain the closed loop transfer function $C(s)/R(s)$.

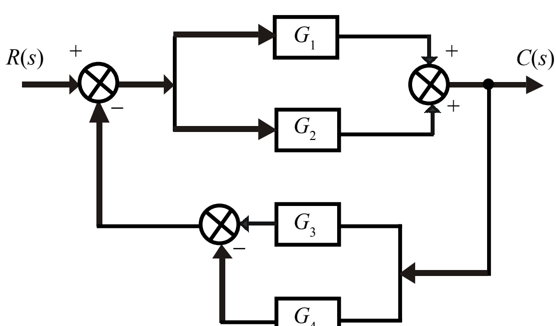


Fig. 2.36. Block diagram of Example 2.21

Solution: The given block diagram of Fig. 2.37 can be reduced as follows. Using Rule 2. (combining blocks in parallel)

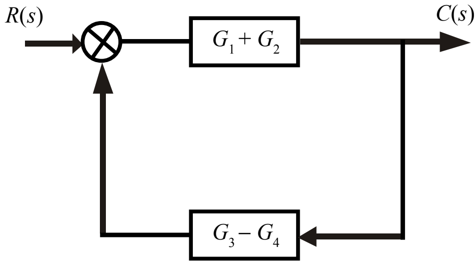


Fig. 2.37. Block diagram of Example 2.21

Fig. 2.37 is in the canonical form of control system and so the transfer function is obtained using rule 7 as

$$\frac{C(s)}{R(s)} = \frac{(G_1 + G_3)}{1 + (G_1 + G_2)(G_3 - G_4)}$$

which is shown in Fig. 2.38.

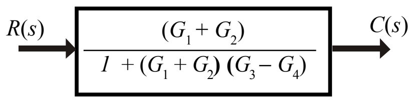


Fig. 2.38. Reduced form of Fig. 2.37

Example 2.22. Evaluate $\dfrac{C(s)}{R(s)}$ from the block diagram shown in Fig. 2.39.

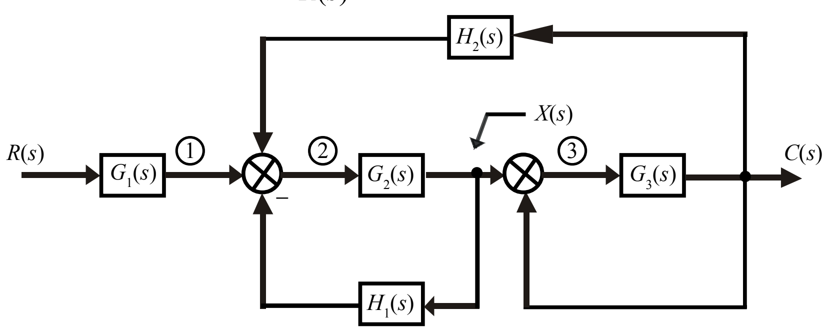


Fig. 2.39. Block diagram of Example 2.22

Solution: Let us define the output of $G_2(s)$ as $X(s)$.

At point (1), we have

$$R(s) \;.\; G_1(s)$$

At point (2), we have

$$RG_1 + CH_2 - XH_1$$

and for $X(s)$.

$$X = G_2\,[RG_1 + CH_2 - XH_1]$$

$$X = \frac{G_1G_2R + CG_2H_2}{1 + HG_2}$$

Step 3: Using rule 2, (combining blocks in parallel), we have Fig. 2.42 reduced to Fig. 2.43

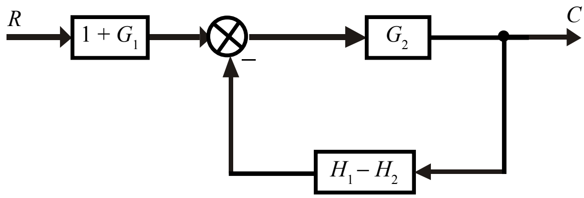


Fig. 2.43. Reduced form of Fig. 2.42

Step 4: Shifting the summing point ahead of the block by using rule 4, we have the block diagram of Fig. 2.43 reduced to Fig. 2.44.

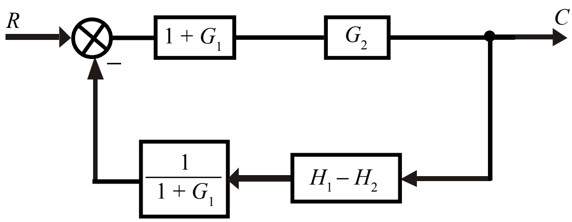


Fig. 2.44. Reduced form of Fig. 2.43

Step 5: Reducing Fig. 2.44 using rule 1, we have Fig. 2.44 reduced to Fig. 2.45.

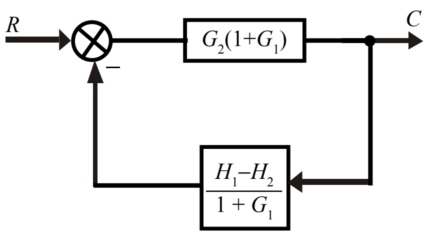


Fig. 2.45. Reduced form of Fig. 2.44

Step 6: Reducing Fig. 2.45 using rule 1, we have Fig. 2.45 reduced to Fig. 2.46.

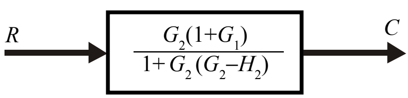


Fig. 2.46. Reduced form of Fig. 2.45

Therefore, the transfer function of the system of Fig. 2.46 is given by

$$\frac{C}{R} = \frac{G_2(1+G_1)}{1+G_2(H_1-H_2)}$$

Example 2.24. Using block diagram reduction techniques, find the closed-loop transfer function of the system whose block diagram is given in Fig. 2.47.

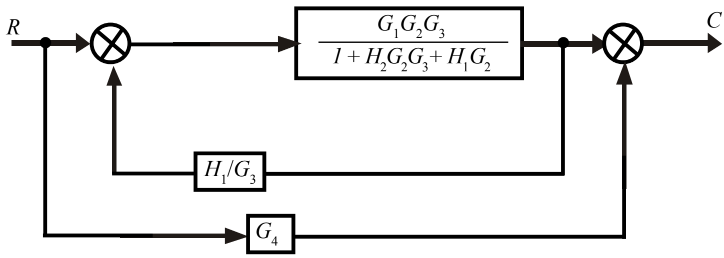


Fig. 2.57. Reduced form of Fig. 2.56

Step 4: Eliminating the feedback loop using rule 7, we have Fig. 2.57 reduced to Fig. 2.58.

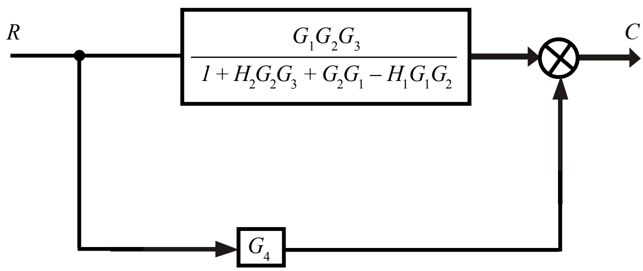


Fig. 2.58. Reduced form of Fig. 2.57

Step 5: Using rule 2 of combining two blocks in parallel, we have Fig. 2.59 reduced to Fig. 2.60.

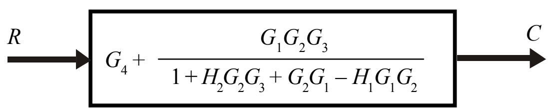


Fig. 2.59. Reduced form of Fig. 2.58

Hence, the transfer function is given by

$$\frac{C}{R} = G_4 + \frac{G_1G_2G_3}{1+H_2G_2G_3+G_2G_1-H_1G_1G_2}$$

Example 2.26: Simplify the block diagram shown in Fig. 2.60

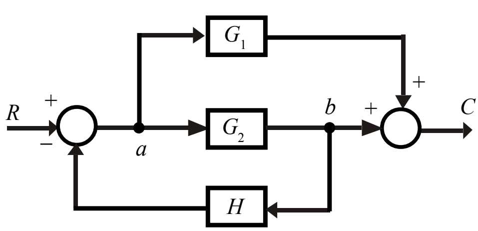


Fig. 2.60. Block diagram of Example 2.26

Solution: First, move the takeoff b to a as shown in Fig. 2.62*a*. Now we can see that G_1 and G_2 are in parallel and the block diagram reduces to that shown in Figure 2.62*b*. The feedback loop with a forward gain of 1 and feedback element H can be reduced as shown in Fig. 2.62*c*. Finally, Figure 2.62*d* shows that the overall transfer between R and C.

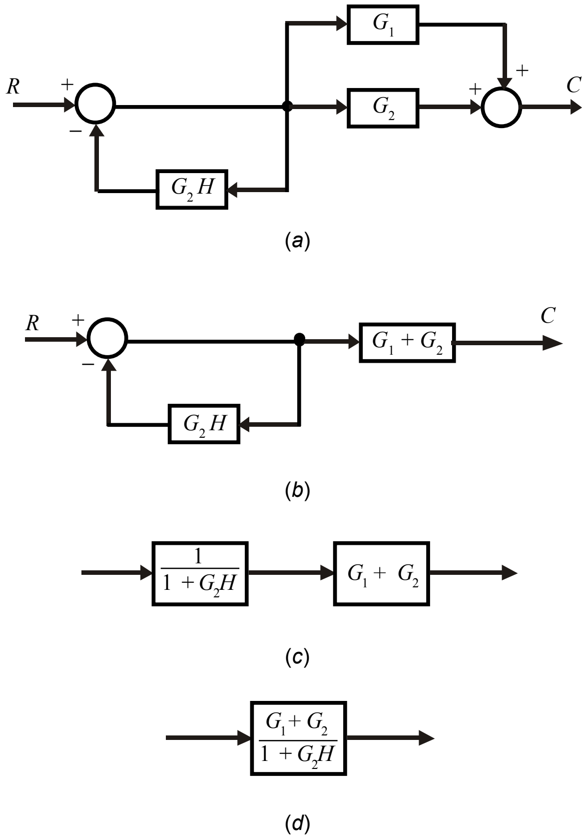


Fig. 2.61. Simplified Block Diagrams

Example 2.27. Use block diagram reduction techniques to obtain the ratio C/R for the system shown in the block diagram of Fig. 2.62.

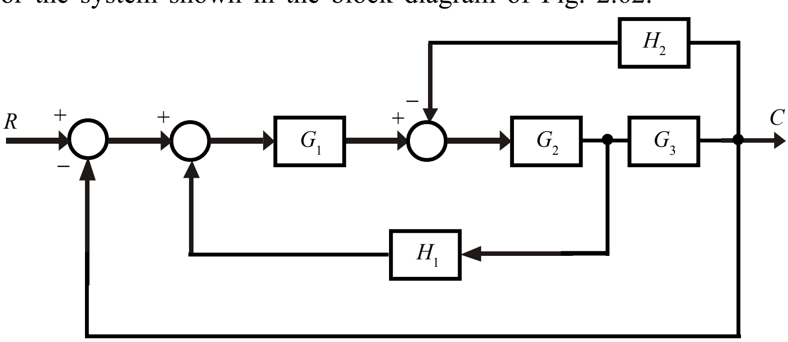


Fig. 2.62. Block diagram of Example 2.27

Solution: Evaluation of C/R_1 Assume $R_2 = 0$. Therefore, summing point No. 5 can be removed. Shift take off point No. 4 beyond block G_3.

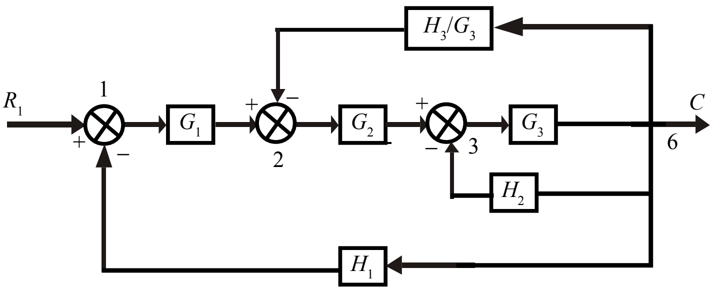


Fig. 2.65.

Eliminate the feedback loop between points 3 and 6.

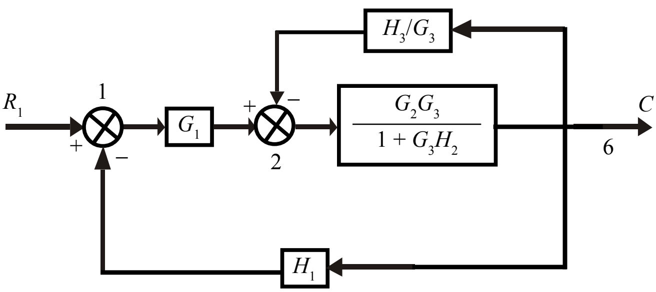


Fig. 2.66.

Eliminating the feedback loop again.

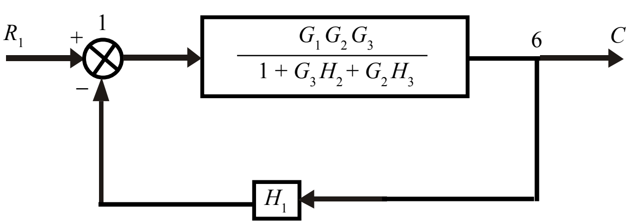


Fig. 2.67.

$$\frac{C}{R_1} = \frac{G_1G_2G_3}{1 + G_3H_2 + H_3G_2 + G_1G_2G_3H_1} \quad \textbf{Ans.}$$

Evaluation of C/R_2

Assume $R_1 = 0$. Thus, summing point No. 1 can be removed.

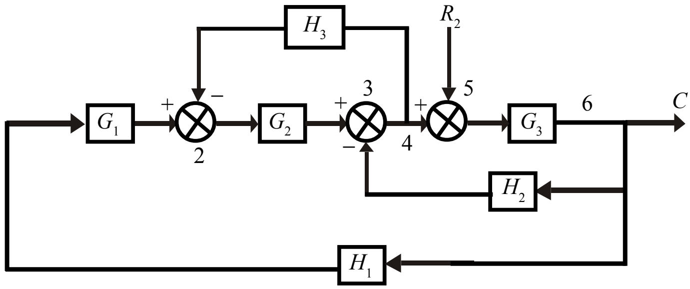


Fig. 2.68.

Shifting the summing point No. 2 and rearranging beyond G_2.

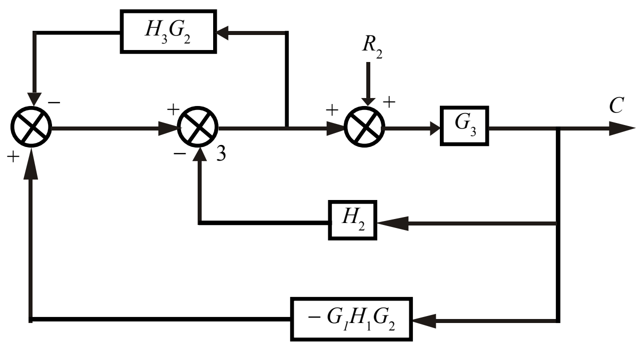


Fig. 2.69.

Rearranging, we get

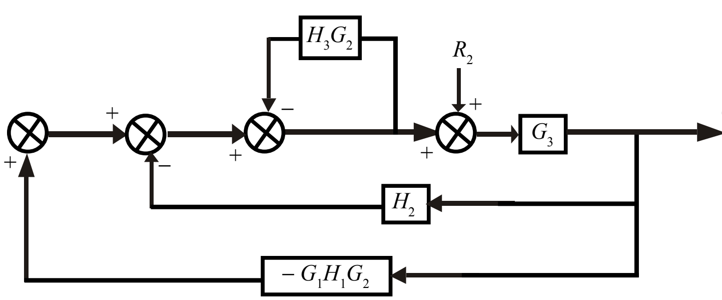


Fig. 2.70.

Rearranging and eliminating the feedback loop

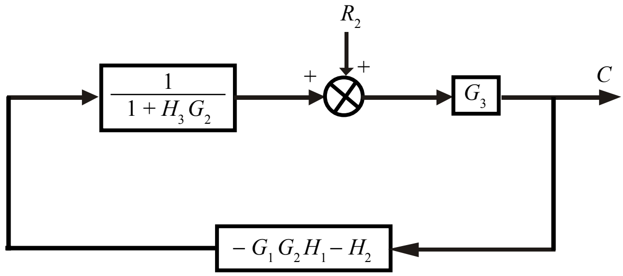


Fig. 2.71.

Rearranging,

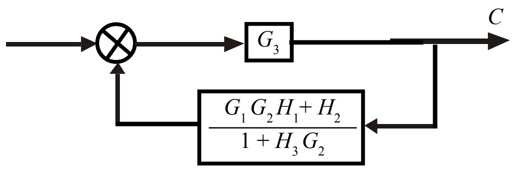


Fig. 2.72.

Eliminating the feedback loop, we get

$$\frac{C}{R_2} = \frac{G_3(1+H_3G_2)}{1+H_3G_2+G_3(G_1G_2H_1+H_2)} \quad \textbf{Ans.}$$

Exmaple 2.29. Find the transfer function for the block diagram shown in Fig. 2.73.

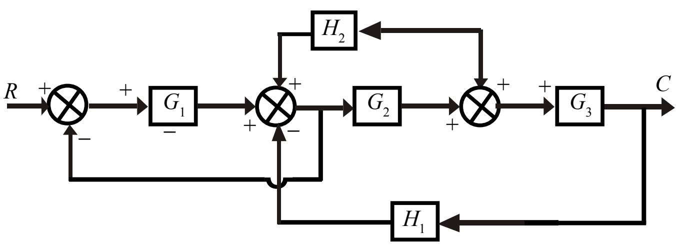


Fig. 2.73.

Solution:

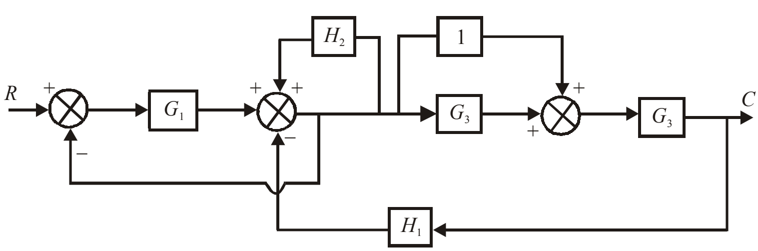


Fig. 2.74.

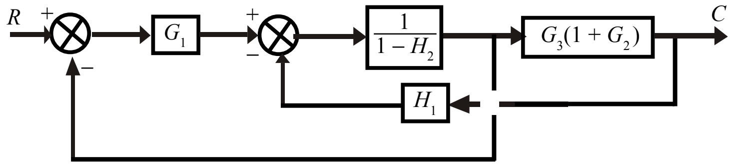


Fig. 2.75.

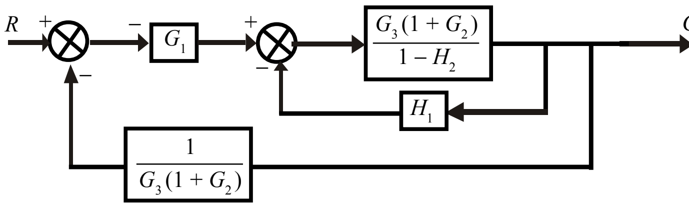


Fig. 2.76.

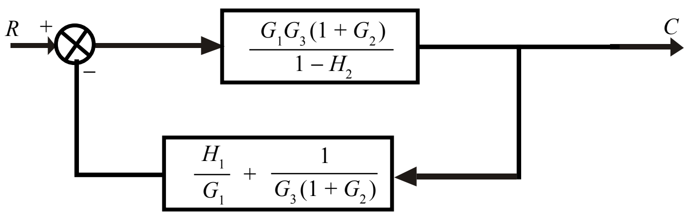


Fig. 2.77.

$$\frac{C}{R} = \frac{\dfrac{G_1G_3(1-G_2)}{1-H_2}}{1+\dfrac{G_1G_3(1+G_2)}{1-H_2}\left[\dfrac{H_1}{G_1}+\dfrac{1}{G_3(1+G_2)}\right]}$$

$\therefore$
$$\frac{C}{R} = \frac{G_1G_3(1+G_2)}{(1-H_2)+H_1G_3(1+G_2)+G_1}$$

Example 2.30: Find closed-loop transfer function of system shown in Fig. 2.78.

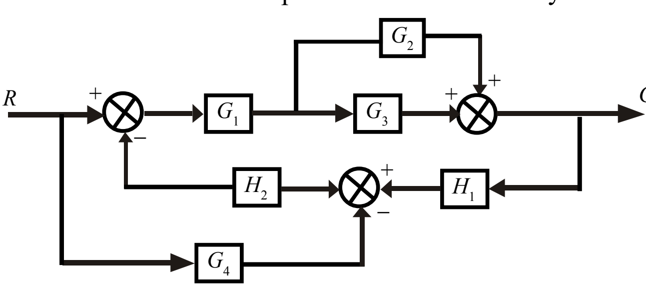


Fig. 2.78.

Solution:

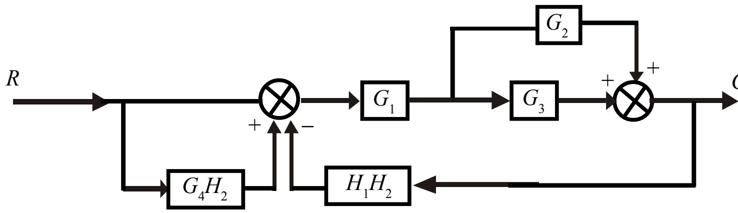


Fig. 2.79.

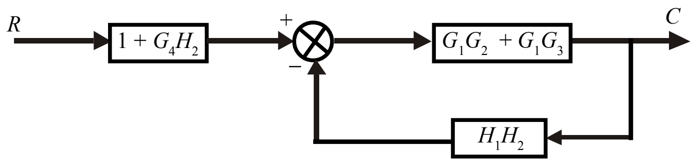


Fig. 2.80.

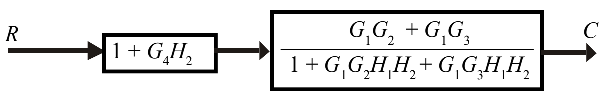


Fig. 2.81.

$$\therefore \qquad \frac{C}{R} = \frac{(1+G_4H_2)(G_1G_2+G_1G_3)}{1+G_1G_2H_1H_2+G_1G_3H_1H_2}$$

or

$$\frac{C}{R} = \frac{G_1(G_2+G_3)(1+G_4H_2)}{1+G_1H_1H_2(G_2+G_3)} \quad \textbf{Ans.}$$

2.8. SIGNAL FLOW GRAPHS

A signal flow graph may be regarded as a simplified notation for a block diagram, although it was originally introduced by S. J. Mason as a cause-and-effect representation of linear systems. In general, besides the difference in the physical appearances of the signal flow graph and the block diagram, the signal flow graph to be constrained by more rigid mathematical relationships, whereas the rules of using the block diagram notation are far more flexible and less stringent.

A signal flow graph may be defined as a graphical means of portraying the input-output relationships between the variables of a set of linear algebraic equations.

Consider that a linear system is described by the set of N algebraic equations

$$y_j = \sum_{k=1}^{N} a_{kj}\, y_k \quad j = 1, 2, ..., N \tag{2.74}$$

It should be pointed out that these N equations are written in the form of cause-and-effect relations:

$$j\text{th effect} = \sum_{k=1}^{N} (\text{gain from } k \text{ to } j)\ (k\text{th cause}) \qquad (2.75)$$

or simply

$$\text{output} = \Sigma\ (\text{gain})(\text{input}) \qquad (2.76)$$

This is the single most important axiom in the construction of the set of algebraic equations from which a signal flow graph is drawn.

This method is believed to provide a faster means for determining the response of multiloop systems than do the block diagram reduction techniques discussed in the previous section.

Consider a set of linear equations having the form

$$y_i = \sum_{j=1}^{n} a_{ij}\, y_j \quad i = 1, 2, ..., n$$

A node is assigned to each variable of interest as shown in Fig. 2.82a. A branch between two nodes relates the variables at both ends. In a fashion similar to block diagrams the gain between the variables is indicated on the branch of

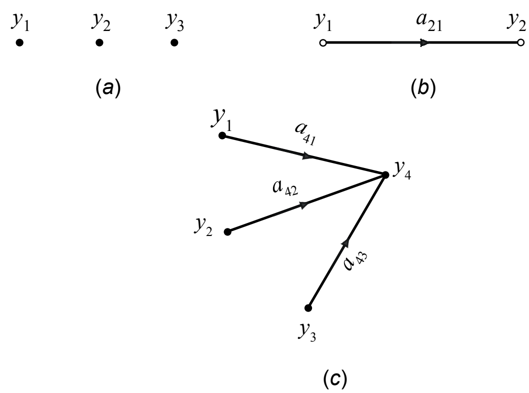


Fig. 2.82. Defining Signal Flow Graphs

(a) Nodes (b) Directed Branch (c) Summation Rule

an associated arrow. Thus, in Figure 2.82b we have

$$y_2 = a_{21}\, y_1$$

The value of a variable at a node is equal to the sum of all incoming signals. Thus, in Fig. 2.82c,

$$y_4 = a_{41}\, y_1 + a_{42}\, y_2 + a_{43}\, y_3$$

A number of definitions is appropriate at this time and will be discussed in next part.

2.8.1. Singal Flow Graph (SFG) Algebra

(a) Addition Rule: The value of the variable designated by a node equals the sum of all signals entering the node. For the equation given by

$$X_i = \sum_{j=1}^{n} a_{ij} X_j \tag{2.77}$$

the SFG is represented by Fig. 2.83.

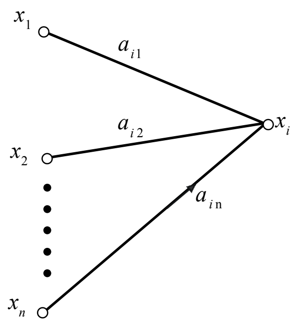


Fig. 2.83. SFG of equation (2.77)

(*b*) Transmission Rule: The value of the variable designated by a node is transmitted on every branch leaving that node.

The equation given by

$$X_1 = a_{ij} X_j, \; i = 1, 2, ..., n, \; j\text{-fixed} \tag{2.78}$$

is represented by SFG as in Fig. 2.84

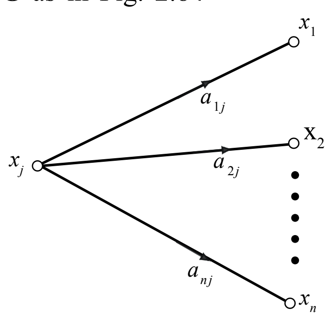


Fig. 2.84. SFG of equation (2.78)

(*c*) Multiplication Rule: A series (cascade) connection of branches with transmittances a_{21} , a_{32} , a_{43} , ... etc., can be replaced by a single branch with a new transmittance equal to the product of the individual transmittances.

$$X_n = [a_{21}, a_{32}, a_{43}, ...] X_1 \tag{2.79}$$

The SFG of equation (2.79) is represented in Fig. 2.70.

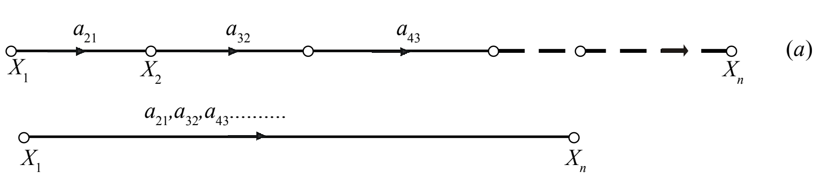


Fig. 2.85. SFG of equation (2.79)

The various terms involved in the SFG are defined as follows.

2.8.2. Definitions in SFG

(*a*) Node: A system variable that equals the sum of all the incoming signals is defined as Node. As such the variables X_i and X_j are represented by a small dot which is called Node.

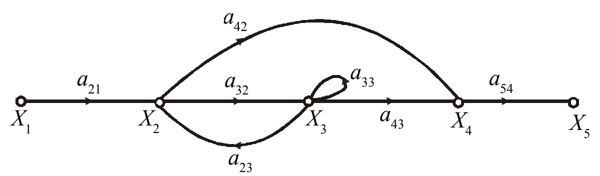


Fig. 2.86. A signal flow graph.

(*b*) Branch: A signal travels along a branch from one node to another in the direction indicated by the branch arrow and the signal gets multiplied by the "transmittance" (transmission function) of the branch. X_1, X_2 , ... X_5 are different nodes which have been connected by branches as shown in Fig. 2.86.

(*c*) Path: A path is a continuous, unidirectional succession of branches along which no node is traversed more than once.

In Fig. 2.86, X_1 to X_2, X_2 to X_3 etc., are paths.

(*d*) Input node: (source node): It is a node with only outgoing branches. In Fig. 2.86, X_1 is an input node.

(*e*) Output node: (sink node): It is a node with only incoming branches. In Fig. 2.86, X_5 is an output (sink) node.

(*f*) Forward Path: It is a path from the input node to the output node. In Fig. 2.86, X_1 to X_2 to X_3 to X_4 to X_5 is a forward path. X_1 to X_2 to X_4 to X_5 is another forward path.

(*g*) Feedback loop: It is a path which originates and terminates on the same node. In Fig. 2.87, X_2 to X_3 and back to X_2 is a feedback path.

(*h*) Self loop: It is a feedback loop consisting of only one branch. In Fig. 2.86, a_{33} is a self loop.

(*i*) Gain: The gain of a branch is the "transmittance" of that branch when the transmittance is a multiplicative operator. For example, a_{33} is the gain of the self-loop.

(*j*) Non-touching loops: Loops which do not have a common node are said to be non-touching.

(*k*) Path gain: It is the product of the branch gains encountered in traversing a path. For example, the path gain of the forward path from X_1 to X_2 to X_3 to X_4 to X_5 is given by $a_{21} \cdot a_{32} \cdot a_{43} \cdot a_{54}$.

(*l*) Loop gain: It is the product of the branch gains of the loop. For example, the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $a_{32} \cdot a_{23}$.

2.8.3. Construction of Signal Flow Graphs

The signal flow graph of a liner feedback control system can be constructed by direct reference to the block diagram of the system. Each variable of the block diagram becomes a node and each block becomes a branch.

Let us consider the block diagram of a canonical feedback control system which is shown in Fig. 2.87

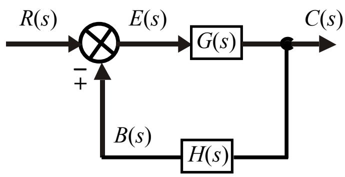


Fig. 2.87. Canonical form of control system.

The signal flow graph (SFG) is easily constructed from Fig. 2.88.

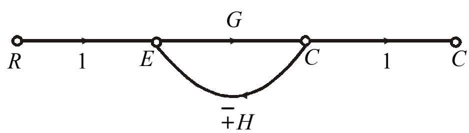


Fig. 2.88. SFG of Fig. 2.87

We see from Fig. 2.88 that the – or + sign of the summing point is associated with H.

The SFG of a system can be constructed from its describing equations. To explain the procedure, let us consider a system described by the following set of simultaneous equations:

$$\left.\begin{aligned} x_2 &= a_{21}\, x_1 + a_{23}\, x_3 \\ x_3 &= a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 \\ x_4 &= a_{42}\, x_2 + a_{43}\, x_3 \end{aligned}\right] \tag{2.80}$$

There are four variables namely, x_1, x_2, x_3 and x_4 and so four nodes are required. We arrange them from left to right and connect them with appropriate branches by which we obtain the signal flow graph of Fig. 2.89.

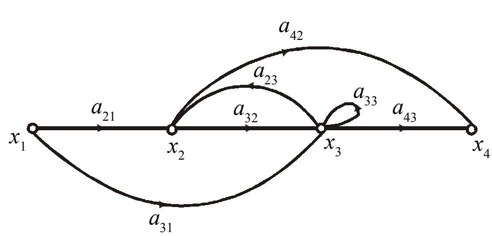


Fig. 2.89. SFG of Equation (2.80)

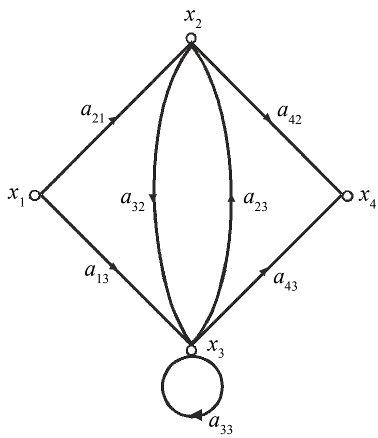


Fig. 2.90. Modified SFG of Fig. 2.89

The overall system gain from input to output may be obtained by Mason's gain formula.

2.8.4. Mason's Gain Formula

It is possible to reduce complicated block diagram to canonical form, from which the control ratio is written as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)\, H(s)} \tag{2.81}$$

It is possible to write the input-output relationship and hence the control ratio of SFG by inspection; which is accomplished by using the Mason's gain formula given by

$$M = \frac{\sum_i P_i \cdot \Delta_i}{\Delta} \tag{2.82}$$

where, P_i = path gain of the *ith* forward path.

Δ = determinant of the graph (SFG)

= 1 – [sum of loop gain of all individual loops]
+ [sum of all gain products of two non-touching loops]
– [sum of all gain products of three non-touching loops] + ...

$$= 1 - \sum_j P_{j1} + \sum_j P_{j2} - \sum_j P_{j3} + \ldots \tag{2.83}$$

Δ_i = the value of Δ for that part of the graph not touching the ith forward path.

P_{jk} = jth possible product of K non-touching loop gains.

M = overall gain of the system

Let us illustrate the Mason's gain rule by finding the overall gain of the signal flow graph given in Example 2.31.

Example 2.31. Determine the overall transfer function C/R from the signal flow graph shown in Fig. 2.91

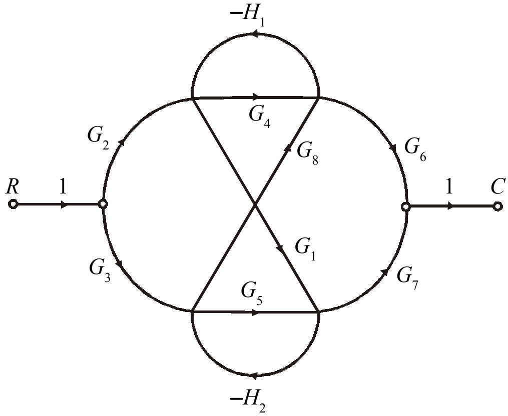


Fig. 2.91. SFG of Example 2.31

Solution: 1. There are six forward paths with path gains

$$P_1 = G_2\ G_4\ G_6$$
$$P_2 = G_3\ G_5\ G_7$$
$$P_3 = G_2\ G_1\ .\ G_7$$
$$P_4 = G_3\ G_8\ G_6$$
$$P_5 = G_2\ G_1\ .\ H_2 G_8\ .\ G_6$$
$$P_6 = G_3\ G_8\ H_1\ G_1\ G_7$$

2. There are three individual loops with loop gains

$$P_{11} = -\ H_1\ G_4$$
$$P_{21} = -\ H_2\ G_5$$
$$P_{31} = G_1\ H_2\ G_8\ H_1$$

3. There is only one possible combination of two non-touching loops with loop gain product.

$$P_{12} = H_1\ H_2\ G_4\ G_5$$

4. There are no combinations of three non-touching loops, four non-touching loops, etc. Therefore, we have

$$P_{j3} = P_{j4} = \ldots = 0$$

Hence, from the equation (2.76), we have

$$\Delta = 1 - [-\ H_1 G_4 - H_2 G_5 + G_1\ H_2\ G_8\ H_1] + [H_1\ H_2\ G_4\ G_5]$$
$$= 1 - G_4 H_2\ G_8 H_1 + H_2\ G_5 - G_1\ H_2\ G_8 H_1 + H_1\ H_2\ G_4\ G_5$$

5. The first forward path is not in touch with one loop (with gain $-H_2\ G_5$). Therefore,

$$\Delta_1 = 1 - (-\ H_2 G_5) = 1 + H_2 G_5 \text{ [written from the value of } \Delta]$$

The second forward path is not in touch with one loop (with gain $-G_4\ H_1$). Therefore

$$\Delta_2 = 1 - (-\ H_1 G_4) = 1 + H_1 G_4 \text{ [written from } \Delta]$$

The other forward paths, namely, third, fourth, fifth and sixth are in touch with all loops individually. Hence, we have

$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$ [written from Δ]

From equation (2.82), the overall gain

$$M = \frac{C}{R} = \frac{[G_2G_4G_6(1+G_5H_2) + G_2G_5G_7(1+H_1G_4) + G_1G_2G_7 + G_3G_6G_8 - G_1G_2G_6G_8H_2 - G_1G_3G_8G_7H_1]}{1 - G_1H_2G_8H_1 + H_2G_5 + H_1G_4 + H_1H_2G_4G_5}$$

The values of Δ, Δ_1, Δ_2, etc. have to be found out very carefully.

Example 2.32. For the system represented by the following equations, find the transfer function $X(s)/U(s)$ by signal flow graph technique.

$$x = x_1 + \alpha_3\, u$$

$$\dot{x}_1 = -\beta_1\, x_1 + x_2 + \alpha_2\, u$$

$$\dot{x}_2 = -\beta_2\, x_1 + \alpha_1\, u$$

Solution: In order to represent differentiated variables, we need to Laplace Transform the given set of equations. Hence, we have the transformed equations as

$$x = x_1 + \alpha_3\, u \tag{2.84}$$

or $$sx_1 = -\beta_1\, x_1 + x_2 + \alpha_2\, u$$

$$x_1 = \frac{x_2}{s+\beta_1} + \frac{\alpha_2}{s+\beta_1}u \tag{2.85}$$

$$sx_2 = \beta_2\, x_1 + \alpha_1\, u$$

or $$x_2 = \frac{\beta_2}{s}\, x_1 + \frac{\alpha_1}{s}\, u \tag{2.86}$$

Making use of the equations (2.84), (2.85) and (2.86) we have the SFG as shown in Fig. 2.92.

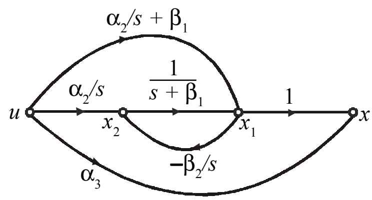


Fig. 2.92. SFG of Example 2.32

Mason's gain rule of equation is given by (2.82) is given by

$$M = \frac{\sum_i P_i\Delta_i}{\Delta}$$

1. There are three forward paths with path gains

$$P_1 = \frac{\alpha_1}{s(s+\beta_1)}$$

$$P_2 = \frac{\alpha_2}{s+\beta_1}$$

$$P_3 = \alpha_3$$

2. There is only one loop with loop gain

$$P_{11} = \left[\frac{1}{s+\beta_1}\right]\left[\frac{-\beta_2}{s}\right] = \frac{\beta_2}{s(s+\beta_1)}$$

3. There is no possibility of having non-touching loops.
Hence, we have

$$\Delta = 1\left[\frac{-\beta_2}{s(s+\beta_1)}\right] = 1 + \frac{-\beta_2}{s(s+\beta_1)}$$

4. The first forward path with gain P_1 touches the loop with gain P_{11}. Therefore,

$$\Delta_1 = 1 \quad \text{(written from } \Delta)$$

The second forward path with gain P_2 touches the loop at node X_1. Therefore,

$$\Delta_2 = 1 \quad \text{(written from } \Delta)$$

The third forward path with gain P_3 does not touch loop. Therefore,

$$\Delta_3 = 1 + \frac{\beta_2}{s(s+\beta_1)} \qquad \text{(written from } \Delta)$$

The overall gain (transfer function) of the system is given by

$$M = \frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

$$= \frac{\left[\left[\frac{\alpha_1}{s(s+\beta_1)}\right] + \left[\frac{\alpha_2}{s+\beta_1}\right] + \alpha_3\left[1 + \frac{\beta_2}{s(s+\beta_1)}\right]\right]}{1 + \frac{\beta_2}{s(s+\beta_1)}}$$

$$\frac{X(s)}{U(s)} = \frac{\alpha_1 + \alpha_2 s + \alpha_3[s^2 + \beta_1 s + \beta_2]}{s^2 + \beta_1 s + \beta_2}$$

Example 2.33. Obtain the transfer function $Y(s)/X(s)$ of the *SFG* shown in Fig. 2.93

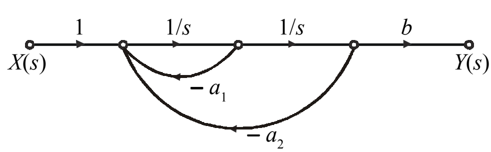


Fig. 2.93. SFG of Example 2.33

Solution: 1. There is only one forward path with path gain.

$$P_1 = 1 \cdot \frac{1}{s} \cdot \frac{1}{s} \cdot b = \frac{b}{s^2}$$

2. There are two feedback loops with loop gains

$$P_{11} = -\frac{a_1}{s}$$

$$P_{21} = -\frac{a_2}{s^2}$$

3. There is no possible combination of two non-touching loops. Therefore,

$$\Delta = 1 - \left[-\frac{a_1}{s} - \frac{a_2}{s^2}\right] = 1 + \frac{a_1}{s} + \frac{a_2}{s^2}$$

4. The forward path touches both the loops. Therefore.

$$\Delta_1 = 1$$

The overall gain is given by

$$M = \frac{Y(s)}{X(s)} = \frac{1}{\Delta}[P_1\Delta_1] = \frac{1}{\left[1 + \frac{a_1}{s} + \frac{a_2}{s^2}\right]}\left[\frac{b}{s^2}\right]$$

$$\frac{Y(s)}{X(s)} = \frac{6}{s^2 + a_1 s + a_2}$$

which is the required transfer function of the system (*SFG*).

Example 2.34. Obtain the transfer function $Y(s)/X(s)$ of the *SFG* shown in Fig. 2.94.

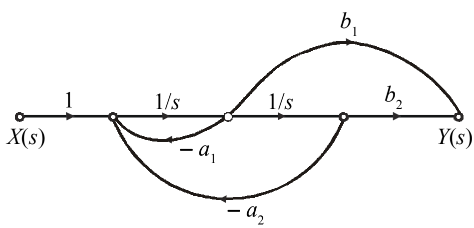


Fig. 2.94. SFG of Example 2.34

Solution: 1. There are two forward paths with path gains

$$P_1 = \frac{b_2}{s^2}$$

$$P_2 = \frac{b_1}{s}$$

2. There are two feedback loops with loop gains

$$P_{11} = -\,a_1/s$$

$$P_{21} = -\,a_2/s^2$$

3. There are no non-touching loops. Therefore, we have

$$\Delta = 1-\left[-\frac{a_1}{s}-\frac{a_2}{s^2}\right]=1+\frac{a_1}{s}+\frac{a_2}{s^2}$$

4. Both the forward paths touch the individual loops. Therefore,

$$\Delta_1 = 1$$
$$\Delta_2 = 1$$

The overall gain is given by

$$M = \frac{Y(s)}{X(s)}=\frac{1}{\Delta}\ [P_1\ \Delta_1 + P_2\ \Delta_2]$$

$$= \frac{\left[\frac{b}{s^2}+\frac{b}{s^2}\right]}{\left[1+\frac{a_1}{s^2}+\frac{a_2}{s^2}\right]}$$

$$\frac{Y(s)}{X(s)} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

which is the transfer function of the SFG shown in Fig. 2.94.

Example 2.35. Consider the block diagram of Example 2.26; a signal flow diagram of the system is shown in Figure 2.95. There are two forward paths with gains G_1 and G_2 and one loop with gain

$$L_1 = -\ G_2 H$$

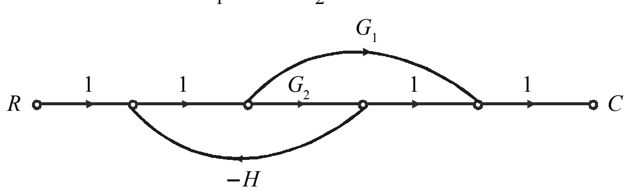


Fig. 2.95. SFG of Example 2.35

The individual determinants are

$$\Delta_1 = \Delta_2 = 1$$

The diagram's determinant is

$$\Delta = 1 + G_2 H$$

In accordance with Mason's gain formula, we get

$$G = \frac{G_1 + G_2}{1+G_2 H}$$

This agrees with our conclusion in Section 2.8.

2.8.5. Signal Flow Graphs from Block Diagams

The easiest method of determining the control ratio of a complicated block diagram is to draw the signal flow graph of the block diagram and then to use

Mason's gain rule to obtain the control ratio. Takeoff points and summing points are separated by a unity-gain branch in the signal flow graph when using Mason's gain rule.

Example 2.36. Draw the SFG and find C/R for the system shown in Fig. 2.96

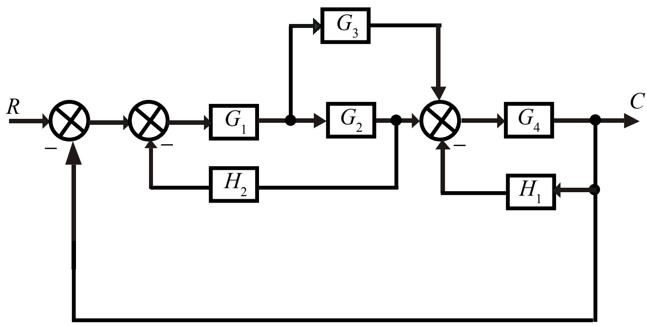


Fig. 2.96. SFG of Example 2.36

Solution: The signal flow graph for the system of Fig. 2.96 is shown in Fig. 2.97.

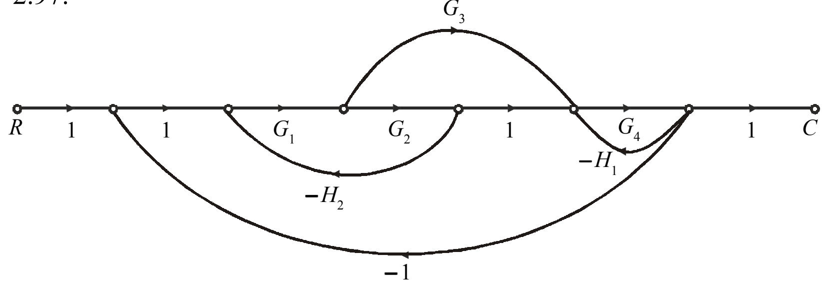


Fig. 2.97. SFG of Fig. 2.96

1. There are two forward paths with path gains

$$P_1 = G_1 G_2 G_4$$
$$P_2 = G_1 G_3 G_4$$

2. There are four individual loops with loop gains

$$P_{11} = -G_1 G_2 H_2$$
$$P_{21} = -G_4 H_1$$
$$P_{31} = -G_1 G_2 G_4$$
$$P_{41} = -G_1 G_3 G_4$$

3. There is only one possible combination of two non-touching loops with loop gain product

$$P_{12} = G_1 G_2 G_4 H_1 H_2$$

4. Therefore, the value of Δ in Mason's gain rule is

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41}] + P_{12}$$
$$= 1 + G_1 G_2 H_2 + G_4 H_1 + G_1 G_2 G_4 + G_1 G_3 G_4 + G_1 G_2 G_4 H_1 H_2$$

5. The first forward path is in touch with all the loops. Therefore,

$$\Delta_1 = 1$$

The second forward path is in touch with all the loops. Therefore,

$$\Delta_2 = 1$$

The overall system gain is given by

$$M = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_s}{\Delta}$$

$$= \frac{G_1G_2G_4 + G_1G_3G_4}{1 + G_1G_2H_2 + G_4H_1 + G_1G_2G_4 + G_1G_3G_4 + G_1G_2G_4H_1H_2}$$

Example 2.37. Draw the singal flow graph and determine C/R for the block diagram shown in Fig. 2.98.

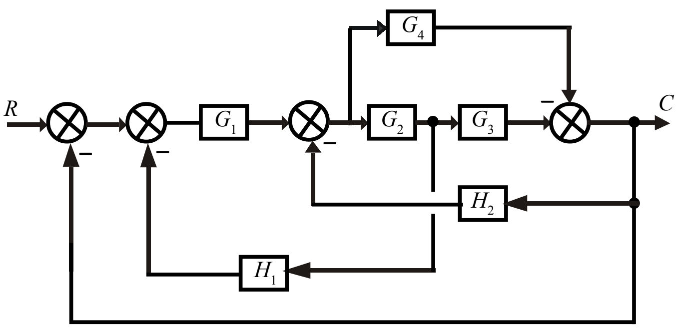


Fig. 2.98. SFG of Example 2.37

Solution: The signal flow graph of Fig. 2.98 is shown in Fig. 2.99.

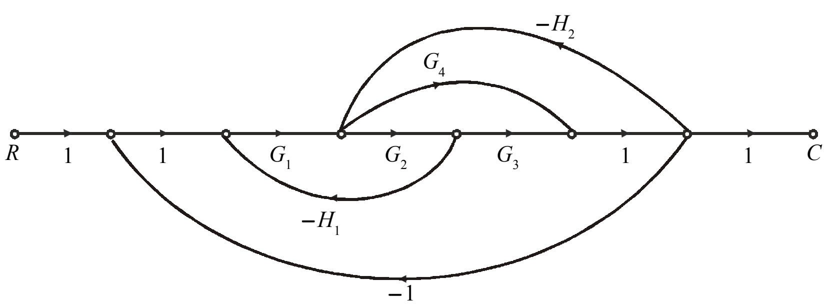


Fig. 2.99. SFG of Fig. 2.98

1. There are two forward paths with path gains

$$P_1 = G_1\ G_2\ G_3$$
$$P_2 = G_1\ G_4$$

2. There are five individual loops with loop gains

$$P_{11} = -\ G_1\ G_2\ G_3$$
$$P_{21} = -\ G_1\ G_2\ H_1$$
$$P_{31} = -\ G_2\ G_3\ H_2$$
$$P_{41} = -\ G_1\ G_4$$
$$P_{51} = -\ G_4\ H_2$$

3. There is no possible combination of two or more non-touching loops.
4. The value of Δ in Mason's gain rule is

$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41} + P_{51}]$$
$$= 1 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_4 + G_4 H_2$$

5. The first forward path is in touch with all the loops. Therefore,

$$\Delta_1 = 1$$

The second forward path is in touch with all the loops. Therefore,

$$\Delta_2 = 1$$

The overall gain is given by

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2}$$

Example 2.38. Find the overall T.F. by using Mason's gain formula for the signal flow graph given in the Fig. 2.100.

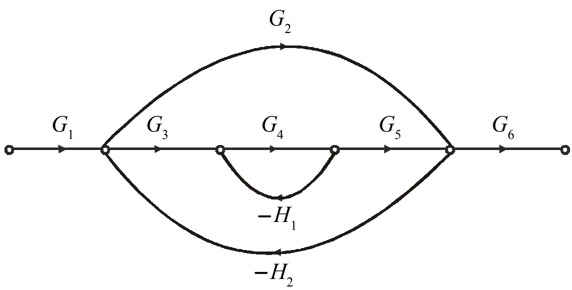


Fig. 2.100.

Solution: Two forward paths, $K = 2$,

$$T_1 = G_1\, G_3\, G_4\, G_5\, G_6$$
$$T_2 = G_1\, G_2\, G_6$$

Loops are,

$$L_1 = -\, G_4\, H_1$$
$$L_2 = -\, G_3\, G_4\, G_5\, H_2$$
$$L_3 = -\, G_2\, H_2$$

Out of these, L_1 and L_3 is combination of two non-touching loops

$$\Delta = 1 - [L_1 + L_2 + L_3] + [L_1\, L_3]$$

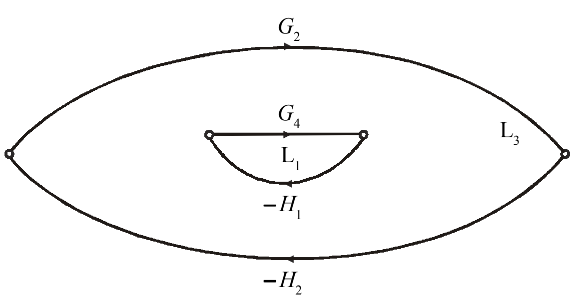


Fig. 2.101. Non-touching loops

Δ_1 = Eliminate L_1, L_2, L_3 as all are touching to T_1 from Δ

$\Delta_1 = 1$

Δ_2 = Eliminate L_1 and L_3, as they are touching to T_2, from Δ · But L_1 is non-touching hence keep it as it is in Δ

$\Delta_2 = 1 - [L_1]$

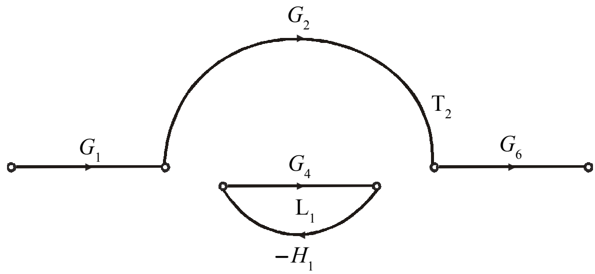


Fig. 2.102. L_1 non-touching to T_2

Substitute in Mason's Gain formula,

$$\text{T.F.} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$\text{T.F.} = \frac{G_1G_3G_4G_5G_6[1] + G_1G_2G_6[1 + G_4H_1]}{1 + G_4H_1 + G_3G_4G_5H_2 + G_2H_2 + G_2G_4H_1H_2}$$

Example 2.39. Calculate $\dfrac{Y_7}{Y_2}$ of the system, whose signal flow graph is given below:

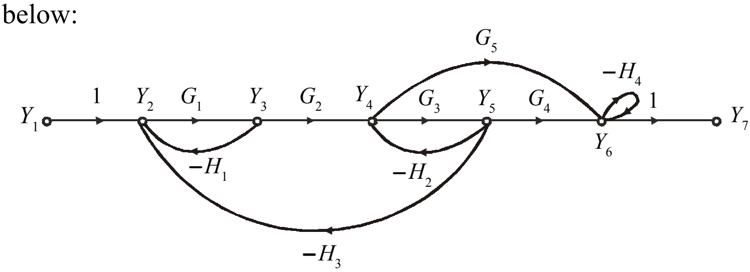


Fig. 2.103. SFG of Example 2.39

Solution: Forward paths for Y_1 to Y_7 are two

$$T_1 = G_1\ G_2\ G_3\ G_4$$
$$T_2 = G_1\ G_2\ G_5$$

Individual feedback loops are:

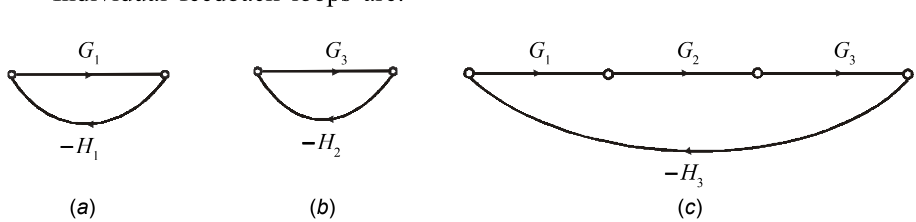


Fig. 2.104. Different Loops of Fig. 2.103

$$L_1 = -G_1H_1,\ L_2 = -G_3H_2,\ L_3 = -G_1G_2G_3H_3,$$

Self-loop $\qquad L_4 = -H_4$

Combinations of two non-touching loops,

$$L_1\ L_2 = +G_1\ G_3\ H_1\ H_2$$
$$L_1\ L_4 = +G_1\ H_1\ H_4$$
$$L_2\ L_4 = +G_3\ H_2\ H_4$$
$$L_3\ L_4 = +G_1\ G_2\ G_3\ H_3\ H_4$$

One combination of three non-touching,

$$L_1\ L_2\ L_4 = -G_1\ G_3\ H_1\ H_2\ H_4$$

$$\therefore\ \Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_1 L_2 + L_1 L_4 + L_2 L_4 + L_3 L_4] - [L_1 L_2 L_4]$$

$\Delta_1 = 1$ all loops are touching

$\Delta_2 = 1 - L_2$ as L_2 is non-touching to forward path.

$$\frac{Y_7}{Y_1} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4 + G_1G_2G_5(1+G_3H_2}{\Delta}$$

Now to find the ratio $\dfrac{Y_2}{Y_1}$.

Forward paths for Y_1 to Y_2 is one. $T_1 = 1$ Now Δ is same

and $\qquad \Delta_1 = 1 - L_2 - L_4 + L_2\ L_4$ as L_2 and L_4 are non-touching to T_1.

$$= 1 + G_3\ H_2 + H_4 + G_3\ H_2\ H_2$$

$$\therefore \qquad \frac{Y_2}{Y_1} = \frac{T_1\Delta_1}{\Delta} = \frac{1+G_3H_2+H_4+G_3H_2H_4}{\Delta}$$

$$\therefore \qquad \frac{Y_7}{Y_1} = \frac{\dfrac{Y_7}{Y_1}}{\dfrac{Y_2}{Y_1}}$$

$$= \frac{\dfrac{G_1G_2G_3G_4 + G_1G_2G_5(1+G_3H_2)}{\Delta}}{\dfrac{1+G_3H_2+H_4+G_3H_2H_4}{\Delta}}$$

$$= \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)}{1+G_3H_2+H_4+G_3H_2H_4}$$

Example 2.40. Find $\dfrac{C(s)}{R(s)}$ by using Mason's gain formula.

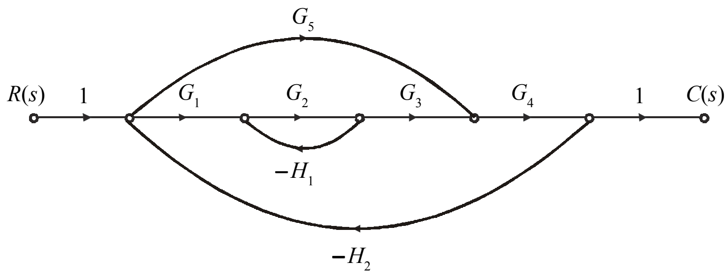


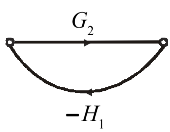
Fig. 2.105. SFG of Example 2.40

Solution: Number of forward paths $K = 2$

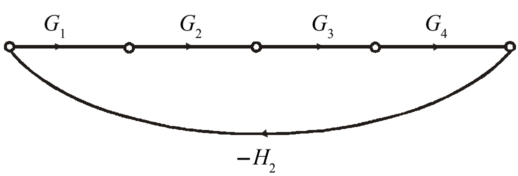
$$\text{Mason's gain formula, T.F.} = \frac{\sum_{k=1}^{2} T_k \Delta_k}{\Delta} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$T_1 = G_1\ G_2\ G_3\ G_4,\ T_2 = G_5\ G_4$$

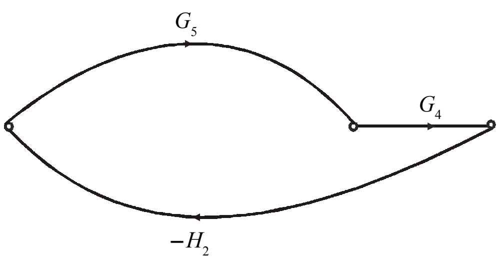
Individual feedback loops are:



(*a*) $\qquad L_1 = -\ G_2\ H_1$



(*b*) $\qquad L_2 = -\ G_1\ G_2\ G_3\ G_4\ H_2$



(*c*) $\qquad L_3 = -\ G_5\ G_4\ H_2$

Fig. 2.106. Different loops of Fig. 2.105

L_1 and L_3 are two non-touching loops.

$$\therefore\quad \Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_1\ L_3]$$

$$= 1 - [-\ G_2\ H_1 - G_1\ G_2\ G_3\ G_4\ H_2 - G_5\ G_4\ H_2] + [G_2\ H_1\ G_5\ G_4\ H_2]$$

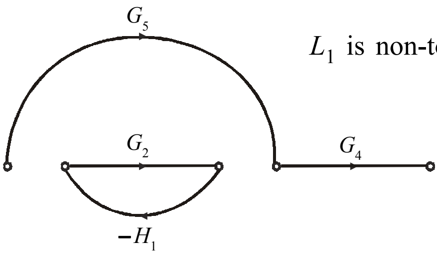
$$= 1 + G_2\ H_1 + G_1\ G_2\ G_3\ G_4\ H_2 + G_5\ G_4\ H_2 + G_2\ G_5\ G_4\ H_1\ H_2$$

Now consider different forward paths,

$$T_2 = G_1\ G_2\ G_3\ G_4$$

All loops are touching to this forward path.

$$\therefore \qquad \Delta_1 = 1$$



L_1 is non-touching to T_2,

Fig. 2.107. Non-touching Loops of Fig. 2.106

Consider $\quad T_2 = G_5\ G_4$

$\therefore \quad \Delta_2 = 1 - [L_1] = 1 - (G_2\ H_1) = 1 + G_2\ H_1$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3G_4 \cdot 1 + G_5G_4(1+G_2H_1)}{\Delta}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4 + G_4G_5(1+G_2H_1)}{1+G_2H_1+G_1G_2G_3G_4H_2+G_5G_4H_2+G_2G_5G_4H_1H_2}$$

Table 2.7. Comparison of Block Diagram and Signal Flow Graph Methods

Sr. No.	Block Diagram	Signal Flow Graph
1.	Basic importance is given to the elements and their transfer functions.	Basic importance is given to the variables of the systems.
2.	Each element is represented by a block.	Each variable is represented by a separated node.
3.	Transfer function of the element is shown inside the corresponding block.	The transfer function is shown along the branches connecting the nodes.
4.	Summing points and takeoff points are separate.	Summing and takeoff points are absent. Any node can have any number of incoming and outgoing branches.
5.	Feedback path is present from output to input.	Instead of feedback path, various feedback loops are considered for the analysis.
6.	For a minor feedback loop present, the formula $\frac{G}{1 \pm GH}$ can be used.	Gains of various forward paths and feedback loops are just the product of associative branch gains. No such formula $\frac{G}{1 \pm GH}$ is necessary.
7.	Block diagram reduction rules can be used to obtain the resultant transfer function.	The Mason's gain formula is available which can be used directly to get resultant transfer function without reduction of signal flow graph.

8.	Method is slightly complicated and time-consuming as block diagram is required to be drawn time to time after each step of reduction.	No need to draw the signal flow graph again and again. Once drawn, use of Mason's Gain Formula gives the resultant transfer function.
9.	Concept of self loop is not existing in block diagram approach.	Self loops can exist in signal flow graph approach.
10.	Applicable only to linear time invariant systems.	Applicable to linear time invariant systems.

2.9. UNSOLVED PROBLEMS

1. Find out the transfer function of the following block diagram as shown in Fig. 2.108.

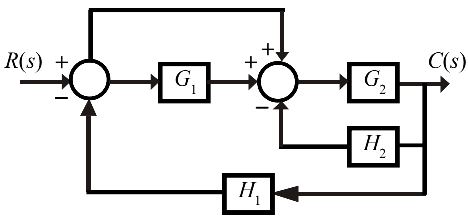


Fig. 2.108

2. The block diagram of a control system is shown in Fig. 2.109 below. Obtain transfer function (*a*) $C(s)/R(s)/_{N=0}$ and (*b*) $C(s)/N(s)/_{R=0.}$

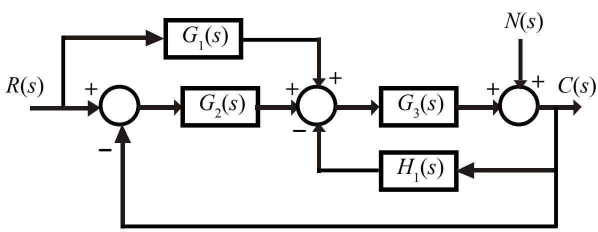


Fig. 2.109

3. For the system shown in Fig. 2.110, obtain the closed-loop transfer function by block reduction method.

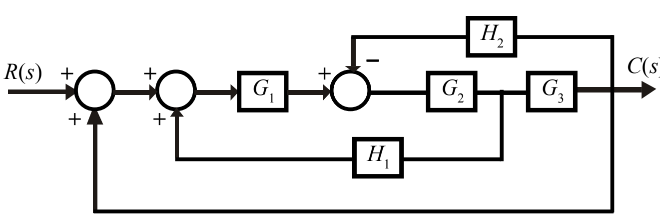


Fig. 2.110

4. Draw signal flow diagram for the system shown in Fig. 2.111. Also find overall transfer function using Mason's gain formula.

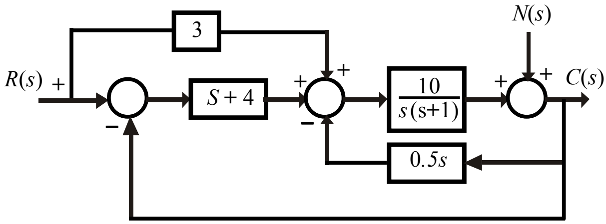


Fig. 2.111

5. Draw the signal flow graph and obtain the transfer function of the system shown in Fig. 2.112.

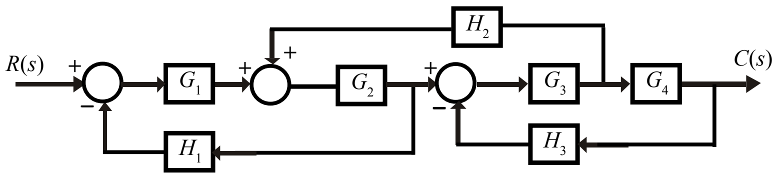


Fig. 2.112

6. Find $C(s)/R(s)$ in the following Fig. 2.113.

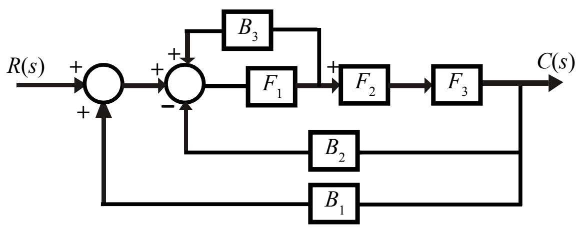


Fig. 2.113

7. Draw signal flow graph for the following block diagram Fig. 2.114.

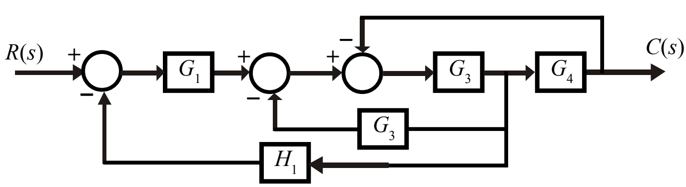


Fig. 2.114

8. Draw the signal flow graph for the following set of equations.

$$x_1 - x_2 - 4x_3 - 6x_4 = 0$$
$$2x_2 - x_3 - 5x_4 = 0$$
$$7x_1 - 3x_3 - x_4 = 0$$

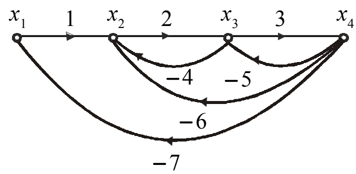


Fig. 2.115

9. Find the transfer function of the signal flow graph shown in Fig. 2.116 using Mason's gain formula.

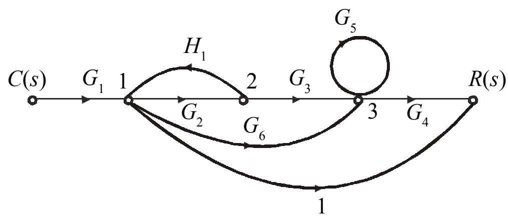


Fig. 2.116

10. Reduce the following block diagram into signal flow graph and then determine the transfer function using Mason's gain formula.

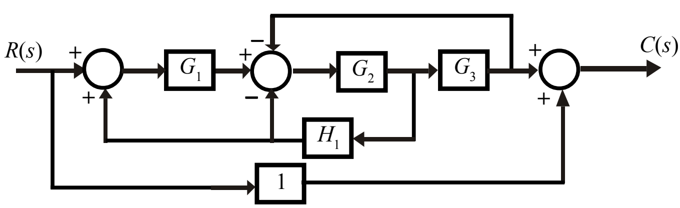


Fig. 2.117

11. Find the transfer function of the signal flow graph shown in Fig. 2.118.

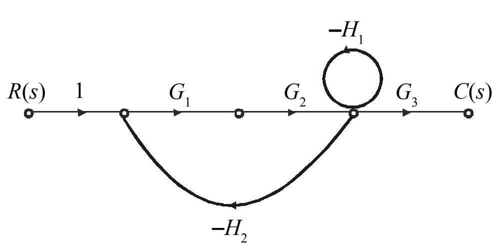


Fig. 2.118

12. Find transfer function for the signal flow graph given below in Fig. 2.119.

15. Draw signal flow graph for the following set of equations.

$$5\frac{d^2x}{dt^2} + 200x - 20y = 0$$

$$5\frac{d^2y}{dt^2} + 200y - 20x = 0$$

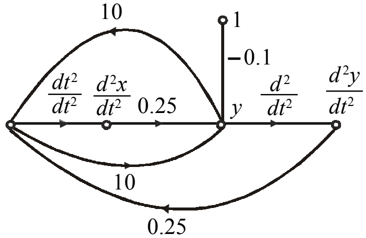


Fig. 2.122

2.10. MULTIPLE CHOICE QUESTIONS

1. Match List I (signals) with List II (Laplace transform) and select the correct answer.

	List I		List II
(*A*)	$e^{-t}\,u\,(t)$	1.	$\frac{1}{s^2}$
(*B*)	$u\,(t)$	2.	$\frac{1}{(s+1)^2}$
(*C*)	$tu\,(t)$	3.	$\frac{1}{s}$
(*D*)	$te^{-t}u\,(t)$	4.	$\frac{1}{s+1}$

$u(t)$ denotes the unit step function.

	A	*B*	*C*	*D*
(*a*)	4	1	3	2
(*b*)	2	3	1	4
(*c*)	4	3	1	2
(*d*)	2	1	3	4

2. The Laplace transform of current in an RLC series circuit with $R = 2$ ohm, $L = 1$H and $C = 1/2$ F is $I(s) = \frac{1}{s^2+2s+2}$. The voltage across the inductor '*L*' will be

(*a*) $e^{-t} \sin tu\,(t)$ (*b*) $e^{-t} \sin tu\,(t)$

(*c*) $e^{-t}(\sin t + \cos t)\,u(t)$ (*d*) $e^{-t}(\cos t - \sin t)\,u(t)$