Linear Programming Problem

2.1 INTRODUCTION

All of us always want to do a particular job in the best possible ways. What are the best or optimal ways to do a job?

For example:

(a) *Assignment:* Let there be m persons and n jobs are to be assigned to these persons. How should these jobs be assigned, so that efficiency of work is maximum?

(b) *Transportation:* This problem arises when the material is to be transported from places of manufacturing to places of requirements. Let there be m places of manufacturing and n places of requirements. How the material should be transported from m places of manufacturing to n places of requirements, so that the total cost of transportation is minimum?

(c) *Inventory:* The problem arises when it is necessary to stock different commodities to meet the demand of customers over a specific period of time. Here one has to decide how much quantity of the commodity and at what time it should be ordered.

These are not only the problems where we do optimisation. In addition to the above, there are many other problems where we optimise time, money, etc. in our day-to-day life under certain restrictions. All of these are formulated mathematically and we call it mathematical programming. Now in the following section, we shall be explaining, what is mathematical programming.

2.2 MATHEMATICAL PROGRAMMING

When we give a precise mathematical language to our thoughts, subject to some conditions and then optimise it, then we call it mathematical programming. Mathematical programming has applications almost everywhere i.e., sciences, engineering, medicines, social sciences, and management, etc. Therefore, mathematical programming or a mathematical model of an optimisation problem consists of an objective function (which is to be optimised) and constraints.

A mathematical programming in general form can be written as follows:

Maximize or Minimize $Z = f(X)$, $X = (x_1, x_2, \ldots x_n)$

Subject to the constraints

$$gi(X) \lesseqgtr b, \; i = 1, 2, \ldots, m$$

(b) *Change of < (>) constraint into a > (<) constraint:* A < (>) constraint can be converted into a > (<) constraint by multiplying both sides by (–1). Thus,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \le b_1$$
$$\equiv - \alpha_1 x_1 - \alpha_2 x_2 - \ldots - \alpha_n x_n \ge -b_1$$

Similarly,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \ge b_1$$
$$\equiv -\alpha_1 x_1 - \alpha_2 x_2 - \ldots - \alpha_n x_n \le -b_1$$

Slack-surplus Variables

An inequation

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \le b_1 \qquad \ldots(1)$$

can be converted into an equation by adding a variable 'S'. So,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + S = b_1.$$

The left hand side of eqn. (1) is smaller than or equal to the right hand side, so we have added a variable 'S'. 'S' is called a *Slack Variable.*

Similarly, an inequation

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \ge b_1$$

can be converted into an equation by subtracting a variable 'S' as left hand side is greater than or equal to right hand side.

So $$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n - S = b_1.$$

In this case 'S' is called a *Surpus Variable.*

Standard form of a LPP: A LPP is said to be in standard form, if it is of the following form.

$$\text{opt. } f(X) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Subject to
$$\alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n = b_1$$
$$\alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n = b_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n = b_m$$
$$x_1, x_2, \ldots, x_n \ge 0;\ b_1, b_2, \ldots, b_m \ge 0$$

i.e,

$$\text{opt. } f(X) = C^T X$$

Subject to
$$AX = b$$
$$X \ge 0,\ b \ge 0.$$

Where,
$$C = (c_1, c_2, \ldots, c_n)^T$$
$$X = (x_1, x_2, \ldots, x_n)^T$$
$$b = (b_1, b_2, \ldots, b_m)^T$$
$$A = (\alpha_{ij})_{m \times n} \text{ matrix.}$$

Thus, if any inequation has b_i as a negative number, it can be changed to positive by multiplying by (–1) and then by adding slack or surplus variable (as the case may be) it may be changed into an equation. Thus, every LPP can be brought into the standard form if all variables follow the non-negativity restrictions *i.e.,* $X \ge 0$.

If k variables are unrestricted in sign, then number of variables increases by k. There is another way to handle such problems. By this method the, number of variables increases by only one irrespective of that k_1 or k_2 number of variables are unrestricted in sign.

Let $x_1, x_2, \ldots, x_k$ variables be unrestricted in sign. Then we introduce the variables $y, y_1, y_2, \ldots, y_k$ such that $y, y_i \geq 0$, $i = 1, 2, \ldots, k$ and

$$x_i = y_i - y.$$

Example 1: Convert the following problem into a maximisation problem.

$$\text{Minimize} \quad Z = f(X) = 2x_1 - x_2 + \frac{1}{2} x_3$$

$$\begin{aligned} \text{Subject to} \quad x_1 + x_2 - x_3 &\leq 5 \\ 2x_1 + 3x_3 &\geq 6 \\ x_1 + 3x_2 &\leq -7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: It is a minimisation problem.

The maximisation problem, we will get by multiplying the objective function by -1 and constraints will remain unchanged. Therefore, the maximisation problem is

$$\text{Maximize} \quad Z = -f(X) = h(X) = -2x_1 + x_2 - \frac{1}{2} x_3$$

$$\begin{aligned} \text{Subject to} \quad x_1 + x_2 - x_3 &\leq 5 \\ 2x_1 + 3x_3 &\geq 6 \\ x_1 + 3x_2 &\leq -7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Example 2: Write the above minimisation problem described in example 1 in standard form.

Solution: Standard form is

$$\text{Minimize } Z = f(X) = 2x_1 - x_2 + \frac{1}{2} x_3$$

$$\begin{aligned} \text{Subject to} \quad x_1 + x_2 - x_3 + s_1 &= 5 \\ 2x_1 + 3x_3 - s_2 &= 6 \\ -x_1 - 3x_2 - s_3 &= 7 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

(Here s_1 is a slack variable and s_2, s_3 are surplus variables).

Example 3: Write the following problem in standard form.

Max. $Z = 3x_1 - x_2 + 7x_3$

$$\begin{aligned} \text{Subject to} \quad 2x_1 - x_2 - x_3 &\leq 7 \\ x_1 - 2x_2 + x_3 &\geq -3 \\ x_1 \geq -3, x_2 \geq 6, x_3 &\geq 0 \end{aligned}$$

Solution: Putting $x_1 = y_1 - 3$, $x_2 = y_2 + 6$ in the given problem we obtain

Max. $Z = 3(y_1 - 3) - (y_2 + 6) + 7x_3$

Then the problem is
Min. $Z = y$
Subject to

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &\le y \\ -x_1 + 2x_2 - 2x_3 &\le y \\ -2x_1 + 3x_2 - 2x_3 &\le y \\ 2x_1 - 3x_2 + 2x_3 &\le y \\ x_1, x_2, x_3, y &\ge 0 \end{aligned}$$

EXERCISE 2.1

1. Convert the following LPP into standard form.
   Minimize $Z = x_1 - 2x_2 + x_3$
   Subject to
   $$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\ge -4 \\ 3x_1 + 5x_2 + 2x_3 &\ge 7 \end{aligned}$$
   $x_1, x_2 \ge 0$, x_3 is unrestricted in sign.
2. Convert the following LPP into standard form.
   (a) Maximize $Z = 3x_1 - 2x_2 + 4x_3$
   Subject to
   $$\begin{aligned} 2x_1 + x_2 + 2x_3 &\le 12 \\ x_1 - 2x_2 - x_3 &\ge -6 \\ 3x_2 - 2x_3 &\le 1 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$
   (b) Part (a) with the requirements $x_2 \ge 0$ changed to the requirements $x_2 \le 0$.
   (Hint: Let $y_2 = -x_2$)
3. In part (a) of exercise 2, the requirements $x_1, x_2, x_3 \ge 0$ are changed to the requirements $x_1 \ge 4$, $x_2 \ge 2$, $x_3 \ge 6$.
   (a) Convert the above LPP into the standard form having six constraints, and
   (b) Convert the above LPP into the standard form having three constraints.
   (Hint: $y_1 = x_1 - 4$, $y_2 = x_2 - 2$, $y_3 = x_3 - 6$).
4. Linearize the following objective function.
   Max. $Z = \text{Min. } \{|3x_1 - 4x_2|, |2x_1 - 6x_2|\}$.

2.4 INTEGER LINEAR PROGRAMMING PROBLEM (ILPP)

An LPP, in addition to non-negativity conditions may also have conditions, that one or more than one, or all decision variables are integers. If at least one variable is integer but not all, we call an LPP as mixed ILPP. If all variables are integers, we call it ILPP.

For example,
Min. $Z = 2x_1 + 3x_2$
Subject to

$$x_1 + x_2 \leq 6$$
$$x_1 - 2x_2 \leq 5$$
$$3x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0 \text{ and integers is an ILPP}$$

while
Min. $Z = 2x_1 + 3x_2$
Subject to

$$x_1 + x_2 \leq 6$$
$$x_1 - 2x_2 \leq 5$$
$$3x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0 \text{ and } x_1 \text{ is an integer is a mixed ILPP.}$$

2.5 FORMULATION OF LP PROBLEM

In general, in optimisation theory, after identification of the problem, collection of relevant data, the given problem should be translated into appropriate mathematical model. This process of translation is called formulation. The mathematical model of LPP includes the following three basic elements.

1. Decision variables that we seek to determine.
2. Objective function that we aim to optimize.
3. Constraints (restrictions) that we need to satisfy.

The first step to develop a model is to identify the decision variables, once decision variables are defined, then, constructing the objective function and constraints are not difficult.

Thus, the mathematical formulation of an LPP consists of the following five major steps.

1. Identifying the objective underlying the problem.
2. Identifying the decision variables.
3. Construction of the objective function.
4. Construction of the constraints using $\lesseqgtr$ sign.
5. Non-negativity restrictions.

Example 1: A manufacturer wishes to determine the number of tables and chairs to be made by him in order to optimise the use of his available resources. These products utilize two different types of timber and he has on hand 2000 board feet of the first type and 1500 board feet of the second type. He has 1000 manhours available for the total job. Each table and chair requires 4 and 2 board feet respectively of the first type of timber, and 3 and 5 board feet of the second type. 5 manhours are required to make a table and 3 manhours are needed to make a chair. The manufacturer makes a profit of Rs. 50 on a table and Rs. 30 on a chair. Formulate the above as an LPP to maximize the profit.

Solution: Let x_1, x_2 be the number of tables and chairs respectively manufactured.

Then the LPP of above problem is

Max. $Z = 50x_1 + 30x_2$

Subject to $4x_1 + 3x_2 \le 2000$

$2x_1 + 5x_2 \le 1500$

$x_1, x_2 \ge 0$ and integers.

Example 2: Ram wants to decide the constituents of a diet which will fulfil his daily requirement of fats, proteins and carbohydrates at the minimum cost. The choice is to be made from three different types of food. The yield per unit of these foods is given in the following table.

Food Type	*Yield/Unit*			*Cost/Unit (Rs.)*
	Fats	*Proteins*	*Carbohydrates*	
1	3	4	8	60
2	2	3	6	50
3	5	6	4	80
Minimum Requirement	180	850	750	

Formulate the above as an LPP.

Solution: Let x_1, x_2, x_3 respectively represent the three food types. Then the LPP of the above problem is

Min. $Z = 60x_1 + 50x_2 + 80x_3$

Subject to $3x_1 + 2x_2 + 5x_3 \ge 180$

$4x_1 + 3x_2 + 6x_3 \ge 850$

$8x_1 + 6x_2 + 4x_3 \ge 750$

$x_1, x_2, x_3 \ge 0$

Example 3: A firm manufactures two types of products P_1 and P_2 and sells them on a profit of Rs. 3 on type P_1 and Rs. 4 on type P_2. Each product is processed on two machines A and B. Type P_1 requires 2 minutes of processing time on A and one minute on B; type P_2 requires 3 minutes on A and 2 minutes on B. The machine A is available for not more than 7 hours 30 minutes while machine B is available for 12 hours during any working day. Formulate the problem as an LPP.

Solution: Let x_1 = number of products of type P_1

and x_2 = number of products of type P_2

then the LPP of the above problem is

Max. $Z = 3x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \le 450$

$3x_1 + 2x_2 \le 720$

$x_1, x_2 \ge 0$

Example 4: The purchasing section of a company has purchased sufficient amount of curtain cloth to meet the requirements of the company. The curtain cloth is in pieces, each of length 15 feet. The curtain requirements is as follows:

Curtain of length (in feet)	*Number required*
6	2000
7	1500
8	3250

The problem is how to cut the pieces to meet the above requirements, so that the trim loss is minimised. The width of required curtains is same as that of available pieces. A piece of 15 feet curtain cloth can be cut in p_1, p_2, p_3, and p_4 patterns according to the following table.

Curtains of length (feet)	*Number of curtains of different sizes in pattern.*			
	p_1	p_2	p_3	p_4
6	2	0	1	0
7	0	2	0	1
8	0	0	1	1
Trim loss (in feet)	3	1	1	0

(This problem is known as Trim loss problem).

Solution: Let x_1, x_2, x_3 and x_4 be the number of pieces cut according to patterns p_1, p_2, p_3 and p_4, respectively.

The constraints of the problem are.

$$2x_1 + x_3 - s_1 \geq 2000$$
$$x_2 + x_4 - s_2 \geq 1500$$
$$x_3 + x_4 - s_3 \geq 3250$$
$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0 \text{ and integers.}$$

The objective function is

Min. $Z = 3x_1 + x_2 + x_3 + 6s_1 + 7s_2 + 8s_3$

Example 5: Peter wants to keep some hens with him and has Rs. 2000. The old hens can be bought for Rs. 50 but young ones cost Rs. 100. The old hens lay 20 eggs per week and the young ones 60 eggs per week, each worth being Re. 1.00. The feed for young and old hens costs Rs. 10 and Rs 6 per hen per week. He can keep 30 hens with him. How many of each kind of hens Peter should buy to get a profit of more than Rs. 500. Formulate as an LPP.

Solution: Let x_1 = number of old hens
and x_2 = number of young hens

Then the LPP is

Max. $Z = (20 \times 1 - 6)\, x_1 + (60 \times 1 - 10)\, x_2$

Max. $Z = 14x_1 + 50x_2$

Subject to
$$50x_1 + 100x_2 \leq 2000$$
$$x_1 + x_2 \leq 30$$
$$14x_1 + 50x_2 \geq 500$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

EXERCISE 2.2

1. A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of clothes namely suitings, shirtings and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre, respectively. One metre suiting requires 3 minutes in weaving 2 minutes in processing and 1 minute in packing. One metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing while one metre of woollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing departments, respectively.

   Formulate as LPP to maximize the profit.

   Ans: (Max. $Z = 2x_1 + 4x_2 + 3x_3$ Subject to $3x_1 + 4x_2 + 3x_3 \leq 3600$, $2x_1 + x_2 + 3x_3 \leq 2400$, $x_1 + 3x_2 + 3x_3 \leq 4800$, $x_1, x_2, x_3 \geq 0$).

2. The owner of Cosmo Sports wishes to determine how many advertisements to place in the selected quarterly magazines *A*, *B* and *C*. His objective is to advertise in such a way that the total exposure to principal buyers of expensive sports good is maximised. Percentage of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used.

	Magazine		
	A	*B*	*C*
Readers	1 Lakh	0.6 Lakh	0.4 Lakh
Principal buyers	20%	15%	8%
Cost per advt. (Rs.)	8000	6000	5000

   The budgeted amount is at most Rs. 1 lakh for the advertisements. The owner has already decided that magazine *A* should have no more than 15 advertisements and that *B* and *C* each have at least 8 advertisements.

   Formulate the problem as an LPP.

   Ans: (Max $Z = 20000\ x_1 + 9000\ x_2 + 3200\ x_3$, Subject to $8000\ x_1 + 6000\ x_2 + 5000, x_3 \leq 100000$, $x_1 \leq 15$, $x_2 \geq 8$, $x_3 \geq 8$ $x_1, x_2, x_3 \geq 0$, where x_1, x_2 and x_3 are number of advertisements in magazines *A*, *B* and *C*, respectively).

3. A farmer has 100 acre farm. He can sell all tomatoes, cabbage or radish he can raise. The price he can obtain Rs. 1.00 per kg for tomatoes, Rs 0.75 per cabbage and Rs. 2.00 per kg for radishes. The average yield per acre is 2000 kg of tomatoes, 3000 heads of cabbage and 1000 kg of radishes. Fertilizer is available at Rs. 0.50 per kg and the amount required per acre is 100 kg each for tomato and cabbage and 50 kg for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days for cabbage. A total of 400 man-days of labour are available at Rs. 20 per man-day. Formulate the problem as an LPP to maximize the farmer's profit.

Ans: (Max. $Z = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$, Subject to $x_1 + x_2 + x_3 \le 100$, $5x_1 + 6x_2 + 5x_3 \le 400$, $x_1, x_2, x_3 \ge 0$ where, x_1, x_2, x_3 is the number of acres to grow tomatoes, cabbages and radishes, respectively).

4. A company has to manufacture the circular tops of cans. Two sizes one of diameter 10 cm and the other of diameter 20 cm are required. They are to be cut from metal sheets of dimensions 20 cm by 50 cm. The requirement of smaller size is 15000 and of larger size is 10000. How to cut the tops from metal sheets so that the number of sheets used is minimised. Formulate as an LPP.

   (Hint: The plates can be cut in three patterns. The first pattern has 10 tops of smaller size, the second pattern has 2 tops of smaller size and 2 tops of larger size, the third pattern has 6 tops of smaller size and 1 top of large size.)

   Ans: (Min $Z = x_1 + x_2 + x_3$, subject to $10x_1 + 2x_2 + 6x_3 \ge 15000$, $2x_2 + x_3 \ge 10000$, $x_1, x_2, x_3 \ge 0$ and integers where x_1, x_2, x_3 be the number of sheets cut according to first, second and third pattern, respectively.)

5. A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4, respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.

Product

		A	*B*	*C*
Machine	M_1	4	3	5
	M_2	2	2	4

   Machines M_1 and M_2 have 2000 and 2500 machine minutes, respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but no more than 150 A's. Formulate the above as an LPP to maximize the profit.

   Ans: (Max. $Z = 3x_1 + 2x_2 + 4x_3$, subject to $4x_1 + 3x_2 + 5x_3 \le 2000$, $2x_1 + 2x_2 + 4x_3 \le 2500$, $100 \le x_1 \le 150$, $200 \le x_2 \ge 0$, $50 \le x_3 \ge 0$, where x_1, x_2 and x_3 be the number of products of A, B and C, respectively).

6. Three grades of coal A, B and C contain ash and phosphorous as impurities. In a particular industrial process a fuel obtained by blending the above grades containing not more than 25% ash and 0.03% phosphorous is required. The maximum demand of the fuel is 100 tonnes. Percentage impurities and costs of the various grades of coal are shown below. Assuming that there is an unlimited supply of each grade of coal and there is no loss in blending. Formulate this as an LPP to minimize the cost.

Coal Grade	*% ash*	*% phosphorous*	*Cost per tonne in Rs.*
A	30	0.02	240
B	20	0.04	300
C	25	0.03	280

   Ans: (Minimize $Z = 240x_1 + 300x_2 + 280x_3$, subject to $x_1 - x_2 + 2x_3 \le 0$, $-x_1 + x_2 \le 0$, $x_1 + x_2 + x_3 \le 100$, $x_1, x_2, x_3 \ge 0$ where x_1, x_2, x_3 are tonnes of grade A, B and C coal, respectively).

7. A ship has three cargo holds: forward, aft and centre; the capacity limits are:

Forward	2000 tonnes	100,000 m^3
Centre	3000 tonnes	135,000 m^3
Aft	1500 tonnes	30,000 m^3

The following cargoes are offered; the ship owners may accept all or any part of each commodity.

Commodity	*Amount (tonnes)*	*Volume per tonne (m^3)*	*Profit per tonne (Rs.)*
A	6,000	60	60
B	4,000	50	80
C	2,000	25	50

In order to preserve the trim of the ship, the weight in each hold must be proportional to the capacity in tonnes. The objective is to maximize the profit. Formulate as an LPP.

Ans: (Max $Z = 60(x_{1A} + x_{2A} + x_{3A}) + 50(x_{2A} + x_{2B} + x_{2C}) + 25(x_{3A} + x_{3B} + x_{3C})$, subject to $x_{1A} + x_{2A} + x_{3A} \le 6000$, $x_{2A} + x_{2B} + x_{2C} \le 4000$, $x_{3A} + x_{3B} + x_{3C} \le 2000$, $x_{1A} + x_{1B} + x_{1C} \le 2000$, $x_{2A} + x_{2B} + x_{2C} \le 3000$, $x_{3A} + x_{3B} + x_{3C} \le 1500$, $60\,x_{1A} + 50\,x_{1B} + 25\,x_{1C} \le 100{,}000$, $60x_{2A} + 50x_{2B} + 25x_{2C} \le 135{,}000$, $60x_{3A} + 50x_{3B} + 25x_{3C} \le 30{,}000$, all *variables* ≥ 0, where x_{iA}, x_{iB}, x_{iC} $(i = 1, 2, 3)$ be the weights (in kg) of commodities *A*, *B* and *C*, respectively).

2.6 SOLUTION OF A LINEAR PROGRAMMING PROBLEM

A solution of an LPP is the set of values of the variables $x_1, x_2, \ldots, x_n$; i.e., the vector $(x_1, x_2, \ldots, x_4)$ that satisfy the conditions and gives the optimal value of the objective function.

There are many vectors X which would satisfy the conditions $AX \gtreqless b$; $X \ge 0$. But only few would give the optimal value of the objective function $f(X)$.

Therefore, in order to find the solution of the LPP we would first find out the set of all solutions of conditions $AX \gtreqless b$; $X \ge 0$. The required optimal solution would be one from it. No point outside this set can be a solution of the problem.

The solution set of the conditions (inequations) $AX \lesseqgtr b$; $X \ge 0$ is called set of feasible solutions and is denoted by S_F. Thus, first we should find S_F and then pick that point of S_F which gives the optimal value of $f(X)$.

The S_F may be empty, that would mean solution does not exist. If S_F is not empty, then S_F may be bounded or unbounded. S_F is bounded means there exists vectors A and B such that $A \le X \le B \;\forall\; X \,\varepsilon\, S_F$. In this case solution of the LPP would exist. In case either A does not exist or B does not exist or none of the two exist, we say S_F is bounded above but not below, or bounded below but not above, or unbounded from both sides. In this case solution of the LPP may or may not exist.

Geometry of S_F, Graphical Solution

Each constraint, non-negative condition, is an equation. This when converted in equation form represents a hyperplane in E_n, a plane in E_3 and a line in E_2.

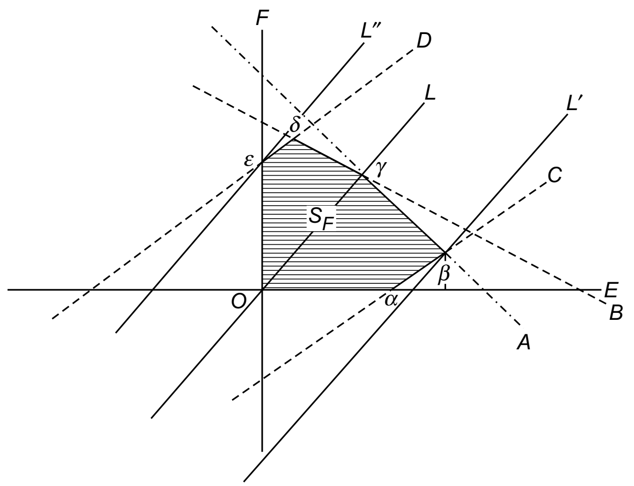
The inequation represents one of two half (hyper) spaces, which is the solution set of the inequations.

The solution set of all the m constraints and non-negativity conditions is the intersection of the half spaces with edges as lines, planes, hyperplanes as the case may be.

In order to understand the inequations in E_2, i.e., in two variables, let the inequations be

$$\begin{aligned} x_1 + x_2 &\le 5 \quad A \\ 4x_1 + 7x_2 &\le 28 \quad B \\ 2x_3 - 3x_2 &\le 6 \quad C \\ -3x_1 + 4x_2 &\le 12, \quad D \\ x_1 \ge 0,\ x_2 &\ge 0\ E,\ F \end{aligned}$$

Then S_F is shaded portion as shown below, where



lines are represented by the so called converted inequations.

Thus, in E_2 it is a region. In general, it is called a polytope. This polytope S_F may be bounded (as in the above case) or unbounded as the case when constraints A and B are not there.

Definition: The set of all convex linear combination (C.L.C.) of finite number of points is called a *polyhedron*.

By definition of C.L.C., a polyhedron is always a convex set.

The points 0, α, β, γ, δ, ε are vertices of S_F. Actually vertices are solutions of two equations.

Now we illustrate a method for solving an optimisation problem. It is known as **Graphical-method**, which can be easily applied in E_2 and to some extent in E_3. Beyond E_3 it is not possible.

Illustration: Find the solution of the following LPP

Max. $Z = 8x_1 - 7x_2$

Subject to $\qquad x_1 + x_2 \le 5$

$$\begin{aligned} 4x_1 + 7x_2 &\le 28 \\ 2x_1 - 3x_2 &\le 6 \\ -3x_1 + 4x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

***Solution*:** We shall first sketch S_F. It is same as above. To find optimum solution, i.e., the maximum value of $f(X) = 8x_1 - 7x_2$, literally and algebraically, we take each and every point of S_F and substitute in $8x_1 - 7x_2$ and pick the maximum value.

It is impossible. Therefore, we first draw the line $f(X) = 8x_1 - 7x_2 = 0$. In figure it is L. Any line parallel to L has the equation of the form

$$8x_1 - 7x_2 = c,\ c \neq 0$$

If $c > 0$, it is towards α, β and if $c < 0$, it is towards δ, ε.

So we move the line L keeping it parallel towards α, β to get maximum value and towards δ, ε to get minimum value, so that at least one point of S_F remains on the line.

In order to get maximum of $8x_1 - 7x_2$, L' is the final position of L and L'' is the final position in order to get minimum value. Thus, the maximum value of the objective function occurs at the vertex β and the minimum value at the vertex ε.

Thus, maximum occurs at $\left(\frac{21}{5}, \frac{4}{5}\right)$ and Max $f(X) = 28$

If it is a minimisation problem, it occurs at (0, 3) and the Min $f(X) = -21$

Looking at the above illustration, we notice that Maximum (Minimum) occurs at a vertex of S_F and S_F is a convex set.

Every half plane represented by an inequation is a convex set. S_F is the intersection of all these convex sets and hence a convex set.

It is not peculiar. But it is always true. We shall prove these results.

Because of non-negativity conditions, S_F is at least bounded below, hence by a theorem it has at least one vertex.

2.7 SOME EXCEPTIONAL CASES IN GRAPHICAL METHOD

In the preceding example we have seen that solution of an LPP occurs at a vertex of S_F. But it is not always true, there may be an LPP for which no solution exists or for which the only solution obtained is an unbounded one or an LPP may have more than one solution. In this section we shall discuss the following three special cases that arise in the application of graphical method.

(a) Alternative optimal solution.
(b) Infeasible (or non-existing) solution.
(c) Unbounded solution.

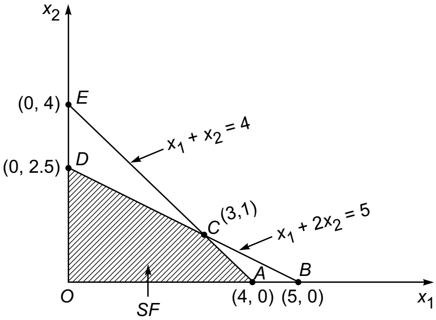
2.7.1 Alternative Optimal Solution

When the objective function assumes the same optimum value at more than one vertex of S_F, then we say that the LPP has an alternative optimal solution.

For example:

Maximize $Z = 2x_1 + 4x_2$

Subject to
$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



Solution: Maximum occurs at C (3, 1) and D (0, 2.5) and the value of $Z = 10$

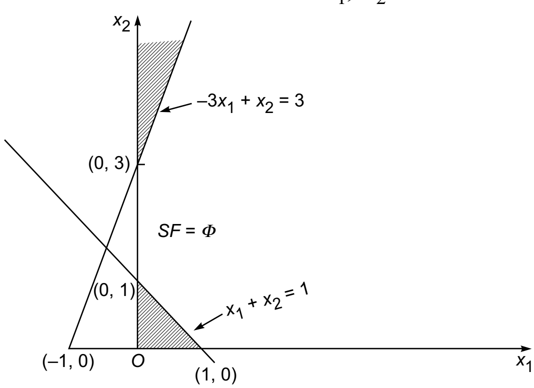
2.7.2 Infeasible Solution

When the constraints are not satisfied simultaneously, the LPP has no feasible solution. This implies if $S_F = \Phi$. This situation can never occur if all the constraints are of the $\leq$ type.

For example:

Maximize $Z = x_1 + x_2$

Subject to
$$x_1 + x_2 \leq 1$$
$$-3x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$



As $S_F = \Phi$, the problem has no feasible solution.

2.7.3 Unbounded Solution

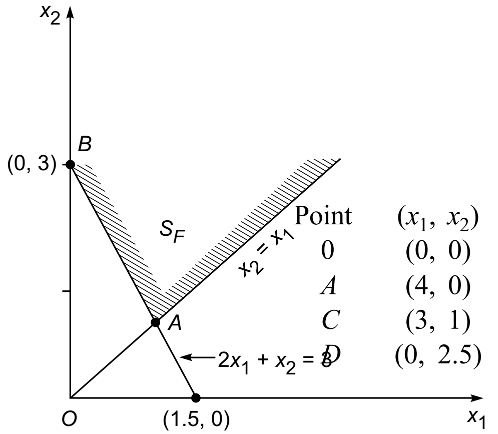
When the value of the decision variables may be increased indefinitely without violating any of the constraints the solution space S_F is unbounded. The value of objective function, in such cases, may increase (for maximization) or decrease (for minimization) indefinitely. Thus, both the solution space and the objective function value are unbounded.

For example:

Maximize $Z = 6x_1 + x_2$

Subject to

$$2x_1 + x_2 \geq 3$$
$$x_2 - x_1 \geq 0$$
$$x_1, x_2 \geq 0$$



Point	(x_1, x_2)	$Z = 2x_1 + 4x_2$
0	(0, 0)	0
A	(4, 0)	8
C	(3, 1)	10 ← Max.
D	(0, 2.5)	10 ← Max.

The graphical solution of the given LPP is depicted in the above figure. The two vertices of the feasible region are *A* and *B*. We observe, that the feasible region S_F is unbounded. The value of the objective function at the vertex *A* (1, 1) and *B* (0, 3) are 7 and 3, respectively.

But there exist number of points in feasible region for which the value of the objective function is more than 7. For example, the point (3, 6) lies in the feasible region and the objective function value at this point is 24 which is more than 7. Thus, both the variables x_1 and x_2 can be made arbitrarily large and the value of *Z* also increases. Hence, the problem has an unbounded solution.

Remark: An unbounded solution means that there exist an infinite number of solutions to the problem.

EXERCISE 2.3

1. Use graphical method to solve

   Maximize $Z = 4x_1 + 3x_2$

   Subject to

   $$2x_1 + x_2 \leq 1000$$
   $$x_1 + x_2 \leq 800$$
   $$x_1 \leq 400$$
   $$x_2 \leq 700$$
   $$x_1, x_2 \geq 0$$

   (**Ans:** $x_1 = 200$, $x_2 = 600$ Max. $Z = 2600$)

2. Solve the following LPP graphically.

Minimize $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \leq 6$

$2x_1 + 3x_2 \geq 3$

$0 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 3$

(**Ans:** $x_1 = 3 = x_2$, Min $Z = 24$)

3. Use graphical method to solve

   Maximize $Z = x_1 + 2x_2$

   Subject to $x_1 - x_2 \leq 1$

   $x_1 + x_2 \geq 3$

   $x_1, x_2 \geq 0$

   (**Ans:** Infeasible sol.)

4. Solve the following LPP graphically

   Maximize $Z = 6x_1 - 3x_2$

   Subject to $x_1 + x_2 \leq 1$

   $2x_1 - x_2 \leq 1$

   $-x_1 + 2x_2 \leq 1$

   $x_1, x_2 \geq 0$

   $\left(\textbf{Ans: } x_1 = \frac{2}{3}, x_2 = \frac{1}{3}, Z = 3 \text{ or } x_1 = \frac{1}{2}, x_2 = 0, Z = 3, \text{Alternative sol.}\right)$

5. Use graphical method to solve

   Minimize Z = $-4x_1 + x_2$

   Subject to $x_1 - 2x_2 \leq 2$

   $-2x_1 + x_2 \leq 2$

   $x_1, x_2 \geq 0$

   (**Ans:** Unbounded solution space and unbounded solution)

6. Solve the problem 5 by changing objective function to maximization.

   (**Ans:** Unbounded solution space but bounded solution)

7. Solve the following graphically

   Minimize $Z = x_1 - 2x_2$

   Subject to $-x_1 + x_2 \leq 1$

   $2x_1 + x_2 \leq 2$

   $x_1, x_2 \geq 0$

   $\left(\textbf{Ans: } x_1 = \frac{1}{3}, x_2 = \frac{4}{3}, Z = -\frac{7}{3}\right)$

2.8 CONVEX SETS AND LINEAR PROGRAMMING PROBLEM

2.8.1 Introduction

It is assumed here that we are familiar with Real Vector Spaces, Inner Product Space, Euclidean Space, linear dependence, linear independence, linear combination, subspaces, etc.

E_n, the set of all n-tuples of real numbers, is an Euclidean space. We shall confine ourselves to E_n only.

A vector $X \varepsilon E_n$ is called a linear combination of vectors $X_1, X_2, \ldots, X_k$, if X can be expressed as

$$X = \alpha_1 X_1 + \ldots + \alpha_k X_k$$

2.8.2 Convex Linear Combination

In order to make a linear combination (l.c.), the choice of scalars too large. A special type of l.c. is called a convex linear combination. To be precise, we define

Definition: Let $X_1, X_2, \ldots, X_k$ be vectors of E_n. A vector $X \varepsilon E_n$ is called a convex linear combination (C.L.C.) of $X_1, X_2, \ldots, X_k$, if

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k,$$
$$\alpha_i \geq 0,\ \alpha_1 + \alpha_2 + \ldots \alpha_k = 1$$

The expression $\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k$, $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$ itself is called C.L.C. of $X_1, X_2, \ldots, X_k$.

A C.L.C. of two vectors X_1, X_2 can also be written as $(1 - \lambda)X_1 + \lambda X_2$, $(\lambda \geq 0, \lambda \leq 1)$ or $0 \leq \lambda \leq 1$

In E_2, $X = (1 - \lambda)X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$ means

$$(x, y) = (1 - \lambda)(x_1, y_1) + \lambda (x_2, y_2)$$

i.e., $$x = \frac{(1-\lambda)x_1 + \lambda x_2}{(1-\lambda)+\lambda},\ y = \frac{(1-\lambda)y_1 + \lambda y_2}{(1-\lambda)+\lambda},\ 0 \leq \lambda \leq 1$$

i.e., points on the line segment joining the points (x_1, y_1) and (x_2, y_2). Thus, set of all C.L.C. of X_1, X_2 in E_2 is a line segment joining X_1, X_2.

In general, in E_n, set of all C.L.C. of X_1, X_2 is the 'line segment' joining the points X_1, X_2. To be precise, in E_n,

$$\{X \varepsilon E_n \mid X = (1 - \lambda)X_1 + \lambda X_2,\ 0 \leq \lambda \leq 1 \mid\}$$
$$= \text{'line segment' joining } X_1, X_2$$
$$\{X \varepsilon E_n \mid X = (1 - \lambda)X_1 + \lambda X_2 \mid \lambda > 0\}$$
$$= \text{'half line' from } X_1 \text{ towards } X_2;\ X_1 \text{ excluded}$$
$$\{X \varepsilon E_n \mid X = (1 - \lambda)X_1 + \lambda X_2,\ \lambda \geq 0\}$$
$$= \text{'ray' from } X_1 \text{ in the direction of } X_2, \text{ and so on.}$$

Example 1: Is $\left(1, -\frac{1}{4}\right)$ a C.L.C. of (1, 0), (1, 1) and (1, –2)?

If yes, express it

Solution: $$\left(1, -\frac{1}{4}\right) = \alpha(1, 0) + \beta(1, 1) + \gamma(1, -2)$$

$$\alpha + \beta + \gamma = 1$$

$$\beta - 2\gamma = -\frac{1}{4}$$

Let $\alpha = \frac{1}{3}, \beta + \gamma = \frac{2}{3}, \beta - 2\gamma = -\frac{1}{4}$

or, $\beta = \frac{13}{36}, \gamma = \frac{11}{36}$

Thus, $\left(1, -\frac{1}{4}\right) = \frac{1}{3}(1, 0) + \frac{13}{36}(1, 1) + \frac{11}{36}(1, -2)$

It is a required C.L.C. and the above is the required expression.

Example 2: Is (1.5, .6) a C.L.C. of (0, 0), (2, 0), (1, 1)? If yes, express it.

Solution: Let $(1.5, .6) = \alpha(0, 0) + \beta(2, 0) + \gamma(1, 1)$

$$2\beta + \gamma = 1.5$$
$$\gamma = .6$$
$$\alpha + \beta + \gamma = 1$$

On solving we get $\gamma = .6$, $\beta = .45$, $\alpha = -.05$.

Thus, it is not a C.L.C. of the required points.

2.8.3 Convex Set

Definition: A non-empty subset $S \subset E_n$ is said to be convex if and only if, the set of all C.L.C. of any two given points of S, is a subset of S, i.e.,

$$\{X = (1 - \lambda) X_1 + \lambda X_2 | X_1, X_2 \varepsilon S, 0 \leq \lambda \leq 1\} \subset S$$

or, iff $X = (1 - \lambda) X_2 + \lambda X_2, 0 \leq \lambda \leq 1, \varepsilon S \ \forall \ X_1, X_2 \varepsilon S.$

Example 3: A line in E_2, E_3 or in E_n is a convex set.

Example 4: A plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta$ is a convex set in E_3.

Example 5: A hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \beta$ is a convex set in E_n.

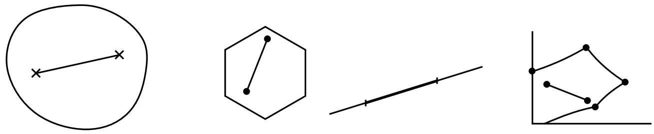
In matrix notations, $\alpha_1 x_1 + \ldots + \alpha_n x_n = \beta$, can be expressed as

$$(\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \beta$$

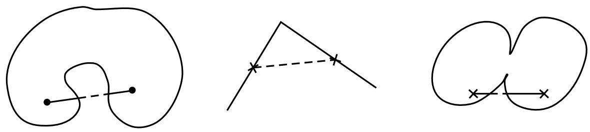
or, $C^T X = \beta$, where $C = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Thus, $C^T X = \beta$ is a hyperplane in E_n.

Example 6: In E_2, E_3, a set S, is a convex set, if any two points of S can be joined by a 'line segment' contained in S. Thus,



are convex sets, while



are not convex sets.

Definition: The sets

$$\{X \,\varepsilon\, E_n \mid C^T X < \alpha\} \qquad (1)$$
$$\{X \,\varepsilon\, E_n \mid C^T X \leq \alpha\} \qquad (2)$$
$$\{X \,\varepsilon\, E_n \mid C^T X > \alpha\} \qquad (3)$$
$$\{X \,\varepsilon\, E_n \mid C^T X \geq \alpha\} \qquad (4)$$

are called hyperspaces in E_n.

In order to understand it, let us come down to E_3. In E_3

$$C^T X = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta$$

is a plane P which divides the whole space in two parts, both called half spaces or simply space. The points in these spaces satisfy the inequalities.

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 < \beta \text{ or } \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 > \beta$$

If the plane P is included in these spaces, then inequalities are

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 < \beta \text{ or } \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \geq \beta.$$

We shall not be going in the depth of topological concepts, but we shall confine ourselves with the understanding of 'boundary' which also has its literal meaning. 'Boundary point' would mean a point on the boundary.

Taking 'boundary' as an undefined term, we define the following.

Definition: A set S, is said to be *open*, if it contains *no* boundary point.

In other words, if no point of boundary of S is in S, then S is open.

Definition: A set S is said to be *closed* if it contains *all* its boundary points, i.e., whole boundary.

Thus, half spaces $C^T X < \alpha$, $C^T X > \alpha$ are open sets while $C^T X \leq \alpha$ and $C^T X \geq \alpha$ are closed sets.

In example 3, we have shown that hyperplane is a convex set and also we prove something more.

Theorem 1: *A hyperplane $S: C^T X = \alpha$ is a closed convex set.*

Proof: No point in $C^T X < \alpha$ and $C^T X > \alpha$ is a boundary point of $C^T X = \alpha$. Thus, all boundary points of the hyperplane are in it. Hence, it is closed.

Now, let $X_1, X_2 \varepsilon S$

Therefore, $C^T X_1 = \alpha$ and $C^T X_2 = \alpha$

Let $X = (1 - \lambda) X_1 + \lambda X_2$, $0 \le \lambda \le 1$ be any C.L.C. of X_1, X_2

Then
$$\begin{aligned} C^T X &= C^T[(1 - \lambda) X_1 + \lambda X_2] \\ &= (1 - \lambda) C^T X_1 + \lambda C^T X_2, \text{ since matrix multiplication is distributive over '+'} \\ &= (1 - \lambda) \alpha + \lambda \alpha = \alpha \end{aligned}$$

Hence $X \varepsilon S$. So S is a convex set.

Hence, the result.

Theorem 2: *A closed half space in E_n is a closed convex set.*

Proof: Let $S = \{X \varepsilon E_n \mid C^T X \le \alpha\}$ be a closed half space.
Let $X_1, X_2 \varepsilon S$. Then $C^T X_1 \le \alpha$, $C^T X_2 \le \alpha$
Let $X = (1 - \lambda) X_1 + \lambda X_2$ be a C.L.C. of X_1, X_2.
Then
$$\begin{aligned} C^T X &= (1 - \lambda) C^T X_1 + \lambda C^T X_2 \\ &\le (1 - \lambda) \alpha + \lambda \alpha = \alpha \end{aligned}$$

$\therefore$ $X \varepsilon S$. So S is convex

Hence, the result.

Lines in E_2 or E_3 are convex sets. By theorem 1, XY-plane and YZ-plane are convex sets but their union is not a convex set as (1, 0, 0) is in XY-plane, (0, 0, 1) is in YZ-plane but their C.L.C., $\frac{1}{2}(1, 0, 0) + \frac{1}{2}(0, 0, 1) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ is in neither. Thus, union of two convex sets is not a convex set. But intersection of two convex sets is a convex set as is proved below.

Theorem 3: *Intersection of two convex sets is a convex set.*

Proof: Let S_1, S_2 be convex sets and $S = S_1 \cap S_2$ be their intersection.

Let $X_1, X_2 \varepsilon S$.

So $X_1, X_2 \varepsilon S_1$ and $X_1, X_2 \varepsilon S_2$

Let $X = (1 - \lambda) X_1 + \lambda X_2$, $0 \le \lambda \le 1$, be a C.L.C. of X_1, X_2 Since S_1, S_2 are convex sets, by definition, $X \varepsilon S_1$ and $X \varepsilon S_2$. Hence, $X \varepsilon S$. Thus, S is convex.

We have defined convex sets in terms of C.L.C. of two points. We have also defined C.L.C. of more than 2 points. Thus, proved

Theorem 4: *A set S is convex iff every C.L.C. of points in S is in S.*

Proof: Let every C.L.C. of points in S be in S.

Therefore, every C.L.C. of two points in S is in S.

Hence, S is convex.

Now, let S be convex and $X_1, X_2, X_3, \ldots, X_m \varepsilon S$

Let $X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_m X_m$, $\alpha_i \ge 0$,

and $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$

be a C.L.C. of $X_1, X_2, \ldots, X_m$. We shall now show that $X \varepsilon S$.

We shall prove it by induction.

Obviously, it holds for $k = 1$, as $X = X_1 \varepsilon S$

Also, it holds for $k = 2$, as $X = \alpha'_1 X_1 + \alpha'_2 X_2$, $\alpha'_1 + \alpha'_2 = 1$, $\alpha'_i \geq 0$ belong to S by definition of convex set.

Let, now, it hold for $k = k$, *i.e.*,

$$\alpha'_1 X_1 + \alpha'_2 X_2 + \ldots + \alpha'_k X_k \varepsilon S$$

$$\alpha'_i \geq 0,\ \alpha'_1 + \alpha'_2 + \ldots + \alpha'_k = 1.$$

Now consider a C.L.C. of $k + 1$ points,

$$\beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k + \beta_{k+1} X_{k+1},\ \beta_1 + \beta_2 + \ldots + \beta_k + \beta_{k+1} = 1$$

$$= \frac{\beta_1 + \beta_2 + \ldots + \beta_k}{\beta_1 + \beta_2 + \ldots + \beta_k} (\beta_1 X_1 + \ldots + \beta_k X_k) + \beta_{k+1} X_{k+1}$$

$$= (\beta_1 + \beta_2 + \ldots + \beta_k)(\alpha_1 X_1 + \ldots + \alpha_k X_k) + \beta_{k+1} X_{k+1}$$

Where, $\alpha_i = \dfrac{\beta_i}{\beta_1 + \ldots + \beta_k}$

Since $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$, $\alpha_1 X_1 + \ldots + \alpha_k X_k = Y$, is a C.L.C. of $X_1, X_2, \ldots, X_k$ which is in S by assumption.

Thus, $\beta_1 X_1 + \ldots + \beta_k X_k + \beta_{k+1} X_{k+1} = (\beta_1 + \ldots + \beta_k) Y + \beta_{k+1} X_{k+1}$ is a C.L.C. of $Y, X_{k+1} \varepsilon S$

Hence, it belongs to S. Hence, the result.

Example 7: The set $S = \{X \varepsilon E_n \mid \|X - X_0\| \leq \alpha\}$ is a convex set, $\|\cdot\|$ is the norm (usual), the 'distance' of X from X_0.

Proof: Let $X_1, X_2 \varepsilon S$. Therefore, $\|X_1 - X_0\| \leq \alpha$ and $\|X_2 - X_0\| \leq \alpha$.

Let $X = (1 - \lambda) X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$, be a C.L.C. of X_1, X_2.

Then

$$\begin{aligned}
\|X - X_0\| &= \|(1 - \lambda) X_1 + \lambda X_2 - X_0\| \\
&= \|(1 - \lambda) X_1 - (1 - \lambda) X_0 + \lambda X_2 - \lambda X_0\| \\
&\leq \|(1 - \lambda)(X_1 - X_0)\| + \|\lambda (X_2 - X_0)\| && \text{(norm property)} \\
&= (1 - \lambda) \|X_1 - X_0\| + \lambda \|X_2 - X_0\| && \text{(norm property)} \\
&\leq (1 - \lambda) \alpha + \lambda\alpha && \text{(given)} \\
&\leq \alpha
\end{aligned}$$

So $X \varepsilon S$. Hence, proved.

In E_2, S is the circle, with interior, centred at X_0 and of radius α, while in E_3, S is the sphere with interior, of radius α and centred at X_0.

Example 8: The solution set of the m inequations in n-variables is a convex set.

Let the inequations be

$$\alpha_{11}\, x_1 + \alpha_{12}\, x_2 + \ldots + \alpha_{1n}\, x_n \le \beta_1$$
$$\alpha_{21}\, x_1 + \alpha_{22}\, x_2 + \ldots + \alpha_{2n}\, x_n \le \beta_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\alpha_{i1}\, x_1 + \alpha_{i2}\, x_2 + \ldots + \alpha_{in}\, x_n \le \beta_i$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\alpha_{m1}\, x_1 + \alpha_{m2}\, x_2 + \ldots + \alpha_{mn}\, x_n \le \beta_m$$

The solution set S_i, of each inequation is a half space $C_i^T X \le \beta_i$ is a convex set.

The solution set of these m inequations is the intersection of $S_1, S_2, \ldots, S_m$, which by theorem is a convex set.

Notation:

$$X = (x_1, x_2, \ldots, x_n) \ge (\alpha_1, \alpha_2, \ldots, \alpha_n) = A$$

means $\quad x_i \ge \alpha_i,\ i = 1, 2, \ldots, n$

Definition:

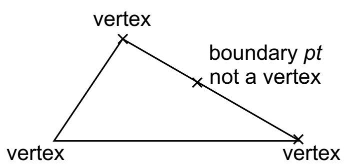
(a) A subset $S \subset E_n$ is said to be bounded below if $\exists$ a point $Y \varepsilon E_n \ni X \ge Y \ \forall \ X \varepsilon S$.

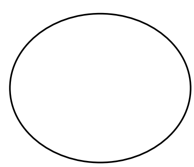
(b) A subset $S \subset E_n$ is said to be bounded above if $\exists$ a point $Y \varepsilon E_n \ni X \le Y \ \forall \ X \varepsilon S$.

(c) A subset $S \subset E_n$ is said to be bounded if it is bounded below as well as bounded above *i.e.*, $\exists \ Y_1, Y_2 \ \varepsilon \ E_n \ni Y_1 \le X \le Y_2 \ \forall \ X \varepsilon S$.

Definition: Let S be a convex subset of E_n. A point $X \varepsilon S$ is called a vertex of S if it cannot be expressed as a C.L.C. of two other (other than X) points of S, *i.e.*, if it is not possible to find X_1, $X_2 \ \varepsilon \ S \ni X = (1 - \lambda)\, X_1 + \lambda X_2$, $0 < \lambda < 1$.

Vertex is a boundary point but a point on boundary need not be a vertex. It is also called an extreme point.





Definition: Let S be a closed subset of E_n. A plane P: $C^T X = \alpha$ is called a separating hyperplane if S is contained in one of the half spaces determined by P, *i.e.*, $C^T X \le \alpha \ \forall \ X \varepsilon S$ or $C^T X \ge \alpha$ $\forall \ S$.

Definition: Let S be a closed convex subset of E_n. A plane P: $C^T X = \alpha$ is called a supporting hyperplane at $X_0 \ \varepsilon \ S$ if $C^T X_0 = \alpha$ and $C^T X \le \alpha \ \forall \ X \varepsilon S$ or $C^T X \ge \alpha \ \forall \ X \varepsilon S$, *i.e.*, $X_0 \ \varepsilon \ S$ lies on the plane P and S is contained in one of the two half spaces determined by P.

Remark: It is obvious that X_0 is a boundary point of S. It cannot be an interior point.

It is obvious that given a closed convex set S, we can find a separating hyperplane passing through a given exterior point, *i.e.*, given point outside S. We shall now give a formal proof.

Theorem 5: Let S be a non-empty closed convex subset of E_n and $X_0 \notin S$. Then there exists a separating hyperplane through X_0, *i.e.*, passing through X_0.

Proof:

$$X_0 \varepsilon S.$$

Consider $T = \{||Y - X_0|| \mid Y \varepsilon S\}$. Since $||\cdot|| \geq 0$, the set T is bounded below and it is a set of real non-negative numbers. Hence, minimum (greatest lower bound) of T exists. Let min T occurs for the point $Z \varepsilon S$. Thus,

$$||Z - X_0|| = \min_{Y \in S} ||Y - X_0||$$

or, $$||Z - X_0|| \leq ||Y - X_0|| \;\forall\; Y \varepsilon S \text{ and a given } Z \varepsilon S.$$

Let X be any point in S. Also $Z \varepsilon S$ and since S is convex,

$$(1 - \lambda) Z + \lambda X \varepsilon S, \; 0 \leq \lambda \leq 1$$

Therefore,

$$||(1 - \lambda) Z + \lambda X - X_0|| \geq ||Z - X_0||$$

or, $$||(1 - \lambda) Z + \lambda X - X_0||^2 \geq ||Z - X_0||^2$$

Using the definition of norm, we have

$$((1 - \lambda) Z + \lambda X - X_0)\cdot(1 - \lambda) Z + \lambda X - X_0) \geq (Z - X_0)\cdot(Z - X_0)$$

'$\cdot$' is the inner product in E_n

or, $$(Z - X_0) + \lambda (X - Z))\cdot((Z - X_0) + \lambda(X - Z)) \geq (Z - X_0)\cdot(Z - X_0)$$

Using the properties of Inner Product in E_n, we get

$$(Z - X_0)\cdot(Z - X_0) + \lambda^2(X - Z)\cdot(X - Z) + 2\lambda(Z - X_0)\cdot(X - Z)$$
$$\geq (Z - X_0)\cdot(Z - X_0)$$

or, $$\lambda^2(X - Z)\cdot(X - Z) + 2\lambda(Z - X_0)\cdot(X - Z) \geq 0$$

Since it is true even for all $\lambda > 0$, we take $\lambda \neq 0$, and

$$\lambda(X - Z)\cdot(X - Z) + 2(Z - X_0)\cdot(X - Z) \geq 0$$

λ can be taken arbitrary small, so it holds even when
$\lambda \to 0$, which in turn gives

$$(Z - X_0)\cdot(X - Z) \geq 0$$

Let $$C = Z - X \text{ and } ||C|| \geq 0 \text{ as } Z \neq X$$

Thus, $$C\cdot(X - Z) \geq 0$$

Using Matrix notations, we obtain

$$C^T(X - Z) \geq 0$$

or, $$C^TX - C^TZ \geq 0$$

or, $$C^TX \geq C^TZ$$

Let $C^TZ = \alpha$. Then, $C^TX \geq \alpha \;\forall\; X \in S$.

Hence, $C^TX = \alpha$ is the required separating hyperplane. Hence, the result.

It is evident that every supporting plane is a separating plane. In other words supporting plane is a limiting case of a separating plane. We shall now prove the existence of a supporting plane through a boundary point of the closed convex set.

Theorem 6: Let S be a non-empty closed convex subset of E_n and X_0 a boundary point of S. Then there exists a supporting hyperplane at X_0.

Proof: As mentioned earlier, it is a limiting case and can be easily proved as a limiting case of the above theorem.

Let X_0 be a boundary point of S. Thus each open ball centred at X_0 has a point of S and also a point exterior (outside) to S, *i.e.*, for each $\in > 0$, $\|X - X_0\| < \in$ has a point not in S.

Let $Y \in$ open ball $\|X - X_0\| < \in$ and $Y \notin S$.

By theorem 5, $\exists$ a separating plane $C^T X = \alpha$ passing through Y, *i.e.*, $C^T Y = \alpha$.

As $\in$ goes on reducing, *i.e.*, $\in \to 0$, the above result holds, *i.e.*, separating plane exists. As $\in \to 0$, points in open ball come nearer to X_0. Therefore, separating plane comes closer to S and in limit it passes through X_0 and hence becomes a supporting plane through X_0. Hence, the theorem.

The above theorem assures that there exists supporting plane passing through a boundary point and hence through a vertex. What about the converse? Whether every supporting plane has a boundary point? The answer is in affirmative by definition. But whether every supporting plane of S has a vertex of X, does not follow from definition. We shall prove this result below under a condition.

Theorem 7: Let S be a non-empty, closed, convex subset of E_n, which is bounded below. Then every supporting plane of S has a vertex (an extreme point) of S.

Proof: Supporting hyperplane is defined with respect to a boundary point. Let X_0 be a boundary point of S. Then by theorem 5, there exists a supporting hyperplane

$$H = \{X \in E_n \mid C^T X = \alpha\}$$

through X_0, *i.e.*, $C^T X_0 = \alpha$.

Let $$T = S \cap H.$$

Since S and H are closed, convex sets, so is T. Since S is bounded below, T is also bounded below. Thus, there exists a $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in E_n \ni$

$$(x_1, x_2, \ldots, x_n) \geq (\alpha_1, \alpha_2, \ldots, \alpha_n) \;\forall\; (x_1, x_2, \ldots, x_n) \in T.$$

Choose a point X^* in T which has smallest first coordinate, smallest 2nd coordinate, …, smallest n^{th} coordinate. Let it be

$$X^* = (x^*_1, x^*_2, \ldots, x^*_{j-1}, x^*_j, x^*_{j+1}, \ldots, x^*_n)$$

This would be an unique point in T because only one point can have all smallest components, *i.e.*, there is no tie in one of the components, say x^*_j.

$$x_i \geq x^*_i,\; i = 1, 2, \ldots, n \;\forall\; X = (x_1, x_2, \ldots, x_n) \in T \qquad (*)$$

We now show that this $X^* \in T$ is an extreme point (vertex).

Let it not be a vertex. Then $\exists\; Y, Z \in T$

Such that

$$X^* = (1 - \lambda)\, Y + \lambda Z,\; 0 < \lambda < 1,\; Y \neq Z$$

i.e., $$x^*_i = (1 - \lambda)\, y_i + \lambda Z_i,\; i = 1, 2, \ldots, n.,\; 0 < \lambda < 1.$$

Let $y_i > Z_i$. Then,

$$x_i^* = Z_i + (1 - \lambda)(y_i - Z_i) > Z_i$$

which contradicts, (*), $x_i^* \le Z_i$,

Let $y_i < Z_i$. Then

$$x_i^* = y_i + \lambda(Z_i - y_i) > y_i$$

which also contradicts, *, $x_i^* \le y_i$.

Thus, $Y_i = Z_i = x_i^*,\ i = 1, 2, \ldots, n$

i.e., $X^* = Y = Z,$

a contradiction. Hence, X^* is a vertex of T.

Now we shall prove that X^* is a vertex of S too. In order to prove this, we shall show that a point of T which is not a vertex of S is not a vertex of T.

Let $X \in T$ and not a vertex of S, *i.e.*, $\exists\ Y$ & $Z \in S$, $\ni$

$$X = (1 - \lambda)\, Y + \lambda Z,\ 0 < \lambda < 1,\ Y \ne Z$$

or, $C^T X = (1 - \lambda)\, C^T Y + \lambda\, C^T Z$

Where C^T corresponds to H.

Since $Y, Z \in S$, $C^T Y$, $C^T Z$ are both $\ge \alpha$ or both $\le \alpha$. Let $C^T Y \ge \alpha$, $C^T Z \ge \alpha$. Then,

$$C^T X \ge \alpha$$

But $X \in H$, $C^T X = \alpha$, therefore $C^T Y = C^T Z = \alpha$

Thus, $Y, Z \in H$; $Y, Z \in T$

Therefore, X is not a vertex of T, which proves the result.

Now as a corollary to the above theorem, we prove the following.

Corollary

Let S be a non-empty, closed, convex subset of E_n. If S is bounded below (above), then S has at least one vertex.

Proof: S is a closed set. So it has a boundary point $X_0 \in S$.

By Theorem 5 there exists a supporting plane through X_0.

By theorem 7 this supporting plane has a vertex as it is bounded below too. Hence, the result.

The theorem 7 can also be proved on similar lines in case S is bounded above and consequently the above corollary will follow with 'below' replaced by 'above'.

In case the set S is bounded then the above result follows but something more also follows. It is proved in the following result.

Theorem 8: Let S be a non-empty, closed, convex subset of E_n. If S is bounded, then it has at least one vertex and every point of S is a C.L.C. of its vertices.

The proof is left to the reader.

Theorem 9: The optimum of $f(X)$, the objective function, of LPP occurs at a vertex of S_F, provided S_F is bounded.

Proof: Let the LPP be a maximization problem. S_F is bounded polytope, so it has finite number of vertices. Let $X_1, X_2, \ldots, X_m$ be m vertices.

Let $f(X)$ assume maximum at $X_0 \in S_F$.

i.e., $f(X_0)$ is maximum, *i.e.*,

$$f(X_0) \geq f(X) \ \forall \ X \in S_F$$

Also $\quad f(X_0) \geq f(X_i) \ \forall \ i = 1, 2, \ldots, m$ (vertices)

We shall now prove that $f(X_0) = f(X_k)$ for some $k = 1, 2, \ldots, m$

Since S_F is non-empty, closed, convex, by a theorem, every point of S_F is a C.L.C. of $X_1, X_2, \ldots, X_m$. So

$$X_0 = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_m X_m, \ 0 \leq \alpha_i \leq 1,$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

or,

$$f(X_0) = f(\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_m X_m)$$
$$= \alpha_1 f(X_1) + \alpha_2 f(X_2) + \ldots + \alpha_m f(X_m)$$

(since f is linear)

Since $\{f(X_1), \ldots, f(X_m)\}$ is a finite set, it has a maximum, let $f(X_k)$ be maximum.

Then $\quad f(X_i) \leq f(X_k) \ \forall \ i = 1, 2, \ldots, k_m$. So

$$f(X_0) \leq \alpha_1 f(X_k) + \alpha_2 f(X_k) + \ldots + \alpha_m f(X_k)$$
$$= (\alpha_1 + \alpha_2 + \ldots + \alpha_k) f(X_k)$$
$$\leq f(X_k)$$

But $\quad f(X_0) \geq f(X_k)$

Therefore, $\quad f(X_0) = f(X_k)$.

Hence, the maximum occurs at X_k, a vertex of S_F.

Hence, the theorem.

From the above theorem optimum occurs at a vertex. We shall now prove that if f is not a constant function, $f(X)$ will not attain maximum at an interior point.

Theorem 10: In an LPP, if the objective function $f(X)$ is non-constant, then f does not attain optimum at an interior point.

Proof: Let f attain maximum at an interior point $X_0 \in S_F$

Then

$$f(X_0) \geq f(X) \ \forall \ X \in S_F$$

Let $X_1 \in S_F$. Since X_0 is an interior point of S_F, $\exists \ X_2 \in S_F \ni X_0 = \lambda X_1 + (1 - \lambda) X_2$, $0 < \lambda < 1$, X_1, X_2 interior points of S_F.

$$f(X_0) = f(\lambda X_1 + (1 - \lambda)(X_2))$$
$$= \lambda f(X_1) + (1 - \lambda) f(X_2)$$

But $\quad f(X_0) \geq f(X_1)$ and $f(X_0) \geq f(X_2)$

$\therefore \quad \lambda f(X_1) + (1 - \lambda) f(X_2) \geq f(X_1)$

or, $\quad f(X_2) \geq f(X_1)$

Similarly, $\quad \lambda f(X_1) + (1 - \lambda) f(X_2) \geq f(X_2)$

$$f(X_1) \geq f(X_2)$$

Which implies $\quad f(X_1) = f(X_2)$

$$f(X_0) = \lambda f(X_1) + (1 - \lambda) f(X_2)$$
$$= \lambda f(X_1) + (1 - \lambda) f(X_2)$$
$$= f(X_1)$$

in standard form and X^* is any optimal solution when C is replaced by C^*, then prove that

$$(C^* - C)^T (X^* - X_0) \geq 0.$$

2.9 ALGEBRAIC FORMULATION OF THE LPP THEORY

From the above theorems 9 and 10 of section 1.8, it is clear that optimum occurs at a vertex. Thus, theoretically speaking, a bounded S_F (polytope) has finite number of vertices, so we can find the values of objective function and then find the optimum value. This would give us optimum value of the function as well as the vertex (coordinates of which are the values of variables) at which optimum occurs. It looks simple in saying and certainly not difficult in E_2, but in higher dimension, it is not simple as there is no geometry.

In this case we proceed algebraically. First we need to equip with certain definitions and concepts.

Normally constraints are inequations. We introduce slack, surplus variables to convert them into equations which increases the number of variables and represent hyperplanes. Vertices are points of intersection of these hyperplanes.

Let constraint equations (after introducing Slack/Surplus variables) be

$$\begin{aligned} \alpha_{11} x_1 + \alpha_{12} x_2 + \ldots + \alpha_{1n} x_n &= b_1 \\ \alpha_{21} x_1 + \alpha_{22} x_2 + \ldots + \alpha_{2n} x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad &\ldots \\ \alpha_{m1} x_1 + \alpha_{m2} x_2 + \ldots + \alpha_{mn} x_n &= b_m \end{aligned}$$

Which can be written in matrix form as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or
$$AX = b.$$

Let $A_1, A_2, \ldots, A_j, \ldots A_n$ denote the columns of A, *i.e.*,

$$A_j = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}$$

Then the matrix A can be expressed as

$$A = [A_1 \ A_2 \ \ldots \ A_n]$$

and the system of equations as

$$[A_1 \ A_2 \ \ldots \ A_n] \ X = b$$

In almost all problems, after introducing slack surplus variables, number of unknowns are more than number of equations, so for our discussion, we can safely take $m < n$.

Case I We have learnt earlier that, if $r(A) \neq r(A, b)$ then the system is inconsistent and there would not be any solution. Thus, S_F would be empty and the LPP will not have any feasible solution.

Case II If $r(A) = r(A, b)$, the system will have a solution and consequently the LPP will have a solution.

If $r(A) = r(A, b) = r < m$, then it means that the system has $m–r$ equations dependent on r equations. These $m–r$ equations are redundant and can be removed without affecting the solution and then $r(A) = r(A, b) = r$ would be equal to the number of equations. Therefore, without loss of generality, we can, for our discussion purposes, assume that $r(A) = r(A, b) = m$, *i.e.*, all the m equations are independent.

Since $r(A) = r(A, b) = m$, A will have a submatrix B of size mxm which would be non-singular and hence invertible.

Also, since $r(A) = r(A, b) = m$, m unknowns can be evaluated in terms of remaining $n–m$ unknowns, values of which can be arbitrarily chosen and can be chosen as zero.

We also know that only those variables can be obtained in terms of others, whose columns make the non-singular submatrix B.

Thus, in order to obtain a solution of the system, we assume $n–m$ variables as zero such that the columns of remaining m variables form a non-singular submatrix B, and solve the remaining m-equations in m-unknowns, *i.e.*, the system $BX_B = b$, which has a unique solution because B is non-singular.

A solution, in which n-m unknowns have been taken to be zero is called a BASIC solution. The variables which have been assumed to be zero are called NON-BASIC variables. The variables which have been obtained by solving $BX_B = b$ are called BASIC variables.

Now, we know that the solution obtained above *i.e.*, basic solution satisfies the constraints. Now considering the non-negativity conditions, we discard those basic solutions in which basic variables have negative values (non-basic variables are already zero) and retain those in which basic variables are ≥ 0. Thus, a basic solution in which all basic variables are ≥ 0 (non-basic variables are already zero) satisfy $AX = b$ and $X \geq 0$ and therefore are called BASIC FEASIBLE SOLUTION, in short BFS.

Theorem 1: A *BFS* of an LPP is a vertex of S_F, the convex set of feasible solution.

Proof: Let X_V be a *BFS*. It has m-basic variables. Let us assume, without loss of generality, first m-variables are basic variables. Let

$$X_V = (v_1, v_2, \ldots, v_m, 0, 0, \ldots, 0)^T,$$

where, $v_1, v_2, \ldots, v_m \geq 0$ and

$$AX_V = b$$

or, $$v_1 A_1 + v_2 A_2, + \ldots + v_m A_m = b,$$

where, $A_1, A_2, \ldots, A_m$ are columns of A corresponding to basic variables, hence linear independent.

Since, X_V is a *BFS*, $X_V \in S_F$.

We shall now prove that X_V is a vertex. Let us assume the contrary.

Let, if possible, X_V be not a vertex. Then, $\exists\ X_1, X_2 \in S_F \ni X_V \neq X_1 \neq X_2$ and

The columns $A_1, A_2, \ldots, A_r$ are vectors of E_m. E_m can have utmost m linearly independent vectors. So, if we can prove that $A_1, A_2, \ldots, A_r$ are linearly independent then it would mean $r \leq m$, *i.e.*, at least n-m variables are zero. Hence, X_V is a BFS.

Thus, it remains to prove that $A_1, A_2, \ldots, A_r$ are linearly independent

Let us assume the contrary. Let $A_1, A_2, \ldots, A_r$ be linearly dependent. So there exists $\alpha_1, \alpha_2, \ldots, \alpha_r$ scalars not all zero such that

$$\alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_r A_r = 0 \tag{1}$$

We also have

$$v_1 A_1 + v_2 A_2 + \ldots + v_r A_r = b. \tag{2}$$

From the above two equations, we obtain

$$(v_1 + c\alpha_1) A_1 + (v_2 + c\alpha_2) A_2 + \ldots + (v_1 + c\alpha_r) A_r = b \tag{3}$$

and,
$$(v_1 - c\alpha_1) A_1 + (v_2 - c\alpha_2) A_2 + \ldots + (v_r - c\alpha_r) A_r = b \tag{4}$$

for any c. If we assume $c \leq \min_i \left\{ \dfrac{v_i}{|\alpha_i|} \right\}$, then

$$v_i \pm c\ \alpha_i \geq 0 \quad \forall \quad i = 1, 2, \ldots, r.$$

Thus, we take $c \leq \min_i \left\{ \dfrac{v_i}{|\alpha_i|} \right\}$.

Take the vectors

$$X_1 = (v_1 + c\alpha_1, v_2 + c\alpha_2, \ldots, v_r + c\alpha_r, 0, 0, \ldots, 0)^T$$

and,
$$X_2 = (v_1 - c\alpha_1, v_2 - c\alpha_2, \ldots, v_r - c\alpha_r, 0, 0, \ldots, 0)^T.$$

In view of (3) and (4) we have,

$$AX_1 = b \text{ and } AX_2 = b$$

and also $X_1, X_2 \geq 0$. Thus, $X_1, X_2 \in S_F$, and $X_1 \neq X_2$.

Also

$$X_V = \frac{1}{2} X_1 + \frac{1}{2} X_2$$

i.e., X_V is a C.L.C. of X_1, X_2 where $X_1, X_2 \in S_F$ and $X_1 \neq X_2$. Therefore, X_V is not a vertex, which is a contradiction. Hence, $A_1, A_2, \ldots, A_r$ are linearly independent So X_V is a BFS. Hence, the theorem.

On summarizing the above two theorems, we find that BFS $\Leftrightarrow$ vertex.

Thus, in order to obtain all vertices, we obtain all its BFS which are ${}^n c_m$ in number. Thus, total number of vertices would also be ${}^n c_m$.

We have noticed that values of non-basic variables in a BFS are zero and values of basic variables a BFS are non-negative which means basic variables can be zero or positive. So we define.

Definition:

(a) A BFS in which all basic variables are > 0 is called a NON-DEGENERATE BFS.

(b) A BFS in which at least one basic variable is zero, is called a DEGENERATE BFS.

In a degenerate case a basic variable which is zero can be treated as non-basic and a non-basic can take its place. These two, though same solution, would be regarded as two different BFS. This implies that a vertex can correspond to more than one BFS.

EXERCISE 2.5

1. Without sketching, find the vertices of the set of feasible solutions S_F for the following LPP

$$\begin{aligned} -x_1 + x_2 &\le 2 \\ 2x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$\left(\textbf{Ans:}\ (0, 0), (0, 1), (1, 0), \left(\frac{1}{3}, \frac{4}{3}\right)\right)$

2. Consider the following system

$$\left.\begin{aligned} 3x_1 + x_2 - x_3 + 2x_4 - x_5 + s_1 &= 2 \\ x_1 - x_2 + 2x_3 - 2x_4 - 3x_5 + s_2 &= -2 \end{aligned}\right\} \quad (1)$$

(a) Can x_2 and x_4 be basic variables. Given reason

[Sol. Equating all variables equal to zero except x_2 & x_4 equation (1) becomes.

$$\left.\begin{aligned} x_2 + 2x_4 &= 2 \\ -x_2 - 2x_4 &= 2 \end{aligned}\right\} \quad (2)$$

The coefficient matrix A of eq. (1) is

$$A = \begin{bmatrix} 3 & 1 & -1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 2 & -2 & -3 & 0 & 1 \end{bmatrix}$$

Submatrix of A

$$B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

The columns of B are linearly dependent or det (B) = 0 or B^{-1} does not exist or B is a singular matrix. Thus, the variables x_2 and x_4 are not basic variables.

Also observe both eqns. in (2) are identical. Hence, there exist infinite number of solutions of system (2) given by $x_2 = 2 - 2x_4$ for each real x_4. But these are not basic solutions of (1).

(b) Answer (a) if the right hand side of 2nd equation in (1) is 2.

(**Ans:** No)

(c) Can x_3 and S_2 be basic variables?

[**Ans:** Yes and the basic solution of (1) is $(x_1, x_2, x_3, x_4, x_5, s_1, s_2)$ = (0, 0, –2, 0, 0, 0, 2)]

(d) Can x_2 and s_1 be basic variables?

[**Ans:** Yes and the basic solution of (1) is $(x_1, x_2, x_3, x_4, x_5, s_1, s_2)$ = (0, 2, 0, 0, 0, 0, 0)]

Here (*) implies either $x_1 - 2x_2 \geq \frac{1}{2}$ or $x_1 - 2x_2 \leq -\frac{1}{2}$

We handle this problem in the following manner.

Let M be a very big positive number. By 'big', we mean a finite number but big enough in comparison to any number involved in the problem.

Then

either $\qquad x_1 - 2x_2 \geq \frac{1}{2}$

or, $\qquad -x_1 + 2x_2 \geq \frac{1}{2}$"

can be replaced by

"$My + x_1 - 2x_2 \geq \frac{1}{2}$

and $M(1 - y) - x_1 + 2x_2 \geq \frac{1}{2}$; $y = 0$ or 1"

If y assumes the value '0', then first inequation becomes active and second inequation

$$M + (-x_1 + 2x_2) \geq \frac{1}{2}$$

becomes redundant as it would hold irrespective of the values of x_1, x_2. If $y = 1$, the second inequation becomes active and first becomes redundant. Hence, the problem becomes

Max $Z = 2x_1 + x_2$

Subject to $\qquad x_1 + x_2 \leq 1$

$$My + x_1 - 2x_2 \geq \frac{1}{2}$$

$$M(1 - y) - x_1 + 2x_2 \geq \frac{1}{2}$$

$$x_1, x_2 \geq 0, \; \boxed{y = 0 \text{ or } 1} \; \text{ or } \; \boxed{y \geq 0, y \leq 1 \text{ and } y \text{ an integer}}$$

where, M is a big number.

2.10.3 Simplex Method

As mentioned in the Introduction, in simplex method, we obtain a starting BFS and move to another BFS, so that value of objective function improves; continue the steps till we get the solution.

But the questions before us are: how to get starting BFS? How to move to another BFS so that value of $f(X)$ improves? When to stop, *i.e.*, how to know, optimum has occurred?

In short, we say, in simplex method keeping feasibility we move towards optimality.

We answer the above questions below and develop the simplex algorithm.

$$[A_1\ A_2\ \ldots\ A_m]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = b$$

or, $\qquad B\,X_B = b$, or $X_B = B^{-1}\,b$.

where, $X_B = (x_1, x_2, \ldots, x_m)^T$ which is practically same as X^* for solution purposes.

Corresponding to this BFS, the value of objective function is

$$f(X_B) = c_1\,x_1 + c_2\,x_2 + \ldots + c_m\,x_m$$
$$= C_B^T\,X_B,$$

Where, $C_B^T = (c_1, c_2, \ldots, c_m)$, the vectors formed by the costs of basic variables.

Since $A_1, A_2, \ldots, A_m$ are linearly independent in E_m, each column A_j of A which is a vector of E_m, can be expressed uniquely as linear combination of $A_1, A_2, \ldots, A_m$. Thus,

$$A_j = \alpha_1^j\,A_1 + \alpha_2^j\,A_2 + \ldots + \alpha_m^j\,A_m,\ j = 1, 2, \ldots, n,$$

where, $\alpha^j = (\alpha_1^j, \alpha_2^j, \ldots, \alpha_m^j)$ is the coordinate vector of A_j relative to the basis $\{A_1, A_2, \ldots, A_m\}$.

Obviously $\alpha^1 = e_1$, $\alpha^2 = e_2$, …, $\alpha^m = e_m$, m-dimensional vectors. But $\alpha^{m+1}, \alpha^{m+2}, \ldots, \alpha^n$ are not e_is. It is clear that, once we know B, *i.e.*, B^{-1}, we can find α^j as

$$A_j = B\,\alpha^j$$

or, $\qquad \alpha^j = B^{-1}\,A_j.$

So far we have assumed that a BFS is known and equipped ourselves with notations and relevant things to take up the question. How to move to another BFS, so that value improves?

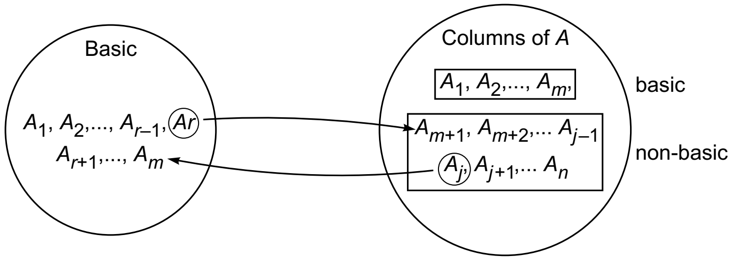
What do we mean by moving to another BFS? It means finding another BFS. We shall proceed step by step. We shall make one non-basic variable as basic variable and consequently one basic as non-basic.

In other words we shall enter one non-basic, in the set of basic variable, so that the solution remains feasible and allow a basic to become non-basic so that solution (value) improves.

Variables can be identified by the columns of A. Therefore, we can also talk in terms of A_j's as follows.

We shall enter one A_j (corresponding to a non-basic variable), $j = m + 1, m + 2, \ldots, n$ in the basis $\{A_1, A_2, \ldots, A_m\}$ so that the solution remains feasible and then take one column out from $A_1, A_2, \ldots, A_m$ so that solution improves.

Let A_j, j is one of the $m + 1, m + 2, \ldots, n$, enters the basis, and the column A_r, r is one of the $1, 2, \ldots, m$ leaves the basis.



$A_j, j = m + 1, m + 2, \ldots, n$ can be expressed as linear combination of $A_1, \ldots, A_m$. Now all columns of non-basic variables would be expressed as a linear combination of new basis, *i.e.*, $A_1, \ldots, A_{r-1}, A_j, A_{r+1}, A_m$. In other words A_r should be expressed in terms of the above new basis. We know that

$$A_j = \alpha_1^j A_1 + \alpha_2^j A_2 + \ldots + \alpha_{r-1}^j A_{r-1} + \alpha_r^j A_r + \alpha_{r+1}^j A_{r+1} + \ldots + \alpha_m^j A_m.$$

A_r can be expressed in terms of new basis $A_1, \ldots, A_{r-1}, A_j, A_{r+1}, \ldots, A_m$ only when $\alpha_r^j \neq 0$. In other words, after entering A_j, only those columns A_r can leave for which r-th coordinate $\alpha_r^j \neq 0$.

Earlier BFS gave

$$x_1 A_1 + x_2 A_2 + \ldots + x_{r-1} A_{r-1} + x_r A_r + x_{r+1} A_{r+1} + \ldots + x_m A_m = b$$

Replacing A_r by (assuming $\alpha_r^j \neq 0$)

$$A_r = -\frac{\alpha_1^j}{\alpha_r^j} A_1 - \frac{\alpha_2^j}{\alpha_r^j} A_2 - \ldots - \frac{\alpha_{r-1}^j}{\alpha_r^j} A_{r-1}$$

$$+ \frac{1}{\alpha_r^j} A_j - \frac{\alpha_{r+1}^j}{\alpha_r^j} A_{r+1} \ldots \frac{\alpha_m^j}{\alpha_r^j} A_m$$

We get

$$\left(x_1 - \frac{\alpha_1^j x_r}{\alpha_r^j}\right) A_1 + \left(x_2 - \frac{\alpha_2^j}{\alpha_r^j} x_r\right) A_2 + \ldots + \left(x_{r-1} - \frac{\alpha_{r-1}^j}{\alpha_r^j} x_r\right) A_{r-1}$$

$$+ \frac{x_r}{\alpha_r^j} A_j + \ldots + \left(x_{r+1} - \frac{\alpha_{r+1}^j x_r}{\alpha_r^j}\right) A_{r+1} + \ldots + \left(x_m - \frac{\alpha_m^j x_r}{\alpha_r^j}\right) A_m = b$$

This gives that

$$\hat{x}_B = \left(\left(x_1 - \frac{\alpha_1^j}{\alpha_r^j} x_r\right), \left(x_2 - \frac{\alpha_2^j x_r}{\alpha_r^j}\right), \ldots, \left(x_{r-1} - \frac{\alpha_{r-1}^j x_r}{\alpha_r^j}\right),\right.$$

$$\left.\frac{x_r}{\alpha_r^j}, \left(x_{r+1} - \frac{\alpha_{r+1}^j x_r}{\alpha_r^j}\right), \ldots \left(x_m - \frac{\alpha_m^j}{\alpha_r^j} x_r\right)\right)$$

is a basic solution. The leaving variable x_r should be so chosen that $\hat{x}_B$ remains a Basic Feasible solution, *i.e.*, $\hat{x}_B \geq 0$ or

$$x_i - \frac{\alpha_i^j}{\alpha_r^j} x_r \geq 0; \; i = 1, 2, \ldots, r-1, r+1, \ldots, m \text{ and } \frac{x_r}{\alpha_r^j} \geq 0.$$

This gives that $\alpha_r^j > 0$ since $x_r \geq 0$. Also, since $x_i' s \geq 0$, if any $\alpha_i^j = 0$, it is okay. So if $\alpha_i^j \neq 0$, we must have

$$\alpha_i^j \left(\frac{x_i}{\alpha_i^j} - \frac{x_r}{\alpha_r^j}\right) \geq 0, \; i = 1, 2, \ldots, r-1, r+1, \ldots, m$$

It is clear, if $\alpha_i^j < 0$, the inequality is satisfied. Only problem are those cases for which $\alpha_i^j > 0$. In this case, we need

$$\frac{x_i}{\alpha_i^j} - \frac{x_r}{\alpha_r^j} \geq 0 \text{ for all those } i \text{ for which } \alpha_i^j > 0.$$

This is possible, if we take

$$\frac{x_r}{\alpha_r^j} = \underset{i}{\text{Min}}\left(\frac{x_i}{\alpha_i^j}, \alpha_i^j > 0\right) = \theta_j \geq 0$$

i.e., we select that variable to leave for which $\frac{x_i}{\alpha_i^j}$, $\alpha_i^j > 0$, $i = 1, 2, \ldots, m$, is minimum. It would guarantee that the new solution would be a BFS.

This tells us that leaving variable affects feasibility. It should be so chosen that the solution remains feasible.

For this, as developed above, after deciding about the entering variable, we look at its coordinate vector. We calculate $\frac{x_i}{\alpha_i^j}$ only for those for which $\alpha_i^j > 0$ (x_i is the value of that basic variable). We pick the one for which $\frac{x_i}{\alpha_i^j}$ is minimum. This decision about leaving variable guarantees feasibility.

Now selection of entering variable. We should enter that variable (*i.e.*, move to that BFS) which improves the value. New BFS is

$$\hat{X}_B = ((x_1 - \alpha_1^j\ \theta_j), (x_2 - \alpha_2^j\ \theta_j), \ldots, (x_{r-1} - \alpha_{r-1}^j\ \theta_j),$$
$$\theta_j, (x_{r+1} - \alpha_{r+1}^j\ \theta_j), \ldots, (x_m - \alpha_m^j\ \theta_j))$$

Earlier the value of the objective function was

$$Z = f(X_B) = C_B^T X_B$$
$$= c_1 x_1 + c_2 x_2 + \ldots + c_{r-1} x_{r-1} + c_r x_r + c_{r+1} x_{r+1} + \ldots + c_m x_m$$

Now the value of the objective function is

$$\hat{Z} = f(\hat{X}_B) =$$
$$c_1 (x_1 - \alpha_1^j\ \theta_j) + c_2 (x_2 - \alpha_2^j\ \theta_j) + \ldots + c_{r-1} (x_{r-1} - \alpha_{r-1}^j\ \theta_j) + \ldots + c_j\ \theta_j + c_{r+1} (x_{r+1} - \alpha_{r+1}^j\ \theta_j) + \ldots + c_m (x_m - \alpha_m^j\ \theta_j)$$

because now the variables are $x_1, x_2, \ldots, x_{r-1}, x_j, x_{r+1}, \ldots, x_m$

Also,

$$\hat{Z} - Z = -\alpha_1^j\ c_1\ \theta_j - \alpha_2^j\ c_2\ \theta_j \ldots \alpha_{r-1}^j\ c_{r-1}\ \theta_j + c_j\ \theta_j - c_r\ x_r - \alpha_{r+1}^j\ c_{r+1}\ \theta_j \ldots c_m\ \alpha_m^j\ \theta_j$$
$$= -\theta_j \left(c_1 \alpha_1^j + c_2 \alpha_2^j + \ldots + c_{r-1} \alpha_{r-1}^j - c_j + c_r \frac{x_r}{\theta_j} + c_{r+1} \alpha_{r+1}^j \ldots + c_m \alpha_m^j\right)$$
$$= -\theta_j (Z_j - c_j),$$

Basic variables	x_{m+1}	x_{m+2} … x_n		x_1 x_2 … … x_m	Sol.
	$Z_{m+1}-c_{m+1}$	$Z_{m+2}-c_{m+2}$ …	Z_n-c_n	Z_1-c_1 Z_2-c_2 … … Z_m-c_m	$f(X)$
x_1 x_2 ⋮ x_m	$\left[\alpha^{m+1}\right]$	$\left[\alpha^{m+2}\right]$	$\left[\alpha^{n}\right]$	$\left[B^{-1}\right]$	$\left[\begin{matrix}B^{-1}b\\=X_B\end{matrix}\right]$

We notice that writing first table would itself be a terse problem. First identify basic variables, then B, then calculate B^{-1}, α^j (= B^{-1} A_j), B^{-1} b, $f(X)$ (= C_B^T B^{-1} b) and also $Z_j - c_j$ for each variables.

The problem of writing starting table would be quite simple, if we choose basic variables in a proper fashion, *i.e.*, the one whose columns in the matrix A form the identity matrix, *i.e.*,

$$B = I$$

∴ $$B^{-1} = I$$

and, $$\alpha^j = B^{-1} A_j = Z A_j = A_j$$

$$X_B = B^{-1}b = Ib = b$$

$$f(X_B) = C_B^T b$$

and, $$Z_j - c_j = C_B^T \alpha^j - c_j \text{ for all variables.}$$

for *basic variable* $$\alpha^j = e_j.$$

∴ $$Z_j - c_j = C_B^T e_j - c_j = c_j - c_j = 0;\ j = 1, 2, \ldots, m$$

for *non-basic variables*

$$Z_j - c_j = C_B^T \alpha^j - c_j$$

In order to simplify it further, we eliminate basic variables from $f(X)$ with the help of constraints, *i.e.*, we make $f(X)$ free of basic variables, so that cost of the basic variables in $f(X)$ are zero, *i.e.*, $C_B^T = \overline{0}$. In this situation

$$Z_j - c_j = -c_j.$$

It will not effect the values of $z_j - c_j$ for basic variables, as $C_B^T = 0$, c_j is also 0, so $Z_j - c_j = 0$.

Also $$f(X_B) = C_B^T X_B = 0$$

In this case table would be as follows

Basic variables	x_{m+1}	x_{m+2} …	x_n	x_1 x_2 … x_m	Solution
Z	$-c_{m+1}$	$-c_{m+2}$ …	$-c_n$	0 0 0	0
x_1 x_2 ⋮ x_m	$\left[A_{m+1}\right]$	$\left[A_{m+2}\right]$ …	$\left[A_n\right]$	$\left[e_1\ e_2\ \ e_m\right]$	$\left[b\right]$

Since, $f(X)$ is taken free of basic variable, we write second row $f(X)$ taking their coefficients and changing their sign and 0 in the column of solution.

From 3rd row to $(m + 3)$th row we write matrix A as it is except that we rearrange the columns if need be so that below basic variable we have B^{-1} *i.e.*, $B = I$ which we have already assumed and b as it is in the last column.

We illustrate the above table writing below:

Example 1: Let the LPP be

$$\text{Min} \quad Z = -6x_1 + 2x_2 - 4x_3$$

$$\text{Subject to} \quad \begin{aligned} 2x_1 - 3x_2 + x_3 &\leq 14 \\ -4x_1 + 5x_2 - 9x_3 &\leq 43 \\ 2x_1 + 2x_2 - 4x_3 &\leq 39 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: This LPP in standard from is

$$\text{Min} \quad Z = -6x_1 + 2x_2 - 4x_3$$

$$\text{Subject to} \quad \begin{aligned} 2x_1 - 3x_2 + x_3 + s_1 &= 14 \\ -4x_1 + 5x_2 - 9x_3 + s_2 &= 43 \\ 2x_1 + 2x_2 - 4x_3 + s_3 &= 39 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	6	–2	4	0	0	0	0
s_1	$\boxed{2}$	–3	1	1	0	0	14
s_2	–4	5	–9	0	1	0	43
s_3	2	2	–4	0	0	1	39

Here all are slack variables. If each equation has slack variable, its corresponding columns will give identity matrix. So we take s_1, s_2, s_3 at the end. Objective function is normally free of slack/surplus variables. So to write 2nd row, we transfer everything on right to left and write it will change the signs of costs. It will give $Z_j - c_j$ below each variable. We write α^j $(= A_j)$ below each variable that amounts same matrix A and below solution we write b.

Thus, we get the starting table. It gives starting BFS as $(x_1, x_2, x_3, s_1, s_2, s_3)$ as (0, 0, 0, 14, 43, 39) as x_1, x_2, x_3 are non-basic variables and s_1, s_2, s_3 are basic variables.

Now to move to other BFS for a better solution, we proceed as follows:

We first decide about entering variable.

Since it is a minimisation problem we pick the variable which has most positive $Z_i - c_i$. It is 'x_1' $Z_1 - c_1$ is 6. So 'x_1' enters and now we decide leaving variable.

$$\underset{i}{\text{Min}}\left(\frac{x_i}{\alpha_i^j}, \alpha_i^j > 0\right) = \text{Min}\left(\frac{14}{2}, \frac{39}{2}\right) = 7$$

which is for 's_1'. Thus, 's_1' leaves.

The entry in 'x_1' column and s_1 row is called pivotal entry. It is enclosed in a square in the table.

Now we have to change the matrix B^{-1}, X_B, α^j, $f(x)$ $z_j - c_j$ in the table. But to do it. Here the basic variables are now x_1, s_2, s_3. So from the problem, B is

$$B = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

In order to find B^{-1}, we proceed as follows

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

and apply row operation to obtain identity matrix on the left. We obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Actually we have converted the column $(2, -4, 2)^T$ into $(1, 0, 0)^T$ by row operations which converted I into B^{-1}.

For new α^j, $(= B^{-1} A_j)$ we have to perform the same row operations on old α^j. For new X_B $(= B^{-1}b)$, we have to perform the same row operations on b, *i.e.*, old X_B. Also for new $Z_j - c_j$, we have to free $f(X)$ from x_1 *i.e.*, to bring $Z_1 - c_1 = 0$. We shall perform the row operations in such a way that entry below x_1 becomes zero, *i.e.*, column of x_1 now becomes (0 : 1, 0, 0). This row operation will bring the new value of $f(X) = C_B^T X_B$ in the column of solution and row of $Z_i - c_i$.

Performing the row operations as mentioned above. We obtain

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	0	7	1	–3	0	0	–42
x_1	1	–3/2	1/2	1/2	0	0	7
s_2	0	–1	–7	2	1	0	71
s_3	0	5	–5	–1	0	1	25

Thus, new BFS is (7, 0, 0, 0, 71, 39) and the value is –42. It is a better solution as the earlier value is '0'. It is important to note that:

$$B^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

appears in the columns of s_1, s_2, s_3 (initial basic variables) though it is the inverse of the matrix of columns of x_1, s_2, s_3. It will hold every time, *i.e.*, new B^{-1} will be in the columns of initial basic variables even if in the new BFS no initial basic variable may appear as a basic variable.

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1 \\ 5 \end{bmatrix}; \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -9 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -7 \\ -5 \end{bmatrix}$$

and,
$$\begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2 \\ -1 \end{bmatrix}$$

are the new columns (which are α^j) for non-basic variables.

Also,
$$\begin{bmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 14 \\ 43 \\ 39 \end{bmatrix} = \begin{bmatrix} 7 \\ 71 \\ 25 \end{bmatrix}$$

is the new X_B.

Also, new
$$C_B^T = (-6, 0, 0)$$

So,
$$f(X) = C_B^T X_B = (-6, 0, 0)\begin{pmatrix} 7 \\ 71 \\ 25 \end{pmatrix} = -42$$

Also, for non-basic variables,

$$Z_2 - c_2 = C_B^T \alpha^2 - c_2$$

$$= [-6, 0, 0]\begin{bmatrix} -3/2 \\ -2 \\ 5 \end{bmatrix} - 2$$

$$= 9 - 2 = 7$$

$$Z_3 - c_3 = [-6, 0, 0]\begin{bmatrix} 1/2 \\ -7 \\ -5 \end{bmatrix} + 4$$

$$= -3 + 4 = 1$$

$$Z_4 - c_4 = [-6, 0, 0]\begin{bmatrix} 1/2 \\ 2 \\ -1 \end{bmatrix} - 0 = -3$$

For basic variables, $Z_i - c_i$ would be zero and columns would be e_i's.

Now, what? Question is, whether the problem is over? If yes, how to know it? If not, how to proceed?

We answer the problem and leave the proof, till this illustrative example is complete. Also, we shall prove that row-operations would change all the columns, X_B, solution and $Z_i - c_i$ to the desired one.

The solution obtained above is certainly a better solution, but not an optimum solution.

In case of a minimisation (maximisation) problem the values of $z_i - c_i$ for all values should be ≤ 0 (≥ 0).

In the above example; the optimum has not yet reached, because $z_i - c_i$ are not all ≤ 0. So we repeat the steps, till optimum is attained *i.e.*, all $z_i - c_i$ are ≤ 0.

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Solution
Z	0	7	1	–3	0	0	–42
x_1	1	–3/2↓	1/2	1/2	0	0	7
s_2	0	–1	–7	2	1	0	71
$\leftarrow s_3$	0	[5]	–5	–1	0	1	25
Z	0	0	$\frac{-16}{5}$	–8/5	0	–7/5	–77
x_1	1	0	–1	4/5	0	3/10	29/2
s_2	0	0	–8	9/5	1	1/5	76
x_2	0	1	–1	–1/5	0	1/5	5

In the last table, we find that all $Z_i - c_i$ are ≤ 0 and since it is a minimisation problem, the optimal has reached and this table, we shall refer to as optimal table. The optimal solution is

$$x_1 = 29/2,\ x_2 = 05,\ x_3 = 0,\ s_1 = 0,\ s_2 = 76,\ s_3 = 0$$

or, $\left(\frac{29}{2}, 5, 0, 0, 76, 0\right)$ and the optimal value is –77.

Now we summarise this method in steps form. This method is known as *simplex method.*

Step 1: Write the LPP in standard form.

Step 2: Check whether the matrix A has the identity matrix as submatrix

If yes, we can start simplex iteration.

If not, we have to search for another method.

In case identity matrix exist,

Rearrange columns so identity matrix appears towards the end and write basic variables in order.

Step 3: Free the objective function from basic variables.

Step 4: Write the starting table. It gives the starting BFS.

Step 5: Decide the entering variable

For maximisation problem – most negative $Z_i - c_i$.

For minimisation problem – most positive $Z_i - c_i$

Step 6: Decide the leaving variable

It is the one for which

$$\zeta = \left(-\frac{\alpha_1^k}{\alpha_r^k}, \frac{-\alpha_2^k}{\alpha_r^k}, \ldots, -\frac{\alpha_{r-1}^k}{\alpha_r^k}, \frac{1}{\alpha_r^k}, \frac{-\alpha_{r+1}^k}{\alpha_r^k}, \ldots, -\frac{\alpha_m^k}{\alpha_r^k}\right)^T$$

and,
$$\hat{B}^{-1} = EB^{-1}$$
$$\hat{X}_B = EX_B$$
$$\hat{Z}_j - C_j = (Z_j - C_j) - \left(\frac{\alpha_r^j}{\alpha_r^k}\right) (Z_k - C_k).$$

Proof:

$$B = (\beta_1, \ldots, \beta_{r-1}, \beta_r, \beta_{r+1}, \ldots, \beta_m)$$
$$\hat{B} = (\beta_1, \ldots, \beta_{r-1}, A_k, \beta_{r+1}, \ldots, \beta_m)$$

The coordinate vector of the column of x_k be

$$(\alpha_1^k, \alpha_2^k, \ldots \alpha_m^k)^T$$

Then, $A_k = B\alpha^k$

$$= \alpha_1^k \beta_1 + \alpha_2^k \beta_2 + \ldots + \alpha_{r-1}^k \beta_{r-1} + \alpha_r^k \beta_r + \alpha_{r+1}^k \beta_{r+1} + \ldots + \alpha_m^k \beta_m$$

Since β_r leaves, $\alpha_r^k \neq 0$, we have

$$\beta_r = \left(-\frac{\alpha_1^k}{\alpha_r^k}\right) \beta_1 + \left(-\frac{\alpha_2^k}{\alpha_r^k}\right) \beta_2 + \ldots + \left(-\frac{\alpha_{r-1}^k}{\alpha_r^k}\right) \beta_{r-1} + \frac{1}{\alpha_r^k} A_k + \left(\frac{-\alpha_{r+1}^k}{\alpha_r^k}\right) \beta_{r+1} + \ldots + \left(-\frac{\alpha_m^k}{\alpha_r^k}\right) \beta_m.$$

Let
$$\zeta = \left(-\frac{\alpha_1^k}{\alpha_r^k}, -\frac{\alpha_2^k}{\alpha_r^k}, \ldots, -\frac{\alpha_{r-1}^k}{\alpha_r^k}, \frac{1}{\alpha_r^k}, -\frac{\alpha_{r+1}^k}{\alpha_r^k}, \ldots, -\frac{\alpha_m^k}{\alpha_r^k}\right)^T$$

Then,
$$\beta_r = [\beta_1, \beta_2 \ldots \beta_{r-1}\ A_k\ \beta_{r+1} \ldots \beta_m]\ \zeta$$
$$= \hat{B}\ \zeta$$

Let
$$E = [e_1\ e_2 \ldots e_{r-1}\ \zeta\ e_{r+1} \ldots e_m]$$

Then,
$$\hat{B}E = [\hat{B}e_1, \hat{B}e_2 \ldots \hat{B}e_{r-1}\ \hat{B}\zeta\ \hat{B}e_{r+1} \ldots \hat{B}e_m]$$
$$= [\beta_1\ \beta_2 \ldots \beta_{r-1}\ \beta_r\ \beta_{r+1} \ldots \beta_m]$$
$$= B$$

Hence,
$$\hat{B}^{-1} = EB^{-1}.$$

Now, let
$$\hat{B} = [\hat{\beta}_1\ \ \hat{\beta}_2\ \ \ldots\ \ \hat{\beta}_m]$$

Where,
$$\hat{\beta}_i = \beta_i,\ i \neq r$$

Starting table

Basic variable	x_1^+	x_2^+	x_1^-	x_2^-	s_1	s_2	s_3	Solution
z	–2	–3	2	3↓	0	0	0	0
s_1	–1	2	1	–2	1	0	0	2
←s_2	2	–1	–2	[1]	0	1	0	2
s_3	–1	–1	1	1	0	0	1	2

Basic	x_1^+	x_2^+	x_1^-↓	x_2^-	s_1	s_2	s_3	Solution
z	–8	0	8	0	0	–3	0	–6
s_1	3	0	–3	0	1	2	0	6
x_2^-	2	–1	–2	1	0	1	0	2
s_3	–3	0	[3]	0	0	–1	1	0
						(a degenerate solution)		
z	0	0	0	0	0	–1/3	–8/3	–6
s_1	0	0	0	0	1	1	1	6
x_2^-	0	–1	0	1	0	1/3	2/3	2
x_1^-	–1	0	1	0	0	–1/3	1/3	0
						(It is an optimal Table)		

Solution: $s_1 = 6, x_2^- = 2, x_1^- = 0, x_1^+ = x_2^+ = s_2 = s_3 = 0$

or, $x_1 = 0, x_2 = -2$

Optimal value = –6

Non-basic variables x_1^+, x_2^+ have $z_i - c_i = 0$. It amounts that it has alternate solution if these are forced to enter the basis, then there is nothing to leave, and the solution will become non-feasible if x_2^+ enters but if we force x_1^+ to enter, we get the following table:

z	0	0	0	0	0	1/3	–8/3	–6
s_1	0	0	0	0	1	1	1	6
x_2^-	0	–1	0	1	0	1/3	2/3	2
x_1^+	1	0	–1	0	0	+1/3	–1/3	0

Here it is again an optimal table.

Solutions are $x_1^+ = 0, x_2^- = 2, s_1 = 6, x_1^- = x_2^+ = s_2 = s_3 = 0$

or $x_1 = 0, x_2 = -2$ and optimal value is –6. Final BFS in terms of x_1, x_2 is same but in terms of $x_1^+, \ldots$ etc., we have

$$(0, 0, 0, 2, 6, 0, 0) \text{ and } (0, 0, 0, 2, 6, 0, 0)$$

Though they are same but in one case x_1^+ is zero as basic variable and $x_1^- = 0$ as non-basic variable while in other case $x_1^+ = 0$ as non-basic and $x_1^- = 0$ as basic.

3. Use simplex method to solve the following LPP

   Maximize $Z = 3x_1 + 5x_2$

   Subject to
$$x_1 + 2x_2 \leq 2000$$
$$x_1 + x_2 \leq 1500$$
$$x_2 \leq 600$$
$$x_1, x_2 \geq 0.$$

(**Ans:** $x_1 = 1000$, $x_2 = 500$, $Z = 5500$)

4. Consider the following LPP

   Minimize $Z = x_1 - 3x_2 + 2x_3$

   Subject to
$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

   one of the simplex iteration table is

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	0	0		$\frac{-1}{5}$	$\frac{-4}{5}$		
x_1	0					$\frac{1}{10}$		
x_2	0					$\frac{3}{10}$		
s_3	0					$-\frac{1}{2}$		

   (Where s_1, s_2, s_3 are slack variables). Without performing simplex iterations, find the following missing entries in the above table.

   (a) x_2 – column
   (b) s_3 – column
   (c) Entry below x_3 in z-row
   (d) The BFS corresponding to above table
   (e) The value of objective function corresponding to above BFS obtained in (d).

$$\left[\textbf{Ans:}\ \text{(a)}; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{(b)}; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{(c)}; 0, \text{(d)}; \begin{pmatrix} x_1 \\ x_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 11 \end{pmatrix} Z = -11\right]$$

$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

What type of solutions we get?

(Degenerate soln. $x_1 = 0$, $x_2 = 2$, $Z = 18$)

11. Maximize $Z = 2x_1 + 4x_2$

Subject to
$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Does this problem has alternate solution, if yes find all the alternate solutions?

(Yes, (a) $x_1 = 0$, $x_2 = 5/2$, $Z = 10$

(b) $x_1 = 3$, $x_2 = 1$, $Z = 10$

$\alpha_1^* = \lambda\,(0) + (1 - \lambda)\,3 = 3 - 3\lambda$

$x_2^* = \lambda\,(3) + (1 - \lambda)\,1 = 2\,\lambda + 1,\ 0 \leq \lambda \leq 1$)

2.12 ARTIFICIAL VARIABLE METHOD

2.12.1 Introduction

In simplex method, we have seen that the presence of an identity matrix is required. So far we have considered only those cases in which identity matrix is present. But it is not always possible. What is to be done in such cases?

One method is to find any non-singular matrix B and proceed as described in the earlier section. But it would be quite cumbersome. So we adopt another method, known as Artificial Variable Method.

2.12.2 BIG M-Method

As we know, normally, the identity matrix is present because of slack variables. If, instead of slack variables, surplus variables are present in some constraints, then identity matrix will not be present.

Therefore, we insert artificial variable R_i with plus sign in each of the constraint having surplus variable. 'i' is the number of constraint in which it was added.

Also, in case of maximisation problem, we add ($-MR_i$) in the objective function corresponding to each artificial variable introduced in the constraints. M is a big positive number. Bigness is defined as biggest so that all numbers involved in steps are smaller than M. And in case of minimisation problem we add ($+MR_i$).

This is done in order to achieve the following:

It is quite evident that since it is an artificial variable, its value must come out to be zero. As, if it is not so, then it means that we are changing the constraints, *i.e.*, the problem. In maximisation problem, since MR_i present in the objective function, with M a big positive number, it would be

Basic	x_1	x_2	s_1	R_1	s_2	s_3	Solution
Z	1	-3	0	M	0	0	0
R_1	1	2	-1	1	0	0	2
s_2	3	1	0	0	1	0	3
s_3	1	0	0	0	0	1	4
z	$1 - M$	$-3 - 2M$	M	0	0	0	$-2M$
R_1	1	[2]	-1	1	0	0	2
s_2	3	1	0	0	1	0	3
s_3	1	0	0	0	0	1	4
z	5/2	0	$-3/2$	3/2 + M	0	0	3
x_2	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1
s_2	$\frac{5}{2}$	0	[$\frac{1}{2}$]	$-\frac{1}{2}$	1	0	2
s_3	1	0	0	0	0	1	4
z	10	0	0	M	3	0	9
x_2	3	1	0	0	1	0	3
s_1	5	0	1	-1	2	0	4
s_3	1	0	0	0	0	1	4

∴ Solution is $x_1 = 0$, $x_2 = 3$, $f(x) = 9$

There is another method of solving such problems involving artificial variables. In this method we solve the problem in two steps. Therefore, this method is known as two-phase methods.

2.12.3 Two-Phase Method

In this method, we introduce artificial variables as above. But the objective function is taken different in phase I. It is done to avoid 'M' which is big enough. Since bigness of M cannot be ascertained, the big M-method is not workable on computers.

In phase I, we solve the following LPP:

$$\text{Min } R = \sum R_i$$

Subject to constraints and non-negativity conditions.

Since $R_i \geq 0$, it is a must that each R_i comes out to be zero as a solution of the above Phase I problem. If it is not so, then it amounts that we have to go out of S_F. Hence, the problem would have no solution. The construction of phase I is to guarantee the existence of a BFS (vertex) of S_F.

After getting the solution from Phase I, we superimpose it on the given objective function of the LPP and apply simplex iteration to get the optimal solution.

Example 3:

Min $Z = f(X) = 2x_1 + x_2$

Subject to
$$3x_1 + x_2 \geq 9$$
$$x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Solution:

Phase-I

Min $R = R_1$

$$3x_1 + x_2 - s_1 + R_1 = 9$$
$$x_1 + x_2 + s_2 = 1$$
$$x_1, x_2, s_1, s_2, R_1 \geq 0.$$

Basic	x_1	x_2	s_1	R_1	s_2	Solution
R	0	0	0	–1	0	0
R_1	3	1	–1	1	0	9
s_2	1	1	0	0	1	1
R	3	1	–1	0	0	9
R_1	3	1	–1	1	0	9
s_2	[1]	1	0	0	1	1
R	0	–02	–01	0	–03	6
R_1	0	–2	–1	1	–3	6
x_1	1	1	0	0	1	1

It is optimal table, but $R_1 = 6 \neq 0$.

Hence, LPP has no solution. Actually there is no feasible solution.

Example 4:

Max $Z = -x_1 + 3x_2$

Subject to
$$x_1 + 2x_2 \geq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1 \leq 4;\ x_1, x_2, x_3 \geq 0$$

Phase I

Min $Z = R_1$

Subject to
$$x_1 + 2x_2 - s_1 + R_1 = 2$$
$$3x_1 + x_2 + s_2 = 3$$
$$x_1 + s_3 = 4$$
$$x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$$

3. If original problem has a solution, big M-method or Phase I method would yield a solution with all R_i's equal to zero.
4. In Phase I, it is always a minimisation problem.
5. The redundancy in the system is possible only if artificial variables are present, as we are assuming that A has the identity matrix as submatrix. For, suppose at any stage R_K appears as ith basic variable then, in the column of R_K, except the ith entry, other entries are zero. If $R_K = 0$, and $\alpha_i^k = 0$ for all non-basic variables, except for $\alpha_i^i = 1$. Hence, all columns are expressible as a linear combination of the remaining basic variables. Thus, the rank of $A = m - 1$ which shows that one row is redundant.

We have already discussed the exceptional cases. The situation of non-existence of feasible solution occurs in case of artificial variable techniques. Here, whenever all R_i's are not zero, it amounts that the problem has no feasible solution.

EXERCISE 2.7

1. Use Big-M method to solve the following LPP.
   Minimize $Z = 3x_1 + 2x_2$
   Subject to
   $$x_1 + x_2 \geq 2$$
   $$x_1 + 3x_2 \leq 3$$
   $$x_1 - x_2 = 1$$
   $$x_1, x_2 \geq 0$$
   (**Ans:** $x_1 = 3/2$, $x_2 = 1/2$, $Z = 11/2$)
2. Solve exercise 1 by two-phase method.
3. Solve the following LPP by two-phase method
   Maximize $Z = 3x_1 - x_2$
   Subject to
   $$2x_1 + x_2 \geq 2$$
   $$x_1 + 3x_2 \leq 3$$
   $$x_2 \leq 4$$
   $$x_1, x_2 \geq 0$$
   (**Ans:** $x_1 = 3$, $x_2 = 0$, $Z = 9$)
4. Use Big-M method to solve the following LPP
   Maximize $Z = 6x_2$
   Subject to
   $$x_1 - x_2 \leq 0$$
   $$2x_1 + 3x_2 \leq -6$$
   x_1, x_2 are unrestricted in sign
   $\left(\textbf{Ans: } x_1 = x_2 = -\dfrac{6}{5}, Z = -\dfrac{36}{5}\right)$
5. Solve exercise 4 by two-phase method.
6. Solve the following problem using Big-M and two-phase methods.
   Maximize $Z = 3x_1 + 2x_2$