2 CHAPTER

Elements of Probability

2.1 INTRODUCTION

In the preceding chapter, we have studied to organize and summarize a data set. However, our interest lies in utilizing the information contained in the sample to infer about the population, from which the sample has been drawn. Theory of probability defined *as the science of decision–making with calculated risks in the face of uncertainty* forms the basis for such statistical inference. Statisticians use probability in two ways. When the population is known, probability is used to describe the likelihood of observing a particular sample outcome. When the population is unknown and only a sample from that is available probability is used in making inference about the population.

In this chapter, we shall learn different ways to calculate probabilities and rules governing them. To start with, in Section 2.2, we consider random experiment and related terminology. In Section 2.3, we discuss the mathematical and statistical concepts of probability and Section 2.4 considers the axiomatic approach. Section 2.5 deals with the addition rule of probability and in Sections 2.6 and 2.7, we consider respectively the concept of conditional probability and multiplication rule of probability. Baye's Rule has been considered in Section 2.7. In the end, a set of review exercises and a problem set is given.

2.2 RANDOM EXPERIMENTS AND RELATED TERMINOLOGY

An *experiment* is the process by which an observation or measurement is obtained. Observation generated may or may not produce a numerical value. For example,

- Tossing a coin and observing the face that falls
- Measuring the minimum room temperature
- Interviewing a student about his or her opinion on co–education.

Next we define random experiment and related terminology.

Trial. Performing an experiment once is called a *trial*. For example, throw of a dice, etc.

If A is the event that both candles are red, then cases favourable to A are R_1R_2 and R_2R_1. Hence,

$$A = \{R_1R_2, \quad R_2R_1\}$$

$$P(A) = \frac{2}{6} = \frac{1}{3}$$

Example 2.3: A five-figure number is formed by the numbers 0, 1, 2, 3, 4, without repetition. Find the probability that number formed is divisible by 4.

Solution: The five digits can be arranged in 5! ways and out of these 4! will begin with zero. Thus, number of five digits formed, that is, exhaustive cases are

$$= 5! - 4! = 96.$$

A number will be divisible by 4 which will have two extreme right digits divisible by 4, that is, the number ending with 04, 12, 20, 24, 32 and 40.

Numbers ending with 04 = 3! = 6

Numbers ending with 12 = 3! – 2! = 4

Numbers ending with 20 = 3! = 6

Numbers ending with 24 = 3! – 2! = 4

Numbers ending with 32 = 3! – 2! = 4

Numbers ending with 40 = 3! = 6.

Thus, total numbers divisible by 4, that is, favourable cases are

$$= 6 + 4 + 6 + 4 + 4 + 6 = 30.$$

Hence, the required probability $= \frac{30}{96} = 0.3125.$

Example 2.4: If 5 of 20 fuses in a box are defective and 5 of them are randomly chosen for inspection, what is the probability that two of the defective fuses will be included?

Solution: Let A be event that out of 5 selected, two will be defective and three will be non-defective.

$$\text{Exhaustive cases} = {}^{20}C_5 = \frac{20!}{15!\,5!} = 15504$$

$$\text{Cases favourable to } A = C_2^5 \times C_3^{15} = \frac{5!}{3!\,2!} \times \frac{15!}{12!\,3!} = 4550$$

Therefore, $$P(A) = \frac{4550}{15504} = 0.293.$$

Example 2.5: A committee of 4 members is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manners

(a) There must be one from each category,

(b) It should have at least one from the purchase department,

(c) The chartered accountant must be in the committee.

Solution: Total number of persons out of which four members are to be selected

$$= 3 + 4 + 2 + 1 = 10.$$

$$\text{Exhaustive number of cases} = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

(a) Favourable number of cases for committee to consist of one member from each category

$$= C_1^3 \times C_1^4 \times C_1^2 \times C_1^1 = 3 \times 4 \times 2 \times 1 = 24.$$

$$\text{Therefore, desired probability} = \frac{24}{210} = 0.114 \text{ (approx.)}.$$

(b) Let A be the event that committee has at least one purchase officer, then A^c is event that committee has no purchase officer.

$$\text{Cases favourable to } A^c = C_4^6 = \frac{6 \times 5}{2 \times 1} = 15$$

Therefore, $$P(A^c) = \frac{15}{210} = 0.071 \text{ (approx)}.$$

Hence, $$P(A) = 1 - P(A^c) = 0.929 \text{ (approx)}.$$

(c) Favourable number of cases to include one chartered accountant out of 4 are

$$1 \times C_3^9 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

$$\text{Hence, the desired probability} = \frac{84}{210} = 0.40.$$

Example 2.6: A class in probability theory consists of 6 boys and 4 girls. An examination is conducted and the students are ranked according to their performance. Assume that no two students obtain the same score, what is the probability that girls receive the top 4 scores?

Solution: Since each ranking corresponds to a particular ordered arrangement of the 10 students, thus total number of different rankings = 10!.

Since there are 4! possible rankings among the girl students and 6! possible rankings among the boy students, so the total number of ways in which the four girls can receive the top rankings

$$= 4! \times 6!$$

$$\text{Hence, the desired probability} = \frac{4! \times 6!}{10!} = \frac{4 \times 3 \times 2 \times 1}{10 \times 9 \times 8 \times 7} = \frac{1}{210}.$$

Example 2.7: From a set of n items a random sample of size k is to be selected. What is the probability a given item will be among the k selected?

Solution: Number of exhaustive cases = C_k^n

Number of different selections that contain the given item = $C_1^1 \times C_{k-1}^{n-1} = C_{k-1}^{n-1}$

Hence, the desired probability $= C_{k-1}^{n-1} \Big/ C_k^n = \dfrac{(n-1)!}{(k-1)!\,(n-k)!} \times \dfrac{(n-k)!}{n!}\,k! = \dfrac{k}{n}$.

Example 2.8: If n people are present in a room, then what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$? Ignore the possibility of someone being born on 29th of Feb.

Solution: Since the birthday of any person can fall on any one of the 365 days, thus the exhaustive number of cases for the birthday of n persons is $(365)^n$.

In case the birthday of all the n persons fall on different days, then the number of favourable cases is

$$(365)\,(364)\,\ldots\,(365 - (n-1))$$

Hence, the probability p that birthdays of all the n persons fall on different days, that is, no two persons celebrate their birthday on the same day

$$= \frac{(365)\,(364)\ldots(365-(n-1))}{(365)^n}$$

$$= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\ldots\left(1-\frac{n-1}{365}\right)$$

We can verify that for $n \geq 23$, this probability is less than $\frac{1}{2}$.

In case the outcomes of a random experiment are not equally likely to occur, the probabilities of occurrence for the different outcomes are assigned on the basis of prior knowledge or experimental evidence. For example, if a coin is not unbiased, then the probabilities of heads and tails can be estimated by tossing the coin a large number of times under identical conditions and recording the outcomes. In such a situation *statistical probability* (or *relative frequency*) *concept* as discussed next is applicable.

2.3.2 Probability: Statistical Concept

The probability of an event is the proportion of times the event occurs in a long run of repeated experiments performed under essentially homogeneous and identical conditions.

Symbolically if in n trials an event A happens $f(A)$ times, then $P(A)$ the probability of happening of the event A is given by

$$P(A) = \lim_{n\to\infty} \frac{f(A)}{n}.$$

If in n trials, A did not occur at all, then $f(A) = 0$, and if A always occurred, then $f(A) = n$. Thus, $0 \leq P(A) \leq 1$.

In accordance with this concept, we estimate the probability of an event by observing what fraction of the time similar events have happened in past. For example, if our record shows that in past out of 1250 flights 1200 had the safe landings under the similar weather conditions then the probability of the safe landing for a flight will be 1200/1250 = 24/25 = 0.96.

Limitations of the statistical probability

1. If an experiment is repeated a large number of times, the experimental conditions may not remain identical.
2. The limit $f(A)/n$ may not exist as $n \to \infty$.

2.4 PROBABILITY: AXIOMATIC CONCEPT

In axiomatic concept, probability is defined as a *set function*. Consider a random experiment E. The set of all possible outcomes of the experiment E is called the *sample space* of the experiment and is denoted by S. Any subset A of the sample space S is known as an *event*. That is, an event is a set consisting of some or all possible outcomes of an experiment. If the outcome of the experiment is contained in A we say that A has happened.

For example, let the experiment be the throw of a dice and the event A be face is even.

Then, $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 4, 6\}$.

If face as a result of the throw is 2 or 4 or 6 we say A has happened.

For any two events A and B of a sample space S the event $A \cup B$, called the *union of the two events A and B*, consists of all outcomes that are either in A or in B or in both A and B, refer Fig. 2.1a. Thus, $A \cup B$ would occur if either A or B or both A and B occur.

The event $A \cap B$ or AB, called the *intersection of the two events A and B*, consists of the outcomes that are in both A and B and occurs if both A and B occur, refer Fig. 2.1b.

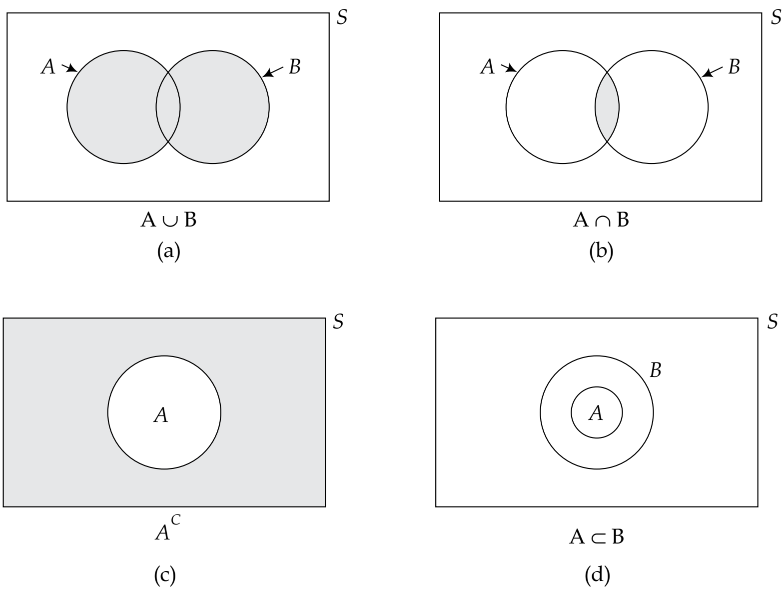

Fig. 2.1

If $A \cap B = \phi$, then the events A and B are said to be *mutually exclusive*, or *mutually disjoint*. The empty set ϕ is said to be the *null event* or *impossible event*. The sample space S defines the *sure event*.

The event A^c, referred to as *complement* of the event A, consists of all the outcomes in the sample space S that are not in A, refer Fig. 2.1c. Thus, A^c occurs if, and only if A does not occur. We note that $S^c = \phi$ and $\phi^c = S$.

For any two events A and B, if all the outcomes of A are also in B, then we say that A is contained in B and we write $A \subset B$. In this case occurrence of A implies the occurrence of B, refer Fig. 2.1d.

We define two events A and B to be *equal* and write $A = B$, if $A \subset B$ and $B \subset A$.

Next, we give the ***axioms of probability***.

Given a finite sample space S and an event $A \in S$ we define $P(A)$, the probability of an event A, as a set function that satisfies the following three axioms:

Axiom 1 (Non-negativity): *$0 \le P(A) \le 1$, for each $A \in S$.*

Axiom 2 (Certainty): *$P(S) = 1$.*

Axiom 3 (Additivity): *If A and B are mutually exclusive events in S, then*

$$P(A \cup B) = P(A) + P(B).$$

The axiomatic probability concept is in consistent with the mathematical and statistical concepts of the probability. On the basis of the axiomatic approach the theory of probability is developed which forms the basis of the statistical inference.

Some results based on the axiomatic concept

1. *Axiom 3* can be extended to include any number of mutually exclusive events $A_1, A_2, ..., A_n$ in S, that is,

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n). \qquad \text{...(2.1)}$$

This can be proved using mathematical induction.

2. *The probability of the impossible event ϕ is zero.*

Since, $S \cup \phi = S$ and $S \cap \phi = \phi$, thus from *Axiom 3*, $P(S) + P(\phi) = P(S)$ which implies $P(\phi) = 0$.

3. *If A is in S, then $P(A^c) = 1 - P(A)$.*

Since, $A \cup A^c = S$ and $A \cap A^c = \phi$, thus from Axiom 3, $P(A) + P(A^c) = P(S)$, which implies $P(A^c) = 1 - P(A)$.

The results **2** and **3** are in conformity with the results already studied in case of classical concept.

4. *If $A \subset B$, then $P(A) \le P(B)$,*

Since $A \subset B$, thus A and $B \cap A^c$ are mutually exclusive events, and $B = A \cup (B \cap A^c)$, refer to Fig. 2.1d. This implies

$$P(B) = P(A) + P(B \cap A^c), \text{ or, } P(B \cap A^c) = P(B) - P(A)$$

Further, $P(B \cap A^c) \ge 0$ implies that $P(A) \le P(B)$.

Example 2.9: In the context of the three arbitrary events A, B, and C write expression for the events

(i) Only A occurs
(ii) Both A and B but not C occur
(iii) At least two occur
(iv) None occur.

Solution: (i) $A \cap \overline{B} \cap \overline{C}$

(ii) $A \cap B \cap \overline{C}$

(iii) $(\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C) \cup (A \cap B \cap \overline{C})$

(iv) $\overline{A} \cap \overline{B} \cap \overline{C}$

Example 2.10: If an experiment has four possible and mutually exclusive outcomes A, B, C and D specify in the following cases whether the assignment of probability is permissible.

(a) $P(A) = 1/3$, $P(B) = 1/6$, $P(C) = 1/4$, $P(D) = 1/4$.
(b) $P(A) = 1/4$, $P(B) = 1/6$, $P(C) = 1/3$, $P(D) = 1/3$.

Solution: (a) All the assignments of probabilities are in the interval [0, 1] and $P(A) + P(B) + P(C) + P(D) = 1$. Hence, the assignments are permissible.
(b) Assignments lie in the interval [0, 1], but $P(A) + P(B) + P(C) + P(D) = 13/12 > 1$. Hence the assignments are not permissible.

Example 2.11: A, B, and C are three mutually exclusive and exhaustive events associated with a random experiment. Find $P(A)$, if $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$.

Solution: Since, A, B and C are mutually exclusive and exhaustive events, thus

$$P(A) + P(B) + P(C) = 1.$$

or, $$P(A) + \frac{3}{2}P(A) + \frac{1}{2}\left(\frac{3}{2}P(A)\right) = 1 \quad \text{or,} \quad \frac{13}{4}P(A) = 1 \text{ or, } P(A) = 4/13.$$

Example 2.12: The proportion of blood phenotypes A, B, AB, and O in the population of a state X are reported as 1.41, 0.10, 0.04 and 0.45 respectively. Find the probability that a person chosen at random from the population will have either type A or type AB blood.

Solution: Since the four sample events A, B, AB, and O are simple events with probabilities (relative frequencies)

$$P(A) = 0.41, \quad P(B) = 0.10, \quad P(AB) = 0.04 \quad \text{and} \quad P(O) = 0.45$$

thus, $$P(A \cup AB) = P(A) + P(AB) = 0.41 + 0.04 = 0.45$$

2.5 ADDITION RULE OF PROBABILITY

The addition rule, also called the *theorem of total probability* is applied to calculate the probability of union of two events, as given below.

Theorem 2.1 (Addition rule): *If A and B are any two events in the sample space S, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \quad ...(2.2)$$

Proof. From Fig. (2.2), we have

$$A \cup B = A \cup (A^c \cap B),$$

where A and $A^c \cap B$ are mutually exclusive.

Therefore, by *Axiom 3*

$$P(A \cup B) = P(A) + P(A^c \cap B) \quad ...(2.3)$$

Also from Fig. 2.2

$$B = (A \cap B) \cup (A^c \cap B),$$

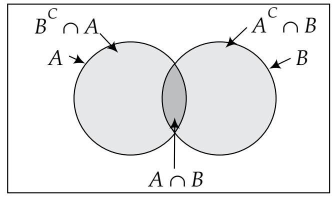

Fig. 2.2

where $A \cap B$ and $A^c \cap B$ are mutually exclusive. Therefore,

$$P(B) = P(A \cap B) + P(A^c \cap B).$$

or,

$$P(A^c \cap B) = P(B) - P(A \cap B).$$

Substituting for $P(A^c \cap B)$ in (2.3), we obtain (2.2), the desired result.

In case the events A and B are mutually exclusive, then $A \cap B = \phi$ and hence (2.2) gives

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom 3.

The result (2.2) can be extended to more than two events, say for three events A, B and C in S, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C), \quad ...(2.4)$$

and in general, for n events $A_1, A_2, ..., A_n$, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i}^{n} \sum_{\substack{j \\ i \neq j}}^{n} P(A_i \cap A_j) + \sum_{i}^{n} \sum_{\substack{j \\ i \neq j \neq k}}^{n} \sum_{k}^{n} P(A_i \cap A_j \cap A_k)$$

$$+ ... + (-1)^{n-1} P(A_1 \cap A_2 \cap ... \cap A_n) \quad ...(2.5)$$

In case all the events A_i are mutually exclusive, then (2.5) gives

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

(2.1), a result already obtained.

In case of n general events, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad \text{(Boole's inequality)} \quad ...(2.6)$$

Remark We note that the probability $1 - P\left(\bigcup_{i=1}^{n} A_i\right)$ is the probability that none of the events A_i's, $i = 1, 2, \ldots, n$ happens.

Example 2.13: A total of 21 per cent male employees of a company smoke cigarettes, 5 per cent smoke cigar and 3 percent smoke both cigar and cigarette. What percentage of males smoke neither cigar nor cigarette?

Solution: Let a male employee of the company is selected at random and A be the event that selected individual smokes cigarette and B be the events that he smokes cigar. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.21 + 0.05 - 0.03 = 0.23.$$

Thus, 0.23 is the probability that employee is a smoker. Hence, 1 – 0.23 = 0.77 is the probability of employee being non-smoker. Thus, 77% employees are non-smokers.

Example 2.14: What is the probability of getting a total of 7 or 11, when a pair of unbiased dice are tossed?

Solution: Let A be the event that 'total is 7' and B be the event that 'total is 11.'

Number of exhaustive cases = 36.

Number of cases favourable to $A = 6$

Number of cases favourable to $B = 2$

Thus, $P(A) = 6/36 = 1/6$ and $P(B) = 2/36 = 1/18$.

Since A and B are mutually exclusive, therefore,

$$P(A \cup B) = P(A) + P(B) = 1/6 + 1/18 = 2/9.$$

Example 2.15: A graduate student applies for a job in two companies X and Y. The probability of being selected in X is 0.6 and being rejected in Y is 0.4. The probability of at least one of his applications being rejected is 0.5. What is the probability of getting job?

Solution: Let A be the event 'getting job in X' and B be the event 'getting job in Y', then $A \cup B$ is the event getting job.

Here $P(A) = 0.6$, $P(\overline{B}) = 0.4$, $P(\overline{A} \cup \overline{B}) = 0.5$,

We have, $P(\overline{A} \cup \overline{B}) = P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cap \overline{B})$.

It gives, $P(\overline{A} \cap \overline{B}) = P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cup \overline{B}) = 0.4 + 0.4 - 0.5 = 0.3$

Thus, $P(A \cup B) = 1 - P(\overline{A} \cap \overline{B}) = 1 - 0.3 = 0.7$.

Example 2.16: A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution: Let A be the event 'getting a king', B be the event 'getting a heart' and, C be the event 'getting a red card'.

Obviously A, B, C are not mutually disjoint events. We are interested in the event $A \cup B \cup C$. We have,

$$P(A) = 4/52 = 1/13, \quad P(B) = 13/52 = 1/4, \quad P(C) = 26/52 = 1/2$$

$$P(A \cap B) = 1/52, \quad P(B \cap C) = P(B) = 1/4, \quad \text{and} \quad P(C \cap A) = 2/52, \quad P(A \cap B \cap C) = P(A \cap B) = 1/52.$$

Thus, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{1}{13} + \frac{1}{4} + \frac{1}{2} - \frac{1}{52} - \frac{1}{4} - \frac{2}{52} + \frac{1}{52} = \frac{1}{13} + \frac{1}{2} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}.$$

Example 2.17: Following table gives the response of 1000 adults about the expense of a college education and relative necessity of some form of financial assistance. The respondents have been classified according to whether they currently had a child in college and whether they think that the loan burden for most college students is too high, reasonable, or too low. Suppose one respondent is chosen at random from this group then

1. What is the probability that respondent has a child in college?
2. What is the probability that respondent has a child in college or thinks that the loan burden is too high?

	Too high (A)	*Reasonable (B)*	*Too little (C)*
Child in college (*D*)	0.35	0.08	0.01
No child in college (*E*)	0.25	0.20	0.11

Solution: 1. Here D is the event that the respondent has a child in the college. Thus,

$$P(D) = 0.35 + 0.08 + 0.01 = 0.44$$

2. The event $A \cup D$ is the event that respondent has a child in the college or thinks that the loan burden is too high. Thus,

$$\begin{aligned} P(A \cup D) &= P(A) + P(D) - P(A \cap D) \\ &= 0.60 + 0.44 - 0.35 \\ &= 0.69 \end{aligned}$$

2.6 CONDITIONAL PROBABILITY

Before going to conditional probability, first we define *compound events* and *independent events*.

When two or more events occur in connection with each other, their simultaneous occurrence is called a compound event.

Events are said to be independent, if the probability of the occurrence of one does not depend on the occurrence or non-occurrence of the others, otherwise, the events are said to be dependent.

For example, let A be the event that first draw from a pack of 52 cards is queen and B be the event that second draw is a king. Then $P(A) = 4/52 = 1/13$ and $P(B) = 3/51$ or $4/51$ depending upon whether the first draw was 'a king' or 'not a king'. Hence, A and B are dependent events.

In case the second card has been drawn after replacing the first, then $P(B) = 1/13$. It does not depend upon whether the first draw was a king or not. Hence, A and B are independent events in this case. However, in both the cases stated above, the event AB is the compound event.

Next, we define the conditional probabiliy.

The probability for the event A to occur, when it is known that the event B has already occurred is called the conditional probability of A given B and is denoted by $P(A \mid B)$.

In this case B serves as a revised (reduced) sample space and the probability is the fraction $P(A \cap B)$ of $P(B)$. Thus,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0 \qquad \ldots(2.7)$$

Similarly, the *conditional probability of B given A* is

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0, \qquad \ldots(2.8)$$

As an example, consider that a pair of unbiased dice is tossed. Then there are 36 possible outcomes given by $S = \{(i, j) : i = 1, 2, ..., 6, j = 1, 2, ..., 6\}$.

Let A be the event that 'sum of the two dice equals 8', then the cases favourable to A are (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) and hence $P(A) = 5/36$.

Let B be the event that 'face of the first dice is 3'. Then A/B is the event that 'sum of the two dice equals 8 when the face of the first dice is 3'.

Exhaustive cases are (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) and favourable case is (3, 5). Hence, the conditional probability of A given B denoted by $P(A \mid B) = 1/6$.

We observe that $P(A \mid B) \neq P(A)$ hence the event A is dependent on B.

If $P(A \mid B) = P(A)$, then event A is independent of the event B.

Example 2.18: If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and high selectivity is 0.18, then what is the probability that a system with high fidelity will also have high selectivity?

Solution: If A is the event that the system has high selectivity and B is the event that it has high fidelity, then A/B will be event that system with high fidelity will also have high selectivity.

We have $P(B) = 0.81$ and $P(A \cap B) = 0.18$

Thus, $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.81} = \frac{2}{9}$$

Example 2.19: A box contain 5 defective, 10 partially defective (that fail after a couple of hours of use) and 25 acceptable transistors. A transistor is chosen at random from the box and put into use. If it does not immediately fails, what is the probability that it is acceptable.

Solution: Since the transistor did not fail immediately, thus it is not defective, and so the desired probability is

$$P(\text{acceptable} \mid \text{not defective}) = \frac{P(\text{acceptable} \mid \text{not defective})}{P(\text{not defective})}$$

$$= \frac{25/40}{35/40} = \frac{5}{7}$$

2.7 MULTIPLICATION RULE OF PROBABILITY

The multiplication rule, also called the *theorem of compound probability* is applied to calculate the probability of intersection of two events as given below.

Example 2.21: A bag contains 4 red and 3 blue balls. Two draws of 2 balls each are made. Find the chance that the first draw gives 2 red balls and the second draw 2 blue balls, if

(a) the balls are returned to the bag after the first draw

(b) the balls are not returned.

Solution: (a) The number of ways in which two balls out of 7 may be drawn = $\overset{7}{C}_2$.

The number of ways in which 2 red balls out of 4 may be drawn = $\overset{4}{C}_2$.

Thus, the probability of drawing two red balls = $C_2^4/C_2^7 = \dfrac{4!}{2!\,2!} \times \dfrac{5!\,2!}{7!} = \dfrac{2}{7}$.

Similarly, the probability of drawing two blue balls at the second draw = 2/7.

Therefore, the desired probability = $\dfrac{2}{7} \times \dfrac{2}{7} = \dfrac{4}{49}$.

(b) As in (a) the probability of drawing two red balls at the first draw = 2/7.

Since the balls are not returned, the probability of drawing two blue balls at the second draw

$$= C_2^3/C_2^5 = 3/10.$$

Therefore, the desired probability in this case = $\dfrac{2}{7} \times \dfrac{3}{10} = \dfrac{3}{35}$.

Example 2.22: Two cards are drawn from a pack of 52 cards. Find the probability that draw includes an ace and a ten.

Solution: Let A be the event that draws are an ace and a ten. Then $A = B \cup C$, where

B : First draws an ace and second draw is a ten

C : First draws a ten and second draw is an ace

Now, $P(B) = \dfrac{4}{52} \times \dfrac{4}{51}$ and $P(C) = \dfrac{4}{52} \times \dfrac{4}{51}$.

Also we observe that B and C are mutually exclusive events, thus applying addition rule

$$P(A) = P(B) + P(C) = \frac{32}{52 \times 51} = \frac{8}{663}.$$

Example 2.23: A system composed of k separate components is said to be a parallel system if it functions when at least one of the k component functions. For such a system, if p_i, independent of others, is the probability that ith component will function $i = 1, 2, ..., k$, then what is the probability that system will function?

Solution: Let A_i be the event that the ith component functions. Then,

$$\begin{aligned} P[\text{System functions}] &= 1 - P[\text{System does not function}] \\ &= 1 - P[\text{No component functions}] \\ &= 1 - P[A_1^c A_2^c \dots A_k^c] \\ &= 1 - \prod_{i=1}^{k} (1 - p_i). \end{aligned}$$

Example 2.24: A system consists of four components as shown in Fig. 2.3. System functions if components A and B both function and at least one of the components C or D functions. If the probabilities of functioning components A, B, C and D, respectively are 0.8, 0.8, 0.6 and 0.6, find the probability that, (a) entire system functions and, (b) the component C does not function given that the system functions. Assume that the components function independently.

Solution: (a) Let A be the event that component A functions, and so on. Then the event that entire system functions is $A \cap B \cap (C \cup D)$.

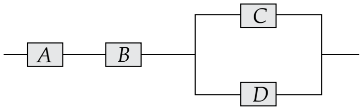

Fig. 2.3

Therefore,

$$P(A \cap B \cap (C \cup D)) = P(A)P(B)P(C \cup D) = P(A)P(B)[1 - P(\overline{C} \cap \overline{D})]$$

$$= P(A)\ P(B)\ [1 - P(\overline{C})\ P(\overline{D})] = (0.8)\ (0.8)\ [1 - (1 - 0.6)\ (1 - 0.6)]$$

$$= (0.64)\ (.84) = 0.5376.$$

(b) $P[C$ does not function | System functions$]$

$$= \frac{P\,[\text{System functions and } C \text{ does not function}]}{P[\text{System functions}]}$$

$$= \frac{P[A \cap B \cap \overline{C} \cap D]}{P[\text{System functions}]} = \frac{P(A)P(B)P(\overline{C})P(D)}{P[\text{System functions}]}$$

$$= \frac{(0.8)(0.8)(0.4)(0.6)}{0.5376} = 0.2857.$$

Example 2.25: The odds that a research monograph will be accepted by 3 independent referees are 3 to 2, 4 to 3, and 2 to 3, respectively. Find the probability that of the three reports,

(a) all will be favourable,

(b) majority of the reports will be favourable,

(c) at least one of the reports will be favourable.

Solution: Let A, B and C be the events that monograph is accepted favourably by the referees I, II and III, respectively. Then

$$P(A) = 3/5, \qquad P(B) = 4/7, \qquad P(C) = 2/5,$$

$$P(\overline{A}) = 2/5, \qquad P(\overline{B}) = 3/7, \qquad P(\overline{C}) = 3/5.$$

(a) $A \cap B \cap C$ is the event all will be favourable.

$$P(A \cap B \cap C) = P(A)P(B)P(C) = 3/5 \times 4/7 \times 2/5 = \frac{24}{175}.$$

(b) The event that majority, that is, at least two will be favourable happens when (i) $A \cap B \cap \overline{C}$, or (ii) $A \cap \overline{B} \cap C$, or (iii) $\overline{A} \cap B \cap C$, or (iv) $A \cap B \cap C$ happens; and all these are mutually exclusive.

Hence, the desired probability = $P(AB\overline{C}) + P(A\overline{B}C) + P(\overline{A}BC) + P(ABC)$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} + \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5}$$

$$= \frac{36}{175} + \frac{18}{175} + \frac{16}{175} + \frac{24}{175} = \frac{94}{175}.$$

(c) $P(A \cup B \cup C) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$

$$= 1 - \frac{2}{5} \times \frac{3}{7} \times \frac{3}{5} = 1 - \frac{18}{175} = \frac{157}{175}.$$

Example 2.26: A message consists of a word that is either a 0 or a 1. Because of noise in the channel there is a small probability of error p, that is,

$$P \text{ (Transmitted 1 is received as 0)} = p$$
$$P \text{ (Transmitted 0 is received as 1)} = p$$

To pass the message reliably, scheme is to repeat the selected digit three times in succession and at the receiving end majority rule will be used to decode.

(a) Evaluate the probability that a transmitted 1 will be received as a 1 under the second scheme.

(b) If a message consisting of the two words, a 1 followed by 0, is to be transmitted using the three-digit scheme, then find the probability that the total message will be correctly decoded under the majority rule.

Solution: (a) A transmitted 111 under the scheme will be decoded as 1 under majority rule if any of the four mutually exclusive messages 101, 110, 011 or 111 is received. The probabilities for the same are respectively

$$(1- p)\, p\, (1 - p), \quad (1 - p)(1 - p)\, p, \quad p\, (1 - p)(1 - p), \quad \text{and} \quad (1 - p)(1 - p)(1 -p)$$

Thus, P (transmitted 1 received as 1)

$$= (1 - p)^3 + 3\, p\, (1 - p)^2$$

(b) We observe that the probability that the transmitted 0 is received 0 is same as obtained in (a) above. Thus, using multiplication rule for two indepent events the probability that the total message consisting of the two words is received correctly under the scheme is $[(1 - p)^3 + 3p\,(1 - p)]^2$.

Example 2.27: Suppose an assembly plant receives its voltage regulators from three different sources, 60% from B_1, 30% from B_2 and 10% from B_3. Let 95%, 80% and 65% of the supply received respectively from the sources B_1, B_2 and B_3 perform as per specifications laid. If A is the event that a voltage regulator received at the plant performs as per specification, then find $P(A)$.

Solution: We can express

$$A = A \cap (B_1 \cup B_2 \cup B_3) = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3).$$

Since B_1, B_2, B_3 are mutually exclusive, therefore, $(A \cap B_1)$, $(A \cap B_2)$ and $(A \cap B_3)$ are also so, and hence

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$
$$= P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)$$
$$= (0.60)(0.95) + (0.30)(0.80) + (0.10)(0.65) = 0.57 + 0.24 + 0.065 = 0.875.$$

Solution: Let A be the event of being repair incomplete and X, Y, Z be events that the repair was made respectively be the technician 'X', 'Y' and 'Z'. Then

$$P(X) = 1/5 \qquad P(Y) = 3/10 \qquad P(Z) = 1/2$$

$$P(A \mid X) = 1/20 \qquad P(A \mid Y) = 1/10 \qquad P(A \mid Z) = 1/15$$

Using Baye's rule

$$P(Z \mid A) = \frac{P(Z)P(A|Z)}{P(X)P(A|X) + P(Y)P(A|Y) + P(Z)P(A|Z)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{15}}{\frac{1}{5}\times\frac{1}{20}+\frac{3}{10}\times\frac{1}{10}+\frac{1}{2}\times\frac{1}{15}} = \frac{1/30}{22/300} = \frac{5}{11}$$

Example 2.29: The probabilities of X, Y and Z becoming managers of a company are 4/9, 2/9 and 1/3, respectively. The probabilities that the Bonus Scheme will be introduced if X, Y and Z becomes managers are 3/10, 1/2 and 4/5, respectively.

(a) Find the probability that Bonus Scheme will be introduced.

(b) If the Bonus Scheme has been introduced, find the probability that manager appointed was X or Y.

Solution: Let B_1, B_2, B_3 be the events that respectively X, Y and Z become manager, and A the event that Bonus Scheme is introduced. We have

$$P(B_1) = 4/9, \qquad P(B_2) = 2/9, \qquad P(B_3) = 1/3,$$

$$P(A \mid B_1) = 3/10, \qquad P(A \mid B_2) = 1/2, \qquad P(A \mid B_3) = 4/5$$

(a) We have, $A = \bigcup_{i=1}^{3} (A \cap B_i)$, where $A \cap B_i$, $i = 1, 2, 3$ are mutually exclusive, thus

$$P(A) = \sum_{i=1}^{3} P(B_i)\, P(A \mid B_i) = \frac{4}{9}\times\frac{3}{10}+\frac{2}{9}\times\frac{1}{2}+\frac{1}{3}\times\frac{4}{5} = \frac{2}{15}+\frac{1}{9}+\frac{4}{15} = \frac{23}{45}.$$

(b) $B_1 \cup B_2$ is the event that manager appointed was X or Y, also $B_1 \cap B_2 = \phi$. Thus,

$$P(B_1 \cup B_2 \mid A) = P(B_1 \mid A) + P(B_2 \mid A)$$

$$= \frac{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}{P(A)} = \frac{\frac{2}{15}+\frac{1}{9}}{\frac{23}{45}} = \frac{22}{90}\times\frac{45}{23} = \frac{11}{23}.$$

Example 2.30: A disease is present in 10% of the population and a diagnostic test designed to detect this disease does not always detect correctly. The table below shows the proportion of times that the test produces various results:

Disease	*Positive (P)*	*Negative (N)*
Present (D)	0.08	0.02
Absent ($\overline{D}$)	0.05	0.85

Using Bayes' rule find the proportion of times the disease is present when test is negative.

Solution: We desire to find $P(D/N)$. Using Bayes' rule, we have

$$P(D/N) = \frac{P(N/D)P(D)}{P(N/D)P(D) + P(N/\overline{D})P(\overline{D})} \quad \ldots(2.14)$$

From the table given

$$P(D) = 0.08 + 0.02 = 0.10,$$

$$P(\overline{D}) = 0.05 + 0.85 = 0.90,$$

$$P(N/D) = \frac{P(ND)}{P(D)} = \frac{0.02}{0.10} = 0.20,$$

and $$P(N/\overline{D}) = \frac{P(N\overline{D})}{P(\overline{D})} = \frac{0.85}{0.90} = 0.94$$

Substituting these values in (2.14), we obtain

Thus, $$P(D/N) = \frac{0.02}{0.87} = 0.023$$

REVIEW EXERCISES

1. Define random experiment, sample space, event, simple event, mutually exclusive events and give examples of each.
2. Give the mathematical and statistical definitions of probability indicating their limitations.
3. Define (i) union and intersection of two events (ii) disjoint events (iii) independent and dependent event, and (iv) compound events.
4. Can events be (i) mutually exclusive and exhaustive (ii) exhaustive and independent (iii) mutually exclusive and independent? Justify your answer in each case by giving an example.
5. State the axioms of probability and prove the various properties satisfied by the probability set function.
6. State and prove the addition rule of probability. Extend this to more than two events.
7. Explain the concept of conditional probability.
8. State and prove the multiplication rule of probability.
9. For the n events $A_1, A_2, ..., A_n$ prove the inequality