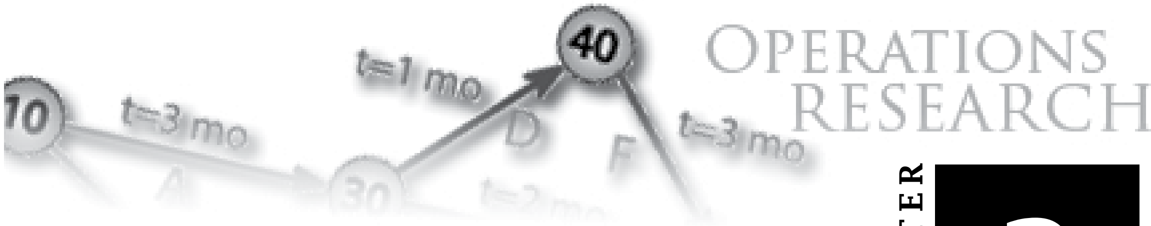

CHAPTER 2

Techniques of Operations Research

2.1 INTRODUCTION

The term, Operations Research was first coined in 1940 by McClosky and Trefthen in a small town called Bowdsey of the United Kingdom. This new science came into existence in a military context. During World War II, military management called on scientists from various disciplines and organised them into teams to assist in solving strategic and tactical problems, relating to air and land defense of the country. Their mission was to formulate specific proposals and plans for aiding the Military Commands to arrive at decisions on optimal utilisation of scarce military resources and efforts and also to implement the decisions effectively. This new approach to the systematic and scientific study is operational research. Hence, OR can be associated with "an art of winning the war without actually fighting it".

2.2 SCOPE OF OPERATIONS RESEARCH

There is a great scope for economists, statisticians, administrators and the technicians working as a team to solve problems of defense by using the OR approach. Besides this, OR is useful in the various other important fields like:

1. Agriculture
2. Finance
3. Industry
4. Marketing and Personnel management
5. Production Management
6. Research and Development etc.

2.3 VARIOUS TECHNIQUES OF OPERATIONS RESEARCH

1. Linear Programming Problem
   - Simplex Algorithm
   - Two-Phase Simplex Method
   - Duality Theory

2.4.2 Applications

- Optimising refinery operations.
- Supply distribution marketing (SDM) system.
- Diet problem.
- Work scheduling problem.
- Capital budgeting problem.
- Short-term financial planning.
- Blending problems.
- Productions process models.
- Inventory model, etc.

2.4.3 Simplex Method

Simplex method is an iterative procedure for solving Linear Programming Problem (LPP) in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

Definition

Let X_b be a basic feasible solution to the LPP,

$$\begin{array}{ll} \text{Max} & Z = C_x \\ \text{Subject to} & A_x = b \\ \text{And} & X \geq 0, \text{ such that it satisfies } X_b = B^{-1}b \end{array}$$

where B is the basic matrix formed by the column of basic variables. The vector $C_b = (C_{b1}, C_{b2} ... C_{bm})$ where C_{ij} are components of C associated with the basic variables is called the cost vector associated with the basic feasible solution X_b.

In the simplex method the following procedure is to be studied to solve the problem and this will be explained in detail in the next chapters:

1. Converting linear programming to standard form
2. Finding basic and non-basic variables
3. Determining the entering and leaving variable
4. Feasible solutions
5. Unbounded solutions and others

The main feature of the simplex method is that it solves the LP in iterations. Each iteration moves the solution to a new corner point that has the potential to improve the value of the objective function. The process ends when no further improvements can be realised. The simplex method involves tedious and voluminous computations, which makes the computer an essential tool for solving LP problems. The computational rules of the simplex method are thus designed to facilitate automatic computations.

2.4.7 Duality Theory

From both the theoretical and practical points of view, the theory of duality is one of the most important and interesting concepts in linear programming. The basic idea behind the duality theory is that every linear programming problem has an associated linear program called its dual such that a solution to the original linear program also gives a solution to its dual. Whenever a linear program is solved by the simplex method, we are actually getting solutions for two linear programming problems. Knowing the relation between a LP and its dual is vital to understand advanced topics in linear and nonlinear programming. This relation is important because it gives us interesting economic insights. Knowledge of duality will also provide additional insights into sensitivity analysis.

The linear programming problem can be viewed as a resource allocation model in which the objective is to maximise revenue or profit subject to limited resources. Looking at the problem from this standpoint, the associated dual problem offers interesting economic interpretations of the LP resource allocation model.

2.4.8 Important Results in Duality

1. The dual of the dual is primal.
2. If one is a maximisation problem then the other is a minimisation one.
3. The necessary and sufficient condition for any LPP and its dual to have an optimal solution is that both must have feasible solution.
4. Fundamental duality theorem states that if either the primal or dual problem has a finite optimal solution, then the other problem also has a finite optimal solution and also the optimal values of the objective function in both the problems are the same i.e., Maximum $Z =$ Minimum Z'. The solution of the other problem can be read from the $Z - C$ row below the columns of slack surplus variables.
5. *Existence theorem* states that, if either problem has an unbounded solution then the other problem has no feasible solution.
6. *Complementary slackness theorem* states that

   i) if a primal variable is positive, then the corresponding dual constraint is an equation at the optimum and vice versa.

   ii) if a primal constraint is a strict inequality then the corresponding dual variable is zero at the optimum and vice versa.

2.4.9 Dual Simplex Method

The dual simplex method is very similar to the regular simplex method, the only difference lies in the criterion used for selecting a variable to enter the basis and to leave the basis. In dual simplex method, we first select the variable to leave the basis and then the variable to enter the basis. This method yields an optimal solution to the given LPP in a finite number of steps provided, no basis is repeated.

The dual simplex method is used to solve problems which start dual feasible (i.e., whose primal is optimal but infeasible). In this method the solution starts optimum, but infeasible and remains infeasible until the true optimum is reached, at which the solution becomes feasible. The advantage of this method is avoiding the artificial variables introduced in the constraints along with the surplus variables as all '$\geq$' constraints are converted into '$\leq$' type.

2.5 TRANSPORTATION

The transportation model is a special case of linear programming that deals with shipping a commodity from sources to destinations (factories to warehouses). The objective of transportation model is to determine the shipping schedule that minimises the total shipping cost while satisfying supply and demand limitations. The model assumes that the shipping cost is proportional to the number of units shipped on a given route. The transportation can be extended to the areas of operation, inventory control, employment scheduling and personnel assignment. The transportation model can be solved as a regular linear programming; its special structure allows the development of a simplex based computational algorithm that makes use of the primal-dual relationships to simplify the computations. The transportation problems actually are special cases of the minimum cost flow problem and can be used for network representation of the transportation. The transportation problem involves 'm' sources each of which are available a_i $(i = 1, 2, \ldots m)$ units of a homogeneous product and 'n' destinations each of which requires b_j $(j = 1, 2, \ldots n)$ units of this product. The numbers 'a' and 'b', are positive integers. The cost of C_i of transporting one unit of product from the ith source to the jth destination is given for each i and j. The objective is to develop an integral transportation schedule that meets all demands from current inventory at a minimum total shipping cost; it is assumed that total supply and total demand are equal.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The transportation algorithm is the simplex method specialised to the format and it involves:

- Finding an initial basic feasible solution.
- Testing the solution for optimality.
- Improving the solution when it is not optimal.

The transportation model can be solved by software like TORA, Excel Solver Solution, LINGO and LINDO and others.

The goal is to allocate the supply available for each origin so as to optimise a criterion while satisfying the demand at each destination. The usual objective function is to minimise the total transportation cost or total weighted distance or to maximise the total profit contribution from the allocation.

2.5.1 Applications

- To minimise the sum of shortage and shipping costs.
- To minimise purchase and shipping cost.
- To minimise the daily cost of processing checks.
- To maximise the revenue.

2.6 ASSIGNMENT MODEL

The best person for the job is a description of the assignment model. The situation can be illustrated by the assignment of workers with varying degrees of skills to jobs. A job that happens to match a worker's skill costs less than that in which the operator is not as skillful. The objective of the model is to determine the least cost (optimum) assignment of workers job.

The assignment model is actually a special of the transportation model in which the workers represented the sources and its jobs represent the destinations. The supply amount at each destination exactly equal one. The cost of transporting worker 'i' to the job 'j' is C_{ij}. The assignment model can be solved directly as a regular transportation model. The fact that all supply and demand amount equal one has led to the development of a simple solution algorithm called the Hungarian method. The new method appears totally unrelated to the transportation model. The algorithm is actually rooted in the simplex method just as the transportation model is. A more efficient way of solving the assignment problem has been developed based on a mathematical property due to the Hungarian method. The basic principle of this method is that the optimal assignment is not affected if a constraint is added or subtracted from any row or column of the standard assignment of cost matrix.

A naïve approach to solve this problem is to enumerate all possible assignments of jobs to machines. For each assignment the total cost may be computed, and the one with the least cost is picked as the best assignment. This will be an inefficient and expensive approach since the number of possible assignments is n! even for n=5, there are 120 possible assignments! It can be thought of formulating to linear programming problem.
The assignment problem can be solved by the Hungarian Method.

Step 1: In the original cost matrix, identify each row's minimum and subtract it from all the entries of the row.

Step 2: For the matrix resulting from step 1, identify each column's minimum and subtract it from all the entries of the column.

Step 3: Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

The Assignment model also, can be solved by softwares like TORA, Excel Solver Solution, LINGO and LINDO and others.

2.6.1 Applications

- To minimise the total time required to complete the jobs.
- To minimise cost assignment of persons to jobs.
- To minimise time for arrival and departure of flights.

The application of assignment technique is assigning people to tasks, airline departure and arrival of the flights and other areas.

2.7 QUEUING THEORY

Queues are a part of everyday's life. We all wait in queues to buy a movie ticket, to make a bank deposit, pay for groceries, mail a package, obtain a food in a cafeteria, to have ride in an amusement park and have become adjusted to wait but still get annoyed by unusually long waits. Queueing theory is the study of waiting in all these various activities. It uses queueing models to represent the various types of queueing systems that arise in practice. Formulas for each model indicate how the corresponding queueing system proceed, perform, including the average amount of waiting that will occur under a variety of circumstances. The queueing models are very helpful for determining how to operate a queueing system in the most effective way if too much service capacity to operate the system involves excessive costs. The models enable finding an appropriate balance between the cost of service and the amount of waiting. The flows of materials through manufacturing operations through a sequence of processing stages represent other forms of queues. In this chapter we learn how to construct and solve equations that describe queueing behaviour for a wide variety of situations. We cannot exhaust even the elementary models, because the number of possible variations is enormous. In the terminology of queueing theory, the first proposal involves changing the parameter values of the models but does not involve structural changes.

The queueing models are very helpful for determining how to operate a queueing system (as shown in Fig. 2.1) in the most effective way if too much service capacity to operate the system involves excessive costs. The models enable finding an appropriate balance between the cost of service and the amount of waiting.

The queueing models can also be solved using software like TORA, Excel Solver Solution, LINGO and LINDO and others.

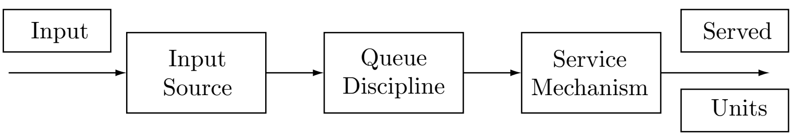

Fig. 2.1 The basic structure of Queueing Model.

The second proposal is for a two-server, single queue system and the third is for a two-server separated queue system.

2.7.1 Applications

- Bank teller
- Bank staffing
- Barber shop
- Tool centre
- Service station.

2.8 GAME THEORY

Life is full of conflicts and competition. The examples of conflicts include parlour games, military battles, political campaigns, advertising and marketing campaigns by competing business firms etc. Game theory deals with decision situation in which two intelligent opponents have competing objectives. In a game conflict, two opponents known as players will each have a number of alternatives or strategies. Associated with each pair of strategies is a pay-off that one player pays to the other, such games are known as two-person zero sum games because the gain by one player equals the loss of the other so that the sum of their net winning is zero. The basic characteristics of two-person zero sum games consider the game called evens and odds.

In general a two-person game is characterised by

1. The strategies of player 1
2. The strategies of player 2
3. The pay-off table.

A primary objective of game theory is the development of rational criteria for selecting a strategy. The following two key assumptions are made:

1. Both players are rational
2. Both players choose their strategies solely to promote their own welfare.

Game theory contrasts with decision analysis where the assumption is that the decision maker is playing a game with a passive opponent which chooses its strategies in some random fashion.

2.8.1 Applications

- Constant sum TV game
- Odds and evens
- Coin toss game with bluffing
- Prisoner's dilemma
- Advertising prisoner's dilemma game
- Chicken game.

2.9 NETWORK METHODS

Management of big projects with a large number of activities pose complex problems in planning, scheduling and controlling, especially when the activities have to be performed in a specialised technological sequence. Fortunately, two operation research techniques, PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) are available to assist the project manager in carrying out these responsibilities. These techniques make heavy use of networks to help plan and display the coordination of all the activities. They also normally use a software package to deal with all the data needed to develop schedule information and then to monitor the progress of the project.

Project management software, namely, MS Project is available for these purposes. Some recent surveys report that as much as 70% of the world's mathematical programming problems can be represented by network related models. The applications of networks are:

- Determination of the shortest route between two cities in a network of roads.
- Determination of the time schedule for the activities of a construction project.
- Determination of the time schedule from oil fields to refineries through a pipeline network.

The solutions of these situations are accomplished through a variety of network optimization algorithms. The following are the algorithms:

- Minimal spanning tree
- Shortest route algorithm
- Maximum flow algorithm
- Minimum cost network algorithm.

The Network model also, can be solved using software like TORA, Excel Solver Solution, LINGO and LINDO and others.

2.13.1 Importance of Integer Programming Problems (IPP)

In LPP all the decision variables were allowed to take any non-negative real values as it is quite possible and appropriate to have fractional values in many situations. There are several frequently occurring circumstances in business and industry that lead to planning models involving integer valued variables. For example, in production, manufacturing is frequently scheduled in terms of batches, lots or runs. In allocation of goods, a shipment must involve a discrete number of trucks or aircraft. In such cases the fractional values of variables like 1/3 may be meaningless in the context of the actual decision problem. This is the main reason why integer programming is so important for managerial decisions.

2.13.2 Applications of Integer Programming

Integer programming is applied both in business and industry. All assignment and transportation problems are integer programming problems, as in assignment and travelling salesmen problems, all the decision variables are either zero or one, i.e.,

$$x_{ij} = 0 \text{ or } 1$$

Other examples are capital budgeting and production scheduling problems. In fact any situation involving decisions of the type "either to do a job or not to do" can be viewed as an IPP in all such situations

$$\begin{aligned} x_{ij} &= 1 \text{ if the } j^{\text{th}}\text{activity is performed} \\ &= 0 \text{ if the } j^{\text{th}}\text{activity is not performed.} \end{aligned}$$

In addition, allocation problems involving the allocation of men and machines give rise to IPP, since such communities can be assigned only in integers and not in fractions.

Note: If the non-integer variable is rounded off, it violates the feasibility and also there is no guarantee that the rounded off solution will be optimal. Due to these difficulties there is a need for developing a systematic and efficient procedure for obtaining the exact optimal integer solution to such problems.

2.14 MARKOV CHAINS

Sometimes we are interested in how a random variable changes over time. For example, we may want to know how the price of a share of a stock or a firm's market share evolves. The study of how a random variable changes over time includes stochastic processes; in particular, we focus on a type of stochastic process known as a Markov chain. A discrete-time stochastic process is a Markov chain.

2.14.1 Applications of Markov Chains

Markov chains have been applied in areas such as

- Education
- Marketing
- Health
- Services

- Finance
- Accounting and
- Production.

2.15 INVENTORY MODELS

A business or an industry usually maintains a reasonable inventory of goods to ensure smooth operation. Traditionally, inventory is viewed as a necessary evil: too little of it causes costly interruptions; too much results in idle capital. The inventory problem determines the inventory level that balances the two extreme cases.

An important factor in the formulation and solution of an inventory model is that the demand (per unit time) of an item may be deterministic (known with certainty) or probabilistic (described by a probability distribution).

Inventory is essential to provide flexibility in operating a system or organisation. An inventory can be classified into raw material, work-in-process inventory and finished goods inventory. The raw material inventory avoids the costly machines being idle and the work in-process inventory removes dependency between various machines of a product line. The finished goods inventory removes dependency between plants and its customers or market. The main functions of an inventory are: smoothing out irregularities in supply, minimising the production cost and allowing organisations to cope up with perishable materials.

2.15.1 Models of Inventory

There are different models of inventory. The inventory models can be classified into deterministic models and probabilistic models.

The various deterministic models are:

1. Purchasing model: No shortages.
2. Manufacturing model: No shortages.
3. Multi-item deterministic model.
4. Purchasing model with shortages.
5. Manufacturing model with shortages.
6. Stochastic models.
7. Purchase inventory model with price breaks.

The various probabilistic modes are

1. Binomial distribution
2. Poisson distribution
3. Negative exponential distribution
4. Normal distribution
5. Dynamic programming
6. Forecasting models
7. Inventory models
8. Queueing system
9. Simulation models.

- Capital budgeting problem
- Reliability improvement problem
- Stage-coach problem (shortest-path problem)
- Cargo loading problem
- Minimising total tardiness in single machine scheduling problem
- Optimal subdividing problem
- Linear programming problem.

2.17 NONLINEAR PROGRAMMING

In previous discussions, we have studied linear programming problems. In a LPP, our goal was to maximise or minimise a linear function subject to linear constraints. In many interesting maximisation and minimisation problems, the objective function may not be a linear function, or some of the constraints may not be linear constraints. Such an optimisation problem is called a nonlinear programming problem (NLP).

The solution methods of nonlinear programming generally can be classified as either direct or indirect algorithms. Examples of direct methods are the gradient algorithms, where the maximum (minimum) of a problem is sought by following the fastest rate of increase (decrease) of the objective function. In indirect methods, the original problem is replaced by another from which the optimum is determined. Examples of these situations include quadratic programming separable programming, and stochastic programming.