Chapter 2

Vector Spaces

2.1 INTRODUCTION

In various practical and theoretical problems, we come across a set V whose elements may be vectors in two or three dimensions or real-valued functions, which can be added and multiplied by a constant (number) in a natural way, the result being again an element of V. Such concrete situations suggest the concept of a vector space. Vector spaces play an important role in many branches of mathematics and physics. In analysis infinite dimensional vector spaces (in fact, normed vector spaces) are more important than finite dimensional vector spaces while in linear algebra finite dimensional vector spaces are used, because they are simple and linear transformations on them can be represented by matrices. This chapter mainly deals with finite dimensional vector spaces.

2.2 DEFINITION AND EXAMPLES

VECTOR SPACE. *An algebraic structure $(V, F, \oplus, \odot)$ consisting of a non-void set V, a field F, a binary operation $\oplus$ on V and an external mapping $\odot : F \times V \to V$ associating each $a \in F, v \in V$ to a unique element $a \odot v \in V$ is said to be a vector space over field F, if the following axioms are satisfied:*

V-1 $(V, \oplus)$ *is an abelian group.*

V-2 For all $u, v \in V$ and $a, b \in F$, we have

(i) $a \odot (u \oplus v) = a \odot u \oplus a \odot v,$

(ii) $(a+b) \odot u = a \odot u \oplus b \odot u,$

(iii) $(ab) \odot u = a \odot (b \odot u),$

(iv) $1 \odot u = u.$

The elements of V are called vectors and those of F are called scalars. The mapping $\odot$ is called scalar multiplication and the binary operation $\oplus$ is termed vector addition.

If there is no danger of any confusion we shall say V is a vector space over a field F, whenever the algebraic structure $(V, F, \oplus, \odot)$ is a vector space. Thus, whenever we say that V is a vector space over a field F, it would always mean that $(V, \oplus)$ is an abelian group and $\odot : F \times V \to V$ is a mapping such that V-2 (i)–(iv) are satisfied.

V is called a *real vector space* if $F = R$ (field of real numbers), and a *complex vector space* if $F = C$ (field of complex numbers).

REMARK-1 *V is called a left or a right vector space according as the elements of a skew-field F are multiplied on the left or right of vectors in V. But, in case of a field these two concepts coincide.*

REMARK-2 *The symbol '+' has been used to denote the addition in the field F. For the sake of convenience, in future, we shall use the same symbol '+' for vector addition $\oplus$ and addition in the field F. But, the context would always make it clear as to which operation is meant. Similarly, multiplication in the field F and scalar multiplication $\odot$ will be denoted by the same symbol '$\cdot$'.*

REMARK-3 *In this chapter and in future also, we will be dealing with two types of zeros. One will be the zero of the additive abelian group V, which will be known as the vector zero and other will be the zero element of the field F which will be known as scalar zero. We will use the symbol 0_V to denote the zero vector and 0 to denote the zero scalar.*

REMARK-4 *Since $(V, \oplus)$ is an abelian group, therefore, for any $u, v, w \in V$ the following properties will hold:*

$$\left.\begin{array}{ll}\text{(i)} & u \oplus v = u \oplus w \Rightarrow \quad v = w \\ \text{(ii)} & v \oplus u = w \oplus u \Rightarrow \quad v = w\end{array}\right\} \text{(Cancellation laws)}$$

(iii) $u \oplus v = 0_v \Rightarrow \quad u = -v$ and $v = -u$

(iv) $-(-u) = u$

(v) $u \oplus v = u \Rightarrow \quad v = 0_v$

(vi) $-(u \oplus v) = (-u) \oplus (-v)$

(vii) 0_v is unique

(viii) For each $u \in V$, $-u \in V$ is unique.

REMARK-5 *If V is a vector space over a field F, then we will write $V(F)$.*

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Every field is a vector space over its any subfield.*

SOLUTION Let F be a field, and let S be an arbitrary subfield of F. Since F is an additive abelian group, therefore, V-1 holds.

Taking multiplication in F as scalar multiplication, the axioms V-2 (i) to V-2 (iii) are respectively the right distributive law, left distributive law and associative law. Also V-2 (iv) is property of unity in F.

Hence, F is a vector space over S.

Since S is an arbitrary sub field of F, therefore, every field is a vector space over its any subfield.

REMARK. *The converse of the above Example is not true, i.e. a subfield is not necessarily a vector space over its over field. For example, R is not a vector space over C, because multiplication of a real number and a complex number is not necessarily a real number.*

EXAMPLE-2 *R is a vector space over Q, because Q is a subfield of R.*

EXAMPLE-3 *C is a vector space over R, because R is a subfield of C.*

EXAMPLE-4 *Every field is a vector space over itself.*

SOLUTION Since every field is a subfield of itself. Therefore, the result directly follows from Example 1.

REMARK. *In order to know whether a given non-void set V forms a vector space over a field F, we must proceed as follows:*

(i) Define a binary operation on V and call it vector addition.

(ii) Define scalar multiplication on V, which associates each scalar in F and each vector in V to a unique vector in V.

(iii) Define equality of elements (vectors) in V.

(iv) Check whether V-1 and V-2 are satisfied relative to the vector addition and scalar multiplication thus defined.

EXAMPLE-5 *The set $F^{m\times n}$ of all $m \times n$ matrices over a field F is a vector space over F with respect to the addition of matrices as vector addition and multiplication of a matrix by a scalar as scalar multiplication.*

SOLUTION For any $A, B, C \in F^{m\times n}$, we have

(i) $(A+B)+C = A+(B+C)$ (Associative law)

(ii) $A+B = B+A$ (Commutative law)

(iii) $A+O = O+A = A$ (Existence of Identity)

(iv) $A+(-A) = O = (-A)+A$ (Existence of Inverse)

Hence, $F^{m\times n}$ is an abelian group under matrix addition.

If $\lambda \in F$ and $A \in F^{m\times n}$, then $\lambda A \in F^{m\times n}$ and for all $A, B \in F^{m\times n}$ and for all $\lambda, \mu \in F$, we have

(i) $\lambda(A+B) = \lambda A + \lambda B$

(ii) $(\lambda+\mu)A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = (\lambda\mu)A$

(iv) $1A = A$

Hence, $F^{m\times n}$ is a vector space over F.

EXAMPLE-6 *The set $R^{m\times n}$ of all $m\times n$ matrices over the field R of real numbers is a vector space over the field R of real numbers with respect to the addition of matrices as vector addition and multiplication of a matrix by a scalar as scalar multiplication.*

SOLUTION Proceed parallel to Example 5.

REMARK. *The set $Q^{m\times n}$ of all $m\times n$ matrices over the field Q of rational numbers is not a vector space over the field R of real numbers, because multiplication of a matrix in $Q^{m\times n}$ and a real number need not be in $Q^{m\times n}$. For example, if $\sqrt{2}\in R$ and $A\in Q^{m\times n}$, then $\sqrt{2}A\notin Q^{m\times n}$ because the elements of matrix $\sqrt{2}A$ are not rational numbers.*

EXAMPLE-7 *The set of all ordered n-tuples of the elements of any field F is a vector space over F.*

SOLUTION Recall that if $a_1,\ a_2,\ \dots,\ a_n\in F$, then $(a_1,a_2,\ \dots,\ a_n)$ is called an ordered n-tuple of elements of F. Let $V=\{(a_1,\ a_2,\dots,a_n): a_i\in F$ for all $i\in \underline{n}\}$ be the set of all ordered n-tuples of elements of F.

Now, to give a vector space structure to V over the field F, we define a binary operation on V, scalar multiplication on V and equality of any two elements of V as follows:

Vector addition (*addition on V*): For any $u=(a_1,\ a_2,\dots,a_n)$, $v=(b_1,\ b_2,\dots,b_n)\in V$, we define

$$u+v=(a_1+b_1,\ a_2+b_2,\dots,a_n+b_n)$$

Since $a_i+b_i\in F$ for all $i\in\underline{n}$. Therefore, $u+v\in V$.

Thus, addition is a binary operation on V.

Scalar multiplication on V: For any $u=(a_1,\ a_2,\dots,a_n)\in V$ and $\lambda\in F$, we define

$$\lambda u=(\lambda a_1,\ \lambda a_2,\dots,\lambda a_n).$$

Since $\lambda a_i\in F$ for all $i\in\underline{n}$. Therefore, $\lambda u\in V$.

Thus, scalar multiplication is defined on V.

Equality of two elements of V: For any $u=(a_1,\ a_2,\dots,a_n), v=(b_1,\ b_2,\dots,b_n)\in V$, we define

$$u=v\Leftrightarrow a_i=b_i\quad \text{for all } i\in\underline{n}.$$

Since we have defined vector addition, scalar multiplication on V and equality of any two elements of V. Therefore, it remains to check whether V-1 and V-2(i) to V-2(iv) are satisfied.

V-1. *V is an abelian group under vector addition.*

Associativity: For any $u = (a_1, a_2, \dots, a_n), \nu = (b_1,\ b_2, \dots, b_n),\ w = (c_1, c_2, \dots, c_n) \in V$, we have

$$\begin{aligned}(u+\nu)+w &= (a_1+b_1, \dots, a_n+b_n)+(c_1,\ c_2, \dots, c_n)\\ &= ((a_1+b_1)+c_1, \dots, (a_n+b_n)+c_n)\\ &= (a_1+(b_1+c_1), \dots, a_n+(b_n+c_n)) \qquad \text{[By associativity of addition on } F\text{]}\\ &= (a_1, \dots, a_n)+(b_1+c_1, \dots, b_n+c_n)\\ &= u+(\nu+w)\end{aligned}$$

So, vector addition is associative on V.

Commutativity: For any $u = (a_1,\ a_2, \dots, a_n),\ \nu = (b_1, \dots, b_n) \in V$, we have

$$\begin{aligned}u+\nu &= (a_1+b_1, \dots, a_n+b_n)\\ &= (b_1+a_1, \dots, b_n+a_n) \qquad \text{[By commutativity of addition on } F\text{]}\\ &= \nu+u\end{aligned}$$

So, vector addition is commutative on V.

Existence of identity (Vector zero): Since $0 \in F$, therefore, $\underline{0} = (0,\ 0, \dots, 0) \in V$ such that

$$u+\underline{0} = (a_1,\ a_2, \dots, a_n)+(0,\ 0, \dots, 0)$$

$$\Rightarrow \quad u+\underline{0} = (a_1+0,\ a_2+0, \dots, a_n+0) = (a_1,\ a_2, \dots, a_n) = u \text{ for all } u \in V$$

Thus, $\underline{0} = (0,\ 0, \dots, 0)$ is the identity element for vector addition on V.

Existence of inverse: Let $u = (a_1,\ a_2, \dots, a_n)$ be an arbitrary element in V. Then, $(-u) = (-a_1,\ -a_2, \dots, -a_n) \in V$ such that

$$u+(-u) = \Big(a_1+(-a_1), \dots, a_n+(-a_n)\Big) = (0,\ 0, \dots, 0) = \underline{0}.$$

Thus, every element in V has its additive inverse in V.
Hence, V is an abelian group under vector addition.

V-2 *For any* $u = (a_1, \dots, a_n),\ \nu = (b_1, \dots, b_n) \in V$ *and* $\lambda,\ \mu \in F$, *we have*

(i)
$$\begin{aligned}\lambda(u+\nu) &= \lambda(a_1+b_1, \dots, a_n+b_n)\\ &= (\lambda(a_1+b_1), \dots, \lambda(a_n+b_n))\\ &= (\lambda a_1+\lambda b_1, \dots, \lambda a_n+\lambda b_n) \qquad \left[\begin{array}{l}\text{By left distributivity of multiplication}\\ \text{over addition in } F\end{array}\right]\\ &= (\lambda a_1, \dots, \lambda a_n)+(\lambda b_1, \dots, \lambda b_n)\\ &= \lambda u+\lambda\nu\end{aligned}$$

(ii) $$\begin{aligned}(\lambda+\mu)u &= \big((\lambda+\mu)a_1,\ldots,(\lambda+\mu)a_n\big)\\ &= (\lambda a_1+\mu a_1,\ldots,\lambda a_n+\mu a_n) && \text{[By right dist. of mult. over add. in } F]\\ &= (\lambda a_1,\ldots,\lambda a_n)+(\mu a_1,\ldots,\mu a_n)\\ &= \lambda u+\mu u.\end{aligned}$$

(iii) $$\begin{aligned}(\lambda\mu)u &= \big((\lambda\mu)a_1,\ldots,(\lambda\mu)a_n\big)\\ &= \big(\lambda(\mu a_1),\ldots,\lambda(\mu a_n)\big) && \text{[By associativity of multiplication in } F]\\ &= \lambda(\mu u)\end{aligned}$$

(iv) $$1u = (1a_1,\ldots,1a_n) = (a_1,\ldots,a_n) = u \qquad [\because\ 1 \text{ is unity in } F]$$

Hence, V is a vector space over F. This vector space is usually denoted by F^n.

REMARK. *If $F=R$, then R^n is generally known as Euclidean space and if $F=C$, then C^n is called the complex Euclidean space.*

EXAMPLE-8 *Show that the set $F[x]$ of all polynomials over a field F is a vector space over F.*

SOLUTION In order to give a vector space structure to $F[x]$, we define vector addition, scalar multiplication, and equality in $F[x]$ as follows:

Vector addition on F[x]: If $f(x)=\sum\limits_i a_i x^i$, $g(x)=\sum\limits_i b_i x^i \in F[x]$, then we define

$$f(x)+g(x)=\sum_i (a_i+b_i)x^i$$

Clearly, $f(x)+g(x)\in F[x]$, because $a_i+b_i\in F$ for all i.

Scalar multiplication on F[x]: For any $\lambda\in F$ and $f(x)=\sum\limits_i a_i x^i \in F[x]$ $\lambda f(x)$ is defined as the polynomial $\sum\limits_i (\lambda a_i)x^i$. i.e,

$$\lambda f(x)=\sum_i (\lambda a_i)x^i$$

Obviously, $\lambda f(x)\in F[x]$, as $\lambda a_i\in F$ for all i.

Equality of two elements of F[x]: For any $f(x)=\sum\limits_i a_i x^i, g(x)=\sum\limits_i b_i x^i$, we define

$$f(x)=g(x)\Leftrightarrow a_i=b_i \quad \text{for all } i.$$

Now we shall check whether V-1 and V-2(i) to V-2(iv) are satisfied.

V-1 *$F[x]$ is an abelian group under vector addition.*

Associativity: If $f(x)=\sum_i a_i x^i,\ g(x)=\sum_i b_i x^i,\ h(x)=\sum_i c_i x^i \in F[x]$, then

$$\begin{aligned}[f(x)+g(x)]+h(x) &= \left(\sum_i (a_i+b_i)x^i\right)+\sum_i c_i x^i \\ &= \sum_i [(a_i+b_i)+c_i]x^i \\ &= \sum_i [a_i+(b_i+c_i)]x^i && \text{[By associativity of + on } F] \\ &= \sum_i a_i x^i+\sum_i (b_i+c_i)x^i \\ &= f(x)+[g(x)+h(x)]\end{aligned}$$

So, vector addition is associative on $F[x]$.

Commutativity: If $f(x)=\sum_i a_i x^i,\ g(x)=\sum_i b_i x^i \in F[x]$, then

$$f(x)+g(x)=\sum_i (a_i+b_i)x^i=\sum_i (b_i+a_i)x^i \qquad \text{[By commutativity of + on } F]$$

$$\Rightarrow \qquad f(x)+g(x)=\sum_i b_i x^i+\sum_i a_i x^i=g(x)+f(x)$$

So, vector addition is commutative on $F[x]$.

Existence of zero vector: Since $\hat{0}(x)=\sum_i 0x^i \in F[x]$ is such that for all $f(x)=\sum_i a_i x^i \in F[x]$

$$\Rightarrow \qquad \hat{0}(x)+f(x)=\sum_i 0x^i+\sum_i a_i x^i=\sum_i (0+a_i)x^i$$

$$\Rightarrow \qquad \hat{0}(x)+f(x)=\sum_i a_i x^i \qquad [\because\ 0\in F \text{ is the additive identity}]$$

$$\Rightarrow \qquad \hat{0}(x)+f(x)=f(x)$$

Thus, $\hat{0}(x)+f(x)=f(x)=f(x)+\hat{0}(x)$ for all $f(x)\in F[x]$.

So, $\hat{0}(x)$ is the zero vector (additive identity) in $F[x]$.

Existence of additive inverse: Let $f(x)=\sum_i a_i x^i$ be an arbitrary polynomial in $F[x]$. Then, $-f(x)=\sum_i (-a_i)x^i \in F[x]$ such that

$$f(x)+(-f(x))=\sum_i a_i x^i+\sum_i (-a_i)x^i=\sum_i [a_i+(-a_i)]x^i=\sum_i 0x^i=\hat{0}(x)$$

Thus, for each $f(x)\in F[x]$, there exists $-f(x)\in F[x]$ such that

$$f(x)+(-f(x))=\hat{0}(x)=(-f(x))+f(x)$$

So, each $f(x)$ has its additive inverse in $F[x]$.
Hence, $F[x]$ is an abelian group under vector addition.

V-2 For any $f(x)=\sum_i a_i x^i,\ g(x)=\sum_i b_i x^i \in F[x]$ *and* $\lambda,\ \mu \in F$*, we have*

(i) $\lambda[f(x)+g(x)]=\lambda\left(\sum_i (a_i+b_i)x^i\right)=\sum_i \lambda(a_i+b_i)x^i$

$=\sum_i (\lambda a_i+\lambda b_i)x^i$ [By left distributivity of multiplication over addition]

$=\sum_i (\lambda a_i)x^i+\Sigma(\lambda b_i)x^i$

$=\lambda f(x)+\lambda g(x)$

(ii) $(\lambda+\mu)f(x)=\sum_i (\lambda+\mu)a_i x^i$

$=\sum_i (\lambda a_i+\mu a_i)x^i$ [By right distributivity of multiplication over addition]

$=\sum_i (\lambda a_i)x^i+\sum_i (\mu a_i)x^i$

$=\lambda f(x)+\mu f(x).$

(iii) $(\lambda\mu)f(x)=\sum_i (\lambda\mu)a_i x^i$

$=\sum_i \lambda(\mu a_i)x^i$ [By associativity of multiplication on F]

$=\lambda\sum_i (\mu a_i)x^i$

$=\lambda(\mu f(x))$

(iv) $1f(x)=\sum_i (1a_i)x^i$

$=\sum_i a_i x^i$ [$\because$ 1 is unity in F]

$=f(x)$

Hence, $F[x]$ is a vector space over field F.

EXAMPLE-9 *The set V of all real valued continuous (differentiable or integrable) functions defined on the closed interval [a, b] is a real vector space with the vector addition and scalar multiplication defined as follows:*

$$(f+g)(x)=f(x)+g(x)$$
$$(\lambda f)(x)=\lambda f(x)$$

For all $f,\ g\in V$ and $\lambda\in R$.

SOLUTION Since the sum of two continuous functions is a continuous function. Therefore, vector addition is a binary operation on V. Also, if f is continuous and $\lambda \in R$, then λf is continuous. We know that for any $f,\ g \in V$

$$f = g \Leftrightarrow f(x) = g(x) \quad \text{for all } x \in [a, b].$$

Thus, we have defined vector addition, scalar multiplication and equality in V.

It remains to verify V-1 and V-2(i) to V-2(iv).

V-1 V is an abelian group under vector addition.

Associativity: Let $f,\ g,\ h$ be any three functions in V. Then,

$$\begin{aligned}[(f+g)+h](x) &= (f+g)(x)+h(x) = [f(x)+g(x)]+h(x)\\ &= f(x)+[g(x)+h(x)] \qquad \text{[By associativity of addition on } R]\\ &= f(x)+(g+h)(x)\\ &= [f+(g+h)](x) \qquad \text{for all } x \in [a,\ b]\end{aligned}$$

So, $\quad (f+g)+h = f+(g+h)$

Thus, vector addition is associative on V.

Commutativity: For any $f,\ g \in V$, we have

$$(f+g)(x) = f(x)+g(x) = g(x)+f(x) \qquad \text{[By commutativity of + on } R]$$

$\Rightarrow \quad (f+g)(x) = (g+f)(x) \qquad$ for all $x \in [a,\ b]$

So, $\quad f+g = g+f$

Thus, vector addition is commutative on V.

Existence of additive identity: The function $\hat{0}(x) = 0$ for all $x \in [a,\ b]$ is the additive identity, because for any $f \in V$

$$(f+\hat{0})(x) = f(x)+\hat{0}(x) = f(x)+0 = f(x) = (\hat{0}+f)(x) \quad \text{for all } x \in [a,\ b].$$

Existence of additive inverse: Let f be an arbitrary function in V. Then, a function $-f$ defined by $(-f)(x) = -f(x)$ for all $x \in [a,\ b]$ is continuous on $[a,\ b]$, and

$$[f+(-f)](x) = f(x)+(-f)(x) = f(x)-f(x) = 0 = \hat{0}(x) \quad \text{for all } x \in [a,\ b].$$

Thus, $\quad f+(-f) = \hat{0} = (-f)+f$

So, each $f \in V$ has its additive inverse $-f \in V$.

Hence, V is an abelian group under vector addition.

V-1. $V_1 \times V_2$ *is an abelian group under vector addition:*

Associativity: For any $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in V_1 \times V_2$, we have

$$\begin{aligned}
[(u_1, u_2)+(v_1, v_2)]+(w_1, w_2) &= \big((u_1+v_1), (u_2+v_2)\big)+(w_1, w_2)\\
&= \big((u_1+v_1)+w_1, (u_2+v_2)+w_2\big)\\
&= \big(u_1+(v_1+w_1), u_2+(v_2+w_2)\big) \quad \begin{bmatrix}\text{By associativity of}\\ \text{vector addition on } V_1\\ \text{and } V_2\end{bmatrix}\\
&= (u_1, u_2)+(v_1+w_1, v_2+w_2)\\
&= (u_1, u_2)+[(v_1, v_2)+(w_1, w_2)]
\end{aligned}$$

So, vector addition is associative on $V_1 \times V_2$.

Commutativity: For any $(u_1, u_2), (v_1, v_2) \in V_1 \times V_2$, we have

$$\begin{aligned}
(u_1, u_2)+(v_1, v_2) &= (u_1+v_1, u_2+v_2)\\
&= (v_1+u_1, v_2+u_2) \quad [\text{By commutativity of vector addition on } V_1 \text{ and } V_2]\\
&= (v_1, v_2)+(u_1, u_2)
\end{aligned}$$

So, vector addition is commutative on $V_1 \times V_2$.

Existence of additive identity: If 0_{V_1} and 0_{V_2} are zero vectors in V_1 and V_2 respectively, then $(0_{V_1}, 0_{V_2})$ is zero vector (additive identity) in $V_1 \times V_2$. Because for any $(u_1,u_2) \in V_1 \times V_2$

$$\begin{aligned}
(u_1, u_2)+(0_{V_1}, 0_{V_2}) &= (u_1+0_{V_1}, u_2+0_{V_2})\\
&= (u_1, u_2) = (0_{V_1}, 0_{V_2})+(u_1, u_2) \quad [\text{By commutativity of vector addition}]
\end{aligned}$$

Existence of additive Inverse: Let $(u_1, u_2) \in V_1 \times V_2$. Then, $(-u_1, -u_2)$ is its additive inverse. Because,

$$\begin{aligned}
(u_1, u_2)+(-u_1, -u_2) &= \big(u_1+(-u_1), u_2+(-u_2)\big)\\
&= (0_{V_1}, 0_{V_2}) = (-u_1, -u_2)+(u_1, u_2) \quad \begin{bmatrix}\text{By commutativity of vector}\\ \text{addition}\end{bmatrix}
\end{aligned}$$

Thus, each element of $V_1 \times V_2$ has its additive inverse in $V_1 \times V_2$.
Hence, $V_1 \times V_2$ is an abelian group under vector addition.

V-2. For any $(u_1, u_2), (v_1, v_2) \in V_1 \times V_2$ *and* $\lambda, \mu \in F$, *we have*

(i)
$$\begin{aligned}
\lambda[(u_1, u_2)+(v_1, v_2)] &= \lambda(u_1+v_1, u_2+v_2)\\
&= \big(\lambda(u_1+v_1), \lambda(u_2+v_2)\big)\\
&= (\lambda u_1+\lambda v_1, \lambda u_2+\lambda v_2) \quad [\text{By } V\text{-2(i) in } V_1 \text{ and } V_2]\\
&= (\lambda u_1, \lambda u_2)+(\lambda v_1, \lambda v_2)\\
&= \lambda(u_1, u_2)+\lambda(v_1, v_2)
\end{aligned}$$

(ii)
$$
\begin{aligned}
(\lambda+\mu)(u_1,u_2) &= \big((\lambda+\mu)u_1,\ (\lambda+\mu)u_2\big)\\
&= \big(\lambda u_1+\mu u_1,\ \lambda u_2+\mu u_2\big) && \text{[By } V\text{-2(ii) in } V_1 \text{ and } V_2]\\
&= (\lambda u_1,\ \lambda u_2)+(\mu u_1,\ \mu u_2)\\
&= \lambda(u_1,\ u_2)+\mu(u_1,\ u_2)
\end{aligned}
$$

(iii)
$$
\begin{aligned}
(\lambda\mu)(u_1,\ u_2) &= \big((\lambda\mu)u_1,(\lambda\mu)u_2\big)\\
&= \big(\lambda(\mu u_1),\lambda(\mu u_2)\big) && \text{[By } V\text{-2(iii) in } V_1 \text{ and } V_2]\\
&= \lambda(\mu u_1,\mu u_2)\\
&= \lambda\{\mu(u_1,u_2)\}
\end{aligned}
$$

(iv)
$$
\begin{aligned}
1(u_1,u_2) &= (1u_1,1u_2)\\
&= (u_1,u_2) && \text{[By } V\text{-2(iv) in } V_1 \text{ and } V_2]
\end{aligned}
$$

Hence, $V_1 \times V_2$ is a vector space over field F.

EXAMPLE-11 *Let V be a vector space over a field F. Then for any $n \in N$ the function space $V^n = \{f : f : \underline{n} \to V\}$ is a vector space over field F.*

SOLUTION We define vector addition, scalar multiplication and equality in V^n as follows:

Vector addition: Let $f, g \in V^n$, then the vector addition on V^n is defined as

$$(f+g)(x) = f(x)+g(x) \text{ for all } x \in \underline{n}.$$

Scalar multiplication: If $\lambda \in F$ and $f \in V^n$, then λf is defined as

$$(\lambda f)(x) = \lambda(f(x)) \text{ for all } x \in \underline{n}.$$

Equality: $f, g \in V^n$ are defined as equal if $f(x) = g(x)$ for all $x \in \underline{n}$.

Now we shall verify V^n-1 and V^n-2(i) to V^n-2 (iv).

$V^n - 1$. *V^n is an abelian group under vector addition:*

Associativity: For any $f, g, h \in V^n$, we have

$$
\begin{aligned}
[(f+g)+h](x) &= (f+g)(x)+h(x) = [f(x)+g(x)]+h(x) \quad \text{for all } x \in \underline{n}\\
&= f(x)+[g(x)+h(x)] && \text{[By associativity of vector addition on } V]\\
&= f(x)+(g+h)(x)\\
&= [f+(g+h)](x) \quad \text{for all } x \in \underline{n}
\end{aligned}
$$

$\therefore \quad (f+g)+h = f+(g+h)$

So, vector addition is associative on V^n.

Commutativity: For any $f, g \in V^n$, we have

$$(f+g)(x) = f(x)+g(x) \quad \text{for all } x \in \underline{n}$$

$\Rightarrow$ $(f+g)(x) = g(x)+f(x)$ [By commutativity of vector addition on V]

$\Rightarrow$ $(f+g)(x) = (g+f)(x)$ for all $x \in \underline{n}$

$\therefore$ $f+g = g+f$

So, vector addition is commutative on V^n.

Existence of additive identity (zero vector): The function $\widehat{0} : \underline{n} \to V$ defined by $\widehat{0}(x) = 0_v$ for all $x \in \underline{n}$ is the additive identity, because

$$(f+\widehat{0})(x) = f(x)+\widehat{0}(x) = f(x)+0_V$$

$\Rightarrow$ $(f+\widehat{0})(x) = f(x)$ for all $f \in V^n$ and for all $x \in \underline{n}$ [$\because$ 0_V is zero vector in V]

Existence of additive inverse: Let f be an arbitrary function in V^n. Then $-f : \underline{n} \to V$ defined by

$$(-f)(x) = -f(x) \quad \text{for all } x \in \underline{n}$$

is additive inverse of f, because

$$[f+(-f)](x) = f(x)+(-f)(x) = f(x)-f(x) = 0_V = \widehat{0}(x) \quad \text{for all } x \in \underline{n}.$$

Hence, V^n is an abelian group under vector addition.

V^n-2. For any $f, g \in V^n$ and $\lambda, \mu \in F$, we have

(i) $[\lambda(f+g)](x) = \lambda[(f+g)(x)] = \lambda[f(x)+g(x)]$

$\Rightarrow$ $[\lambda(f+g)](x) = \lambda f(x)+\lambda g(x)$ [By V-2(i) in V]

$\Rightarrow$ $[\lambda(f+g)](x) = (\lambda f+\lambda g)(x)$ for all $x \in \underline{n}$

$\therefore$ $\lambda(f+g) = \lambda f+\lambda g.$

(ii) $[(\lambda+\mu)f](x) = (\lambda+\mu)f(x)$

$\Rightarrow$ $[(\lambda+\mu)f](x) = \lambda f(x)+\mu f(x)$ [By V-2(ii) in V]

$\Rightarrow$ $[(\lambda+\mu)f](x) = (\lambda f)(x)+(\mu f)(x)$

$\Rightarrow$ $[(\lambda+\mu)f](x) = (\lambda f+\mu f)(x)$ for all $x \in \underline{n}$

$\therefore$ $(\lambda+\mu)f = \lambda f+\mu f$

(iii) $[(\lambda\mu)f](x) = (\lambda\mu)f(x)$

$\Rightarrow$ $[(\lambda\mu)f](x) = \lambda\big(\mu f(x)\big)$ [By V-2(iii) in V]

$\Rightarrow$ $[(\lambda\mu)f](x) = \lambda\big((\mu f)(x)\big)$

9. Let U and W be two vector spaces over a field F. Let V be the set of ordered pairs (u,w) where $u \in U$ and $w \in W$. Show that V is a vector space over F with addition in V and scalar multiplication V defined by
$$(u_1,\ w_1)+(u_2,\ w_2)=(u_1+u_2,\ w_1+w_2)$$
and
$$a(u,\ w)=(au,\ aw)$$
(This space is called the external direct product of u and w).
10. Let V be a vector space over a field F and $n \in N$. Let $V^n = \{(v_1,\ v_2,\dots,v_n) : v_i \in V;\ i=1,\ 2,\dots,n\}$. Show that V^n is a vector space over field F with addition in V and scalar multiplication on V defined by
$$(u_1,\ u_2,\dots,u_n)+(v_1,\ v_2,\dots,v_n)=(u_1+v_1,\ u_2+v_2,\dots,u_n+v_n)$$
and
$$a(v_1,\ v_2,\dots,v_n)=(av_1,\ av_2,\dots,av_n)$$
11. Let $V=\{(x,\ y) : x,\ y\in R\}$. Show that V is not a vector space over R under the addition and scalar multiplication defined by
$$(a_1,\ b_1)+(a_2,\ b_2)=(3b_1+3b_2,\ -a_1-a_2)$$
$$k(a_1,\ b_1)=(3kb_1,\ -ka_1)$$
for all $(a_1,\ b_1),\ (a_2,\ b_2)\in V$ and $k\in R$.
12. Let $V=\{(x,\ y) : x,y\in R\}$. For any $\alpha=(x_1,\ y_1), \beta=(x_2,\ y_2)\in V,\ c\in R$, define
$$\alpha\oplus\beta=(x_1+x_2+1,\ y_1+y_2+1)$$
$$c\odot\alpha=(cx_1,\ cy_1)$$
   (i) Prove that $(V,\oplus)$ is an abelian group.
   (ii) Verify that $c\odot(\alpha\oplus\beta)\neq c\odot\alpha\oplus c\odot\beta$.
   (iii) Prove that V is not a vector space over R under the above two operations.
13. For any $u=(x_1,\ x_2,\ x_3),\ v=(y_1,\ y_2,\ y_3)\in R^3$ and $a\in R$, let us define
$$u\oplus v=(x_1+y_1+1,\ x_2+y_2+1,\ x_3+y_3+1)$$
$$a\odot u=(ax_1+a-1,\ ax_2+a-1,\ ax_3+a-1)$$
Prove that R^3 is a vector space over R under these two operations.
14. Let V denote the set of all positive real numbers. For any $u,\ v\in V$ and $a\in R$, define
$$u\oplus v=uv$$
$$a\odot u=u^a$$
Prove that V is a vector space over R under these two operations.
15. Prove that R^3 is not a vector space over R under the vector addition and scalar multiplication defined as follows:
$$(x_1,\ x_2,\ x_3)+(y_1,\ y_2,\ y_3)=(x_1+y_1,\ x_2+y_2,\ x_3+y_3)$$
$$a(x_1,\ x_2,\ x_3)=(ax_1,\ ax_2,\ ax_3)$$

16. Let $V = \{(x,\ 1) : x \in R\}$. For any $u = (x,\ 1),\ v = (y,\ 1) \in V$ and $a \in R$, define

$$u \oplus v = (x+y, 1)$$
$$a \odot u = (ax, 1)$$

Prove that V is a vector space over R under these two operations.

17. Let V be the set of ordered pairs (a, b) of real numbers. Let us define

$$(a,\ b) \oplus (c,\ d) = (a+c,\ b+d)$$

and
$$k(a,\ b) = (ka,\ 0)$$

Show that V is not a vector space over R.

18. Let V be the set of ordered pairs (a, b) of real numbers. Show that V is not a vector space over R with addition and scalar multiplication defined by

(i) $(a,\ b) + (c,\ d) = (a+d,\ b+c)$ and $k(a,\ b) = (ka,\ kb)$

(ii) $(a,\ b) + (c,\ d) = (a+c,\ b+d)$ and $k(a,\ b) = (a,\ b)$

(iii) $(a,\ b) + (c,\ d) = (0,\ 0)$ and $k(a,\ b) = (ka,\ kb)$

(iv) $(a,\ b) + (c,\ d) = (ac,\ bd)$ and $k(a,\ b) = (ka,\ kb)$

19. The set V of all convergent real sequences is a vector space over the field R of all real numbers.

20. Let p be a prime number. The set $Z_p = \{0,\ 1,\ 2, \dots, (p-1)\}$ is a field under addition and multiplication modulo p as binary operations. Let V be the vector space of polynomials of degree at most n over the field Z_p. Find the number of elements in V.

[Hint: Each polynomial in V is of the form $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_0,\ a_1,\ a_2, \dots, a_n \in Z_p$. Since each a_i can take any one of the p values $0,\ 1,\ 2, \dots, (p-1)$. So, there are p^{n+1} elements in V]

2.3 ELEMENTARY PROPERTIES OF VECTOR SPACES

THEOREM-1 *Let V be a vector space over a field F. Then*

(i) $a \cdot 0_V = 0_V$ *for all* $a \in F$

(ii) $0 \cdot v = 0_V$ *for all* $v \in V$

(iii) $a \cdot (-v) = -(a \cdot v) = (-a) \cdot v$ *for all* $v \in V$ *and for all* $a \in F$

(iv) $(-1) \cdot v = -v$ *for all* $v \in F$

(v) $a \cdot v = 0_V \Rightarrow$ *either* $a = 0$ *or* $v = 0_V$

PROOF. (i) We have,

$$a \cdot 0_V = a \cdot (0_V + 0_V) \quad \text{for all } a \in F \qquad [\because \; 0_V + 0_V = 0_V]$$

$$\Rightarrow \quad a \cdot 0_V = a \cdot 0_V + a \cdot 0_V \quad \text{for all } a \in F \qquad [\text{By } V\text{-2(i)}]$$

$$\therefore \quad a \cdot 0_V + 0_V = a \cdot 0_V + a \cdot 0_V \quad \text{for all } a \in F \qquad [0_V \text{ is additive identity in } V]$$

$$\Rightarrow \quad 0_V = a \cdot 0_V \quad \text{for all } a \in F \qquad \left[\begin{array}{l}\text{By left cancellation law for addition}\\ \text{on } V\end{array}\right]$$

(ii) For any $v \in V$, we have

$$0 \cdot v = (0+0) \cdot v \qquad [\because \; 0 \text{ is additive identity in } F]$$

$$\Rightarrow \quad 0 \cdot v = 0 \cdot v + 0 \cdot v \qquad [\text{By } V\text{-2(ii)}]$$

$$\therefore \quad 0 \cdot v + 0_V = 0 \cdot v + 0 \cdot v \qquad [\because \; 0_V \text{ is additive identity in } V]$$

$$\Rightarrow \quad 0 \cdot v = 0_V \qquad [\text{By left cancellation law for addition in } V]$$

So, $0 \cdot v = 0_V$ for all $v \in V$.

(iii) For any $v \in V$ and $a \in F$, we have

$$a \cdot [v + (-v)] = a \cdot v + a \cdot (-v) \qquad [\text{By } V\text{-2(i)}]$$

$$\Rightarrow \quad a \cdot 0_V = a \cdot v + a \cdot (-v) \qquad [\because \; v + (-v) = 0_V \text{ for all } v \in V]$$

$$\Rightarrow \quad 0_V = a \cdot v + a \cdot (-v) \qquad [\text{By (i)}]$$

$$\Rightarrow \quad a \cdot (-v) = -(a \cdot v) \qquad [\because \; V \text{ is an additive group}]$$

Again, $$[a + (-a)] \cdot v = a \cdot v + (-a) \cdot v \qquad [\text{By } V\text{-2(ii)}]$$

$$\Rightarrow \quad 0 \cdot v = a \cdot v + (-a) \cdot v \qquad [\because \; a + (-a) = 0]$$

$$\Rightarrow \quad 0_V = a \cdot v + (-a) \cdot v \qquad [\text{By (ii)}]$$

$$\Rightarrow \quad (-a) \cdot v = -(a \cdot v) \qquad [\because \; V \text{ is an additive group}]$$

Hence, $a \cdot (-v) = -(a \cdot v) = (-a) \cdot v$ for all $a \in F$ and for all $v \in V$.

(iv) Taking $a = 1$ in (iii), we obtain

$$(-1) \cdot v = -v \quad \text{for all } v \in V.$$

(v) Let $a \cdot v = 0_V$, and let $a \neq 0$. Then $a^{-1} \in F$.
Now

$$a \cdot v = 0_V$$

$$\Rightarrow \quad a^{-1} \cdot (a \cdot v) = a^{-1} \cdot 0_V$$

$$\Rightarrow \quad (a^{-1}a) \cdot v = a^{-1} \cdot 0_V \qquad [\text{By } V\text{-2(iii)}]$$

$$\Rightarrow \quad (a^{-1}a) \cdot v = 0_V \qquad [\text{By (i)}]$$

$$\Rightarrow \quad 1 \cdot v = 0_V$$

$$\Rightarrow \quad v = 0_V \qquad [\text{By } V\text{-2(iv)}]$$

Again let $a \cdot v = 0_V$ and $v \neq 0_V$. Then to prove that $a = 0$, suppose $a \neq 0$. Then $a^{-1} \in F$. Now

$$a \cdot v = 0_V$$
$$\Rightarrow \quad a^{-1} \cdot (a \cdot v) = a^{-1} \cdot 0_V$$
$$\Rightarrow \quad (a^{-1}a) \cdot v = a^{-1} \cdot 0_V \qquad \text{[By } V\text{-2(iii)]}$$
$$\Rightarrow \quad 1 \cdot v = 0_V \qquad \text{[By (i)]}$$
$$\Rightarrow \quad v = 0_V, \text{ which is a contradiction.} \qquad \text{[By } V\text{-2(iv)]}$$

Thus, $a \cdot v = 0_V$ and $v \neq 0_V \Rightarrow a = 0$.
Hence, $a \cdot v = 0_V \Rightarrow$ either $a = 0$ or $v = 0_V$.

Q.E.D.

It follows from this theorem that the multiplication by the zero of V to a scalar in F or by the zero of F to a vector in V always leads to the zero of V.

Since it is convenient to write $u - v$ instead of $u + (-v)$. Therefore, in what follows we shall write $u - v$ for $u + (-v)$.

REMARK. *For the sake of convenience, we drop the scalar multiplication '$\cdot$' and shall write av in place of $a \cdot v$.*

THEOREM-2 *Let V be a vector space over a field F. Then,*

(i) If $u, v \in V$ and $0 \neq a \in F$, then

$$au = av \Rightarrow u = v$$

(ii) If $a, b \in F$ and $0_V \neq u \in V$, then

$$au = bu \Rightarrow a = b$$

PROOF. (i) We have,

$$au = av$$
$$\Rightarrow \quad au + (-(av)) = av + (-(av))$$
$$\Rightarrow \quad au - av = 0_V$$
$$\Rightarrow \quad a(u - v) = 0_V$$
$$\Rightarrow \quad u - v = 0_V \qquad \text{[By Theorem 1(v)]}$$
$$\Rightarrow \quad u = v$$

(ii) We have,

$$au = bu$$
$$\Rightarrow \quad au + (-(bu)) = bu + (-(bu))$$
$$\Rightarrow \quad au - bu = 0_V$$
$$\Rightarrow \quad (a - b)u = 0_V$$
$$\Rightarrow \quad a - b = 0 \qquad \text{[By Theorem 1(v)]}$$
$$\Rightarrow \quad a = b$$

Q.E.D.

EXERCISE 2.2

1. Let V be a vector space over a field F. Then prove that

$$a(u-v) = au - av \quad \text{for all } a \in F \text{ and } u, v \in V.$$

2. Let V be a vector space over field R and $u,\ v \in V$. Simplify each of the following:

   (i) $4(5u-6v)+2(3u+v)$ (ii) $6(3u+2v)+5u-7v$

   (iii) $5(2u-3v)+4(7v+8)$ (iv) $3(5u+\frac{2}{v})$

3. Show that the commutativity of vector addition in a vector space V can be derived from the other axioms in the definition of V.
4. Mark each of the following as true or false.
   (i) Null vector in a vector space is unique.
   (ii) Let $V(F)$ be a vector space. Then scalar multiplication on V is a binary operation on V.
   (iii) Let $V(F)$ be a vector space. Then $au = av \Rightarrow \ u = v$ for all $u,\ v \in V$ and for all $a \in F$.
   (iv) Let $V(F)$ be a vector space. Then $au = bu \Rightarrow \ a = b$ for all $u \in V$ and for all $a,\ b \in F$.
5. If $V(F)$ is a vector space, then prove that for any integer n, $\lambda \in F$, $v \in V$, prove that $n(\lambda v) = \lambda(nv) = (n\lambda)v$.

ANSWERS

2. (i) $26u - 22v$ (ii) $23u + 5v$
   (iii) The sum $7v + 8$ is not defined, so the given expression is not meaningful.
   (iv) Division by v is not defined, so the given expression is not meaningful.
4. (i) T (ii) F (iii) F (iv) F

2.4 SUBSPACES

SUB-SPACE *Let V be a vector space over a field F. A non-void subset S of V is said to be a subspace of V if S itself is a vector space over F under the operations on V restricted to S.*

If V is a vector space over a field F, then the null (zero) space $\{0_V\}$ and the entire space V are subspaces of V. These two subspaces are called trivial (improper) subspaces of V and any other subspace of V is called a non-trivial (proper) subspace of V.

THEOREM-1 *(Criterion for a non-void subset to be a subspace) Let V be a vector space over a field F. A non-void subset S of V is a subspace of V iff or all $u,\ v \in S$ and for all $a \in F$*

(i) $u - v \in S$ and, (ii) $au \in S$.

PROOF. First suppose that S is a subspace of vector space V. Then S itself is a vector space over field F under the operations on V restricted to S. Consequently, S is subgroup of the additive abelian group V and is closed under scalar multiplication.

Hence, (i) and (ii) hold.

Conversely, suppose that S is a non-void subset of V such that (i) and (ii) hold. Then,

(i) $\Rightarrow$ S is an additive subgroup of V and therefore S is an abelian group under vector addition.

(ii) $\Rightarrow$ S is closed under scalar multiplication.

Axioms V-2(i) to V-2(iv) hold for all elements in S as they hold for elements in V.

Hence, S is a subspace of V. Q.E.D.

THEOREM-2 *(Another criterion for a non-void subset to be a subspace) Let V be a vector space over a field F. Then a non-void subset S of V is a subspace of V iff $au+bv \in S$ for all $u,\ v \in S$ and for all $a,\ b \in F$.*

PROOF. First let S be a subspace of V. Then,

$$au \in S,\ bv \in S \quad \text{for all } u, v \in S \text{ and for all } a, b \in F \qquad \text{[By Theorem 1]}$$

$$\Rightarrow \quad au+bv \in S \quad \text{for all } u, v \in S \text{ and for all } a, b \in F [\because \ S \text{ is closed under vector addition}]$$

Conversely, let S be a non-void subset of V such that $au+bv \in S$ for all $u,\ v \in S$ and for all $a,\ b \in F$.

Since $1,\ -1 \in F$, therefore

$$1u+(-1)v \in S \quad \text{for all } u,\ v \in S.$$

$$\Rightarrow \quad u-v \in S \quad \text{for all } u,\ v \in S \qquad \text{(i)}$$

Again, since $0 \in F$. Therefore,

$$au+0v \in S \quad \text{for all } u \in S \text{ and for all } a \in F.$$

$$\Rightarrow \quad au \in S \quad \text{for all } u \in S \text{ and for all } a \in F \qquad \text{(ii)}$$

From (i) and (ii), we get

$u-v \in S$ and $au \in S$ for all $u,\ v \in S$ and for all $a \in F$.

Hence, by Theorem 1, S is a subspace of vector space V.

Q.E.D.

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Show that $S = \{(0,\ b,\ c) : b,\ c \in R\}$ is a subspace of real vector space R^3.*

SOLUTION Obviously, S is a non-void subset of R^3. Let $u = (0,\ b_1,\ c_1),\ v = (0,\ b_2,\ c_2) \in S$ and $\lambda,\ \mu \in R$. Then,

$$\lambda u + \mu v = \lambda(0,\ b_1,\ c_1) + \mu(0,\ b_2,\ c_2)$$

$$\Rightarrow \qquad \lambda u + \mu v = (0,\ \lambda b_1 + \mu b_2,\ \lambda c_1 + \mu c_2) \in S.$$

Hence, S is a subspace of R^3.

REMARK. *Geometrically-R^3 is three dimensional Euclidean plane and S is yz-plane which is itself a vector space. Hence, S is a sub-space of R^3. Similarly, $\{(a,\ b,\ 0) : a,\ b \in R\}$, i.e. xy-plane, $\{(a,0,c) : a,c \in R\}$, i.e. xz plane are subspaces of R^3. In fact, every plane through the origin is a subspace of R^3. Also, the coordinate axes $\{(a,0,0) : a \in R\}$, i.e. x-axis, $\{(0,a,0) : a \in R\}$, i.e. y-axis and $\{(0,0,a) : a \in R\}$, i.e. z-axis are subspaces of R^3.*

EXAMPLE-2 *Let V denote the vector space R^3, i.e. $V = \{(x,y,z) : x,y,z \in R\}$ and S consists of all vectors in R^3 whose components are equal, i.e. $S = \{(x,x,x) : x \in R\}$. Show that S is a subspace of V.*

SOLUTION Clearly, S is a non-empty subset of R^3 as $(0,0,0) \in S$.

Let $u = (x,x,x)$ and $v = (y,y,y) \in S$ and $a,b \in R$. Then,

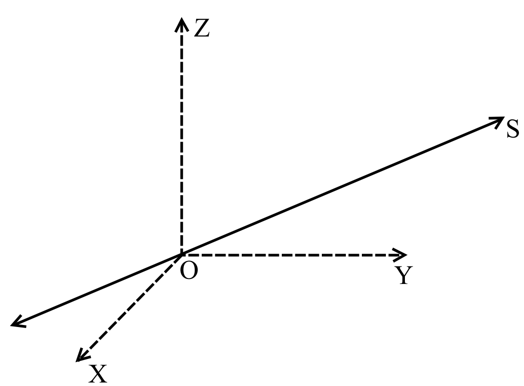
$$au + bv = a(x,x,x) + b(y,y,y)$$

$$\Rightarrow \qquad au + av = (ax + by, ax + by, ax + by) \in S.$$

Thus, $\quad au + bv \in S$ for all $u,\ v \in S$ and $a,b \in R$.

Hence, $\quad S$ is *a* subspace of V.

REMARK. *Geometrically S, in the above example, represents the line passing through the origin O and having direction ratios proportional to 1,1,1 as shown in the following figure.*



If there is no danger of any confusion we shall say V is a vector space over a field F, whenever the algebraic structure $(V,\ F,\ \oplus,\ \odot)$ is a vector space. Thus, whenever we say that V is a vector space over a field F, it would always mean that $(V,\ \oplus)$ is an abelian group and $\odot : F \times V \to V$ is a mapping such that V-2 (i)–(iv) are satisfied.

V is called a *real vector space* if $F = R$ (field of real numbers), and a *complex vector space* if $F = C$ (field of complex numbers).

REMARK-1 *V is called a left or a right vector space according as the elements of a skew-field F are multiplied on the left or right of vectors in V. But, in case of a field these two concepts coincide.*

REMARK-2 *The symbol '+' has been used to denote the addition in the field F. For the sake of convenience, in future, we shall use the same symbol '+' for vector addition $\oplus$ and addition in the field F. But, the context would always make it clear as to which operation is meant. Similarly, multiplication in the field F and scalar multiplication $\odot$ will be denoted by the same symbol '$\cdot$'.*

REMARK-3 *In this chapter and in future also, we will be dealing with two types of zeros. One will be the zero of the additive abelian group V, which will be known as the vector zero and other will be the zero element of the field F which will be known as scalar zero. We will use the symbol 0_V to denote the zero vector and 0 to denote the zero scalar.*

REMARK-4 *Since $(V,\oplus)$ is an abelian group, therefore, for any $u, v, w \in V$ the following properties will hold:*

$$\left.\begin{array}{ll}\text{(i)} & u \oplus v = u \oplus w \Rightarrow \ v = w \\ \text{(ii)} & v \oplus u = w \oplus u \Rightarrow \ v = w\end{array}\right\} \text{(Cancellation laws)}$$

(iii) $u \oplus v = 0_v \Rightarrow \ u = -v$ and $v = -u$
(iv) $-(-u) = u$
(v) $u \oplus v = u \Rightarrow \ v = 0_v$
(vi) $-(u \oplus v) = (-u) \oplus (-v)$
(vii) 0_v is unique
(viii) For each $u \in V,\ -u \in V$ is unique.

REMARK-5 *If V is a vector space over a field F, then we will write $V(F)$.*

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Every field is a vector space over its any subfield.*

SOLUTION Let F be a field, and let S be an arbitrary subfield of F. Since F is an additive abelian group, therefore, V-1 holds.

Taking multiplication in F as scalar multiplication, the axioms V-2 (i) to V-2 (iii) are respectively the right distributive law, left distributive law and associative law. Also V-2 (iv) is property of unity in F.

$$\lambda u + \mu v = \lambda(0,\ b_1,\ c_1) + \mu(0,\ b_2,\ c_2)$$

$$\Rightarrow \qquad \lambda u + \mu v = (0,\ \lambda b_1 + \mu b_2,\ \lambda c_1 + \mu c_2) \in S.$$

Hence, S is a subspace of R^3.

REMARK. *Geometrically-R^3 is three dimensional Euclidean plane and S is yz-plane which is itself a vector space. Hence, S is a sub-space of R^3. Similarly, $\{(a,\ b,\ 0) : a,\ b \in R\}$, i.e. xy-plane, $\{(a,0,c) : a,c \in R\}$, i.e. xz plane are subspaces of R^3. In fact, every plane through the origin is a subspace of R^3. Also, the coordinate axes $\{(a,0,0) : a \in R\}$, i.e. x-axis, $\{(0,a,0) : a \in R\}$, i.e. y-axis and $\{(0,0,a) : a \in R\}$, i.e. z-axis are subspaces of R^3.*

EXAMPLE-2 *Let V denote the vector space R^3, i.e. $V = \{(x,y,z) : x,y,z \in R\}$ and S consists of all vectors in R^3 whose components are equal, i.e. $S = \{(x,x,x) : x \in R\}$. Show that S is a subspace of V.*

SOLUTION Clearly, S is a non-empty subset of R^3 as $(0,0,0) \in S$.

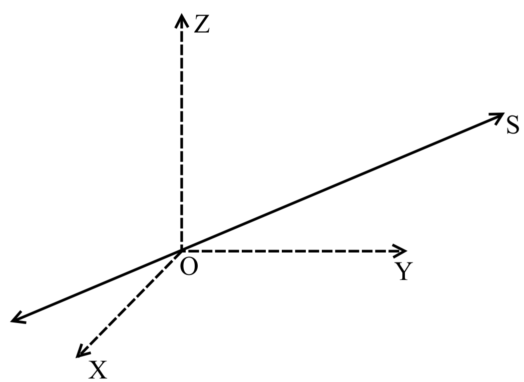
Let $u = (x,x,x)$ and $v = (y,y,y) \in S$ and $a,b \in R$. Then,

$$au + bv = a(x,x,x) + b(y,y,y)$$

$$\Rightarrow \qquad au + av = (ax + by, ax + by, ax + by) \in S.$$

Thus, $au + bv \in S$ for all $u,\ v \in S$ and $a,b \in R$.
Hence, S is a subspace of V.

REMARK. *Geometrically S, in the above example, represents the line passing through the origin O and having direction ratios proportional to 1,1,1 as shown in the following figure.*



EXAMPLE-3 *Let a_1, a_2, a_3 be fixed elements of a field F. Then the set S of all triads (x_1, x_2, x_3) of elements of F, such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$, is a subspace of F^3.*

SOLUTION Let $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3)$ be any two elements of S. Then $x_1, x_2, x_3, y_1, y_2, y_3 \in F$ are such that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \quad \text{(i)}$$

and,

$$a_1y_1 + a_2y_2 + a_3y_3 = 0 \quad \text{(ii)}$$

Let a, b be any two elements of F. Then,

$$au + bv = a(x_1, x_2, x_3) + b(y_1, y_2, y_3) = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3).$$

Now,

$$\begin{aligned} & a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3) \\ &= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3) \\ &= a0 + b0 = 0 \end{aligned} \quad \text{[From (i) and (ii)]}$$

$\therefore \quad au + bv = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \in S.$

Thus, $au + bv \in S$ for all $u, v \in S$ and all $a, b \in F$. Hence, S is a subspace of F^3.

EXAMPLE-4 *Show that the set S of all $n \times n$ symmetric matrices over a field F is a subspace of the vector space $F^{n \times n}$ of all $n \times n$ matrices over field F.*

SOLUTION Note that a square matrix A is symmetric, if $A^T = A$.

Obviously, S is a non-void subset of $F^{n \times n}$, because the null matrix $0_{n \times n}$ is symmetric.

Let $A, B \in S$ and $\lambda, \mu \in F$. Then,

$$(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$$

$$\Rightarrow \quad (\lambda A + \mu B)^T = \lambda A + \mu B \quad [\because \ A, B \in S \ \therefore \ A^T = A, B^T = B]$$

$$\Rightarrow \quad \lambda A + \mu B \in S.$$

Thus, S is a non-void subset of $F^{n \times n}$ such that $\lambda A + \mu B \in S$ for all $A, B \in S$ and for all $\lambda, \mu \in F$.

Hence, S is a subspace of $F^{n \times n}$.

EXAMPLE-5 *Let V be a vector space over a field F and let $v \in V$. Then $F_v = \{av : a \in F\}$ is a subspace of V.*

SOLUTION Since $1 \in F$, therefore $v = 1\,v \in F_v$. Thus, F_v is a non-void subset of V.

Let $\alpha, \beta \in F_v$. Then, $\alpha = a_1 v, \ \beta = a_2 v$ for some $a_1, a_2 \in F$.

EXAMPLE-3 *Let a_1, a_2, a_3 be fixed elements of a field F. Then the set S of all triads (x_1, x_2, x_3) of elements of F, such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$, is a subspace of F^3.*

SOLUTION Let $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3)$ be any two elements of S. Then $x_1, x_2, x_3, y_1, y_2, y_3 \in F$ are such that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \quad \text{(i)}$$

and,

$$a_1y_1 + a_2y_2 + a_3y_3 = 0 \quad \text{(ii)}$$

Let a, b be any two elements of F. Then,

$$au + bv = a(x_1, x_2, x_3) + b(y_1, y_2, y_3) = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3).$$

Now,

$$\begin{aligned} & a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3) \\ &= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3) \\ &= a0 + b0 = 0 \qquad \text{[From (i) and (ii)]} \end{aligned}$$

$\therefore \quad au + bv = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \in S.$

Thus, $au + bv \in S$ for all $u, v \in S$ and all $a, b \in F$. Hence, S is a subspace of F^3.

EXAMPLE-4 *Show that the set S of all $n \times n$ symmetric matrices over a field F is a subspace of the vector space $F^{n \times n}$ of all $n \times n$ matrices over field F.*

SOLUTION Note that a square matrix A is symmetric, if $A^T = A$.

Obviously, S is a non-void subset of $F^{n\times n}$, because the null matrix $0_{n\times n}$ is symmetric.

Let $A, B \in S$ and $\lambda, \mu \in F$. Then,

$$\begin{aligned} & (\lambda A + \mu B)^T = \lambda A^T + \mu B^T \\ \Rightarrow \quad & (\lambda A + \mu B)^T = \lambda A + \mu B \qquad [\because \ A, B \in S \ \therefore \ A^T = A, B^T = B] \\ \Rightarrow \quad & \lambda A + \mu B \in S. \end{aligned}$$

Thus, S is a non-void subset of $F^{n\times n}$ such that $\lambda A + \mu B \in S$ for all $A, B \in S$ and for all $\lambda, \mu \in F$.

Hence, S is a subspace of $F^{n\times n}$.

EXAMPLE-5 *Let V be a vector space over a field F and let $v \in V$. Then $F_v = \{av : a \in F\}$ is a subspace of V.*

SOLUTION Since $1 \in F$, therefore $v = 1\ v \in F_v$. Thus, F_v is a non-void subset of V.

Let $\alpha, \beta \in F_v$. Then, $\alpha = a_1 v,\ \beta = a_2 v$ for some $a_1, a_2 \in F$.

Let $\lambda, \mu \in F$. Then,

$$\lambda\alpha + \mu\beta = \lambda(a_1 v) + \mu(a_2 v)$$

$$\Rightarrow \quad \lambda\alpha + \mu\beta = (\lambda a_1)v + (\mu a_2)v \qquad \text{[By } V\text{-2(iii)]}$$

$$\Rightarrow \quad \lambda\alpha + \mu\beta = (\lambda a_1 + \mu a_2)v \qquad \text{[By } V\text{-2(ii)]}$$

$$\Rightarrow \quad \lambda\alpha + \mu\beta \in F_v \qquad [\because \quad \lambda a_1 + \mu a_2 \in F]$$

Thus, F_v is a non-void subset of V such that $\lambda\alpha + \mu\beta \in F_v$ for all $\alpha, \beta \in F_v$ and for all $\lambda, \mu \in F$.

Hence, F_v is a subspace of V.

EXAMPLE-6 *Let $AX = O$ be homogeneous system of linear equations, where A is an $n \times n$ matrix over R. Let S be the solution set of this system of equations. Show S is a subspace of R^n.*

SOLUTION Every solution U of $AX = O$ may be viewed as a vector in R^n. So, the solution set of $AX = O$ is a subset of R^n. Clearly, $AX = O$. So, the zero vector $O \in S$. Consequently, S is non-empty subset of R^n.

Let $X_1, X_2 \in S$. Then,

X_1 and X_2 are solutions of $AX = O$

$\Rightarrow \; AX_1 = O$ and $AX_2 = O$ (i)

Let a, b be any two scalars in R. Then,

$$A(aX_1 + bX_2) = a(AX_1) + b(AX_2) = aO + bO = O \qquad \text{[Using(i)]}$$

$\Rightarrow \quad aX_1 + bX_2$ is a solution of $AX = O$

$\Rightarrow \quad aX_1 + bX_2 \in S.$

Thus, $aX_1 + bX_2 \in S$ for $X_1, X_2 \in S$ and $a, b \in R$.

Hence, S is a subspace of R^n.

REMARK. *The solution set of a non-homogeneous system $AX = B$ of linear equations in n unknowns is not a subspace of R^n, because zero vector O does not belong to its solution set.*

EXAMPLE-7 *Let V be the vector space of all real valued continuous functions over the field R of all real numbers. Show that the set S of solutions of the differential equation*

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$$

is a subspace of V.

SOLUTION We have,

$$S = \left\{ y : 2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0 \right\}, \qquad \text{where } y = f(x).$$

Now,

$$\begin{aligned} & (aa_1+ba_2)+(ab_1+bb_2)+2(ac_1+bc_2) \\ &= a(a_1+b_1+2c_1)+b(a_2+b_2+2c_2) \\ &= a\times 0+b\times 0=0 \qquad \text{[Using (i)]} \end{aligned}$$

$$\therefore \quad au+bv=(aa_1+ba_2,\ ab_1+bb_2,\ ac_1+bc_2)\in S$$

Thus, $au+bv\in S$ for all $u,v\in S$ and $a,b\in R$.

Hence, S is a subspace of R^3.

EXAMPLE-9 *Let S be the set of all elements of the form $(x+2y,\ y,\ -x+3y)$ in R^3, where $x,y\in R$. Show that S is a subspace of R^3.*

SOLUTION We have, $S=\{(x+2y,\ y,\ -x+3y): x,y\in R\}$

Let u,v be any two elements of S. Then,

$$u=(x_1+2y_1,y_1,-x_1+3y_1), v=(x_2+2y_2,y_2,-x_2+3y_2)$$

where $x_1,y_1,x_2,y_2\in R$.

Let a,b be any two elements of R. In order to prove that S is a subspace of V, we have to prove that $au+bv\in S$, for which we have to show that $au+bv$ is expressible in the form $(\alpha+2\beta,\beta,-\alpha+3\beta)$.

Now,

$$\begin{aligned} & au+bv=a(x_1+2y_1,y_1,-x_1+3y_1)+b(x_2+2y_2,y_2,-x_2+3y_2) \\ \Rightarrow\quad & au+bv=\big((ax_1+bx_2)+2(ay_1+by_2),ay_1+by_2,-(ax_1+bx_2)+3(ay_1+by_2)\big) \\ \Rightarrow\quad & au+bv=(\alpha+2\beta,\beta,-\alpha+3\beta), \quad \text{where } \alpha=ax_1+bx_2 \quad \text{and} \quad \beta=ay_1+by_2. \\ \Rightarrow\quad & au+bv\in S \end{aligned}$$

Thus, $au+bv\in S$ for all $u,v\in S$ and $a,b\in R$.

Hence, S is a subspace of R^3.

EXAMPLE-10 *Let V be the vector space of all 2×2 matrices over the field R of all real numbers. Show that:*

(i) the set S of all 2×2 singular matrices over R is not a subspace of V.

(ii) the set S of all 2×2 matrices A satisfying $A^2=A$ is not a subspace of V.

SOLUTION (i) We observe that $A=\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 0\\ b & 0\end{bmatrix}$, $a\neq 0\ b\neq 0$ are elements of S.

But, $A+B=\begin{bmatrix}0 & a\\ b & 0\end{bmatrix}$ is not an element of S as $|A+B|=-ab\neq 0$.

So, S is not a subspace of V.

(ii) We observe that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$ as $I^2 = I$. But, $2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin S$, because

$$(2I)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq 2I.$$

So, S is not a subspace of V.

SMALLEST SUBSPACE CONTAINING A GIVEN SUBSET. *Let V be a vector space over a field F, and let W be a subset of V. Then a subspace S of V is called the smallest subspace of V containing W, if*

(i) $W \subset S$

and (ii) S' is a subspace of V such that $W \subset S' \Rightarrow S \subset S'$.

The smallest subspace containing W is also called the subspace generated by W or subspace spanned by W and is denoted by $[W]$. If W is a finite set, then S is called a finitely generated space.

In Example 5, F_v is a finitely generated subspace of V containing $\{v\}$.

EXERCISE 2.3

1. Show that the set of all upper (lower) triangular matrices over field C of all complex numbers is a subspace of the vector space V of all $n \times n$ matrices over C.
2. Let V be the vector space of real-valued functions. Then, show that the set S of all continuous functions and set T of all differentiable functions are subspaces of V.
3. Let V be the vector space of all polynomials in indeterminate x over a field F and S be the set of all polynomials of degree at most n. Show that S is a subspace of V.
4. Let V be the vector space of all $n \times n$ square matrices over a field F. Show that:

   (i) the set S of all symmetric matrices over F is a subspace of V.

   (ii) the set S of all upper triangular matrices over F is a subspace of V.

   (iii) the set S of all diagonal matrices over F is a subspace of V.

   (iv) the set S of all scalar matrices over F is a subspace of V.

5. Let V be the vector space of all functions from the real field R into R. Show that S is a subspace of V where S consists of all:

   (i) bounded functions.

   (ii) even functions.

   (iii) odd functions.

6. Let V be the vector space R^3. Which of the following subsets of V are subspaces of V?

   (i) $S_1 = \{(a,b,c) : a+b=0\}$

   (ii) $S_2 = \{(a,b,c) : a=2b+1\}$

12. Which of the following sets of vectors $\alpha = (a_1, a_2, \ldots, a_n)$ in R^n are subspaces of R^n? $(n \geq 3)$.

    (i) all α such that $a_1 \geq o$

    (ii) all α such that $a_1 + 3a_2 = a_3$

    (iii) all α such that $a_2 = a_1^2$

    (iv) all α such that $a_1 a_2 = 0$

    (v) all α such that a_2 is rational.

13. Let V be the vector space of all functions from $R \to R$ over the field R of all real numbers. Show that each of the following subsets of V are subspaces of V.

    (i) $S_1 = \{f : f : R \to R \text{ satisfies } f(0) = f(1)\}$

    (ii) $S_2 = \{f : f : R \to R \text{ is continuous }\}$

    (iii) $S_3 = \{f : f : R \to R \text{ satisfies } f(-1) = 0\}$

14. Let V be the vector space of all functions from R to R over the field R of all real numbers. Show that each of the following subsets of V is not a subspace of V.

    (i) $S_1 = \left\{f : f \text{ satisfies } f(x^2) = \{f(x)\}^2 \text{ for all } x \in R\right\}$

    (ii) $S_2 = \{f : f \text{ satisfies } f(3) - f(-5) = 1\}$

15. Let $R^{n\times n}$ be the vector space of all $n \times n$ real matrices. Prove that the set S consisting of all $n \times n$ real matrices which commutate with a given matrix A in $R^{n\times n}$ form a subspace of $R^{n\times n}$.

16. Let C be the field of all complex numbers and let n be a positive integer such that $n \geq 2$. Let V be the vector space of all $n \times n$ matrices over C. Show that the following subsets of V are not subspaces of V.

    (i) $S = \{A : A \text{ is invertible}\}$ (ii) $S = \{A : A \text{ is not invertible}\}$

17. Prove that the set of vectors $(x_1, x_2, \ldots, x_n) \in R^n$ satisfying the m equations is a subspace of $R^n(R)$:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0,
\end{aligned}
$$

    aij 's are fixed reals.

If $S_1 \subset S_2$, then $S_1 \cup S_2 = S_2$, which is a subspace of V. Again, if $S_2 \subset S_1$, then $S_1 \cup S_2 = S_1$ which is also a subspace of V.

Hence, in either case $S_1 \cup S_2$ is a subspace of V.

Conversely, suppose that $S_1 \cup S_2$ is a subspace of V. Then we have to prove that either $S_1 \subset S_2$ or $S_2 \subset S_1$.
If possible, let $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$. Then,

$$S_1 \not\subset S_2 \Rightarrow \text{ There exists } u \in S_1 \text{ such that } u \notin S_2$$

$$\text{and, } S_2 \not\subset S_1 \Rightarrow \text{ There exists } v \in S_2 \text{ such that } v \notin S_1.$$

Now,

$$u \in S_1, v \in S_2$$

$$\Rightarrow \quad u, v \in S_1 \cup S_2$$

$$\Rightarrow \quad u + v \in S_1 \cup S_2 \qquad [\because \ S_1 \cup S_2 \text{ is a subspace of } V]$$

$$\Rightarrow \quad u + v \in S_1 \text{ or, } u + v \in S_2.$$

$$\text{If} \quad u + v \in S_1$$

$$\Rightarrow \quad (u + v) - u \in S_1 \qquad [\because u \in S_1 \text{ and } S_1 \text{ is a subspace of } V]$$

$$\Rightarrow \quad v \in S_1, \text{ which is a contradiction} \qquad [\because v \notin S_1]$$

$$\text{If} \quad u + v \in S_2$$

$$\Rightarrow \quad (u + v) - v \in S_2 \qquad [\because v \in S_2 \text{ and } S_2 \text{ is a subspace of } V]$$

$$\Rightarrow \quad u \in S_2, \text{ which is a contradiction} \qquad [\because u \notin S_2]$$

Since the contradictions arise by assuming that $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$.

Hence, either $S_1 \subset S_2$ or, $S_2 \subset S_1$.

Q.E.D.

REMARK. *In general, the union of two subspaces of a vector space is not necessarily a subspace. For example,* $S_1 = \{(a, 0, 0) : a \in R\}$ *and* $S_2 = \{(0, b, 0) : b \in R\}$ *are subspaces of* R^3. *But, their union* $S_1 \cup S_2$ *is not a subspace of* R^3, *because* $(2, 0, 0), (0, -1, 0) \in S_1 \cup S_2$, *but* $(2, 0, 0) + (0, -1, 0) = (2, -1, 0) \notin S_1 \cup S_2$.

As remarked above the union of two subspaces is not always a subspace. Therefore, unions are not very useful in the study of vector spaces and we shall say no more about them.

THEOREM-4 *The inter section of a family of subspaces of* $V(F)$ *containing a given subset S of V is the (smallest subspace of V containing S) subspace generated by S.*

PROOF. Let $\{S_i : i \in I\}$ be a family of subspaces of $V(F)$ containing a subset S of V. Let $T = \bigcap_{i \in I} S_i$. Then by Theorem 2, T is a subspace of V. Since $S_i \supset S$ for all i. Therefore, $T = \bigcap_{i \in I} S_i \supset S$.

Hence, T is a subspace of V containing S.

Now it remains to show that T is the smallest subspace of V containing S.

Let W be a subspace of V containing S. Then W is one of the members of the family $\{S_i : i \in I\}$. Consequently, $W \supset \bigcap_{i\in I} S_i = T$.

Thus, every subspace of V that contains S also contains T. Hence, T is the smallest subspace of V containing S.

Q.E.D.

THEOREM-5 *Let $A = \{v_1, v_2, \ldots, v_n\}$ be a non-void finite subset of a vector space $V(F)$. Then the set $S = \left\{ \sum_{i=1}^{n} a_i v_i : a_i \in F \right\}$ is the subspace of V generated by A, i.e. $S = [A]$.*

PROOF. In order to prove that S is a subspace of V generated by A, it is sufficient to show that S is the smallest subspace of V containing A.

We have, $v_i = 0 \cdot v_1 + 0v_2 + \cdots + 0v_{i-1} + 1 \cdot v_i + 0v_{i+1} + \cdots + 0v_n$

So, $v_i \in S$ for all $i \in \underline{n}$

$\Rightarrow \quad A \subset S.$

Let $u = \sum_{i=1}^{n} a_i v_i,\ v = \sum_{i=1}^{n} b_i v_i \in S$, and let $\lambda, \mu \in F$. Then,

$$\lambda u + \mu v = \lambda \left(\sum_{i=1}^{n} a_i v_i \right) + \mu \left(\sum_{i=1}^{n} b_i v_i \right)$$

$$\Rightarrow \quad \lambda u + \mu v = \sum_{i=1}^{n} (\lambda a_i) v_i + \sum_{i=1}^{n} (\mu b_i) v_i \qquad \text{[By } V\text{-2(i) and } V\text{-2(iii)]}$$

$$\Rightarrow \quad \lambda u + \mu v = \sum_{i=1}^{n} (\lambda a_i + \mu b_i) v_i \in S \qquad [\because\ \lambda a_i + \mu b_i \in F \text{ for all } i \in \underline{n}]$$

Thus, $\lambda u + \mu v \in S$ for all $u, v \in S$ and for all $\lambda, \mu \in F$.

So, S is a subspace of V containing A.

Now, let W be a subspace of V containing A. Then, for each $i \in \underline{n}$

$$a_i \in F, v_i \in A$$

$$\Rightarrow \quad a_i \in F, v_i \in W \qquad [\because\ A \subset W]$$

$$\Rightarrow \quad a_i v_i \in W \qquad [\because\ W \text{ is a subspace of } V]$$

$$\Rightarrow \quad \sum_{i=1}^{n} a_i v_i \in W$$

Thus, $S \subset W$.

Hence, S is the smallest subspace of V containing A.

Q.E.D.

REMARK. *If φ is the void set, then the smallest subspace of V containing φ is the null space $\{0_V\}$. Therefore, the null space is generated by the void set.*

SUM OF SUBSPACES. *Let S and T be two subspaces of a vector space $V(F)$. Then their sum (linear sum) $S + T$ is the set of all sums $u + v$ such that $u \in S, v \in T$.*

Thus, $S + T = \{u + v : U \in S, v \in T\}$.

THEOREM-6 *Let S and T be two subspaces of a vector space $V(F)$. Then,*

(i) $S+T$ is a subspace of V.

(ii) $S+T$ is the smallest subspace of V containing $S\cup T$, i.e. $S+T=[S\cup T]$.

PROOF. (i) Since S and T are subspaces of V. Therefore,

$$0_V \in S,\ 0_V \in T \Rightarrow 0_V = 0_V + 0_V \in S+T.$$

So, $S+T$ is a non-void subset of V.
Let $u=u_1+u_2, v=v_1+v_2 \in S+T$ and let $\lambda,\mu \in F$. Then, $u_1,v_1 \in S$ and $u_2,v_2 \in T$.
Now,

$$\lambda u+\mu v=\lambda(u_1+u_2)+\mu(v_1+v_2)$$
$$\Rightarrow \quad \lambda u+\mu v=(\lambda u_1+\lambda u_2)+(\mu v_1+\mu v_2) \qquad \text{[By } V\text{-2(i)]}$$
$$\Rightarrow \quad \lambda u+\mu v=(\lambda u_1+\mu v_1)+(\lambda u_2+\mu v_2) \qquad \text{[Using comm. and assoc. of vector addition]}$$

Since S and T are subspaces of V. Therefore,

$$u_1,v_1 \in S \text{ and } \lambda,\mu \in F \Rightarrow \lambda u_1+\mu v_1 \in S$$
$$\text{and,} \quad u_2,v_2 \in T \text{ and } \lambda,\mu \in F \Rightarrow \lambda u_2+\mu v_2 \in T.$$

Thus,

$$\lambda u+\mu v=(\lambda u_1+\mu v_1)+(\lambda u_2+\mu v_2) \in S+T.$$

Hence, $S+T$ is a non-void subset of V such that $\lambda u+\mu v \in S+T$ for all $u,v \in S+T$ and $\lambda,\mu \in F$. Consequently, $S+T$ is a subspace of V.

(ii) Let W be a subspace of V such that $W \supset (S\cup T)$.
Let $u+v$ be an arbitrary element of $S+T$. Then $u \in S$ and $v \in T$.
Now,

$$u \in S, v \in T \Rightarrow u \in S\cup T, v \in S\cup T \Rightarrow u,v \in W \qquad [\because\ W \supset S\cup T]$$
$$\Rightarrow u+v \in W \qquad [\because\ W \text{ is a subspace of } V]$$

Thus, $\quad u+v \in S+T \Rightarrow u+v \in W$

$\Rightarrow \quad S+T \subset W$.
Obviously, $S\cup T \subset S+T$.
Thus, if W is a subspace of V containing $S\cup T$, then it contains $S+T$. Hence, $S+T$ is the smallest subspace of V containing $S\cup T$. Q.E.D.

DIRECT SUM OF SUBSPACES. *A vector space $V(F)$ is said to be direct sum of its two subspaces S and T if every vector u in V is expressible in one and only one as $u=v+w$, where $v \in S$ and $w \in T$.*

If V is direct sum of S and T, then we write $V=S\oplus T$.

EXAMPLE-4 *Consider the vector space $V = R^3$. Let $S = \{(x, y, 0) : x, y \in R\}$ and $T = \{(x, 0, z) : x, z \in R\}$. Prove that S and T are subspaces of V such that $V = S + T$, but V is not the direct sum of S and T.*

SOLUTION Clearly, S and T represent xy-plane and xz-plane respectively. It can be easily checked that S and T are subspaces of V.

Let $v = (x, y, z)$ be an arbitrary point in V. Then,

$$v = (x, y, z) = (x, y, 0) + (0, 0, z)$$

$$\Rightarrow \quad v \in S + T \qquad [\because (x, y, 0) \in S \text{ and } (0, 0, z) \in T]$$

$$\therefore \quad V \subset S + T$$

Also, $S + T \subset V$.

Hence, $V = S + T$ i.e. R^3 is a sum of the xy-plane and the yz-plane.

Clearly, $S \cap T = \{(x, 0, 0) : x \in R\}$ is the x-axis.

$$\therefore \quad S \cap T \neq \{(0, 0, 0)\}.$$

So, V is not the direct sum of S and T.

COMPLEMENT OF A SUBSPACE. *Let $V(F)$ be a vector space and let S be a subspace of V. Then a subspace T of V is said to be complement of S, if $V = S \oplus T$.*

INDEPENDENT SUBSPACES. *Let $S_1, S_2, \ldots, S_n$ be subspaces of a vector space V. Then $S_1, S_2, \ldots, S_n$ are said to be independent, if*

$$S_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j = \{0_V\} \text{ for all } i = 1, 2, \ldots, n.$$

THEOREM-8 *Let $S_1, S_2, \ldots, S_n$ be subspaces of a vector space $V(F)$ such that $V = S_1 + S_2 + \cdots + S_n$. Then, the following are equivalent:*

(i) $V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$.

(ii) $S_1, S_2, \ldots, S_n$ *are independent.*

(iii) For each $v_i \in S_i$, $\sum_{i=1}^{n} v_i = 0_V \Rightarrow v_i = 0_V$.

PROOF. Let us first prove that (i) implies (ii).

Let $u \in S_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j$. Then there exist $v_1 \in S_1, v_2 \in S_2, \ldots, v_i \in S_i, \ldots, v_n \in S_n$ such that

$$u = v_i = v_1 + v_2 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_n$$

$$\Rightarrow \quad u = v_1 + v_2 + \cdots + v_{i-1} + 0_V + v_{i+1} + \cdots + v_n = 0v_1 + 0v_2 + \cdots + 1v_i + \cdots + 0v_n$$

$$\Rightarrow \quad v_1 = 0_V = v_2 = \cdots = v_i = \cdots = v_n \qquad [\because V = S_1 \oplus S_2 \oplus \cdots \oplus S_n]$$

$$\Rightarrow \quad u = 0_V$$

Thus, $u \in S_i \cap \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} S_j \Rightarrow u = 0_V$.

Hence, $S_i \cap \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} S_j = 0_V$. Consequently, $S_1, S_2, \ldots, S_n$ are independent.

(ii) $\Rightarrow$(iii).

Let $v_1 + v_2 + \cdots + v_n = 0_V$ with $v_i \in S_i, i \in \underline{n}$. Then,

$$\Rightarrow \quad v_i = -(v_1 + v_2 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_n)$$

$$\Rightarrow \quad v_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} v_j.$$

Since $v_i \in S_i$ and $-\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} v_j \in \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} S_j$. Therefore,

$$v_i \in S_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j \quad \text{for all } i \in \underline{n}.$$

$$\Rightarrow \quad v_i = 0_V \quad \text{for all } i \in \underline{n} \qquad [\because S_1, S_2, \ldots, S_n \text{ are independent}]$$

(iii) $\Rightarrow$ (i)

Since $V = S_1 + S_2 + \cdots + S_n$ is given. Therefore, it is sufficient to show that each $v \in V$ has a unique representation as a sum of vectors in $S_1, S_2, \ldots, S_n$.

Let, if possible, $v = v_1 + \cdots + v_n$ and $v = w_1 + \cdots + w_n$ be two representations for $v \in V$. Then,

$$v_1 + v_2 + \cdots + v_n = w_1 + w_2 + \cdots + w_n$$

$$\Rightarrow \quad (v_1 - w_1) + \cdots + (v_n - w_n) = 0_V$$

$$\Rightarrow \quad v_1 - w_1 = 0_V, \ldots, v_n - w_n = 0_V \qquad \text{[Using (iii)]}$$

$$\Rightarrow \quad v_1 = w_1, \ldots, v_n = w_n.$$

Thus, each $v \in V$ has a unique representation. Hence, $V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$.

Consequently, (iii) $\Rightarrow$(i).

Hence, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) i.e. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Q.E.D.

LINEAR VARIETY. *Let S be a subspace of a vector space $V(F)$, and let $v \in V$. Then,*

$$v + S = \{v + u : u \in S\}$$

is called a linear variety of S by v or a translate of S by v or a parallel of S through v.

S is called the base space of the linear variety and v a leader.

THEOREM-9 *Let S be a subspace of a vector space $V(F)$ and let $P = v + S$ be the parallel of S through v. Then,*

(i) every vector in P can be taken as a leader of P, i.e $u + S = P$ for all $u \in P$.

(ii) two vectors $v_1, v_2 \in V$ are in the same parallel of S iff $v_1 - v_2 \in S$

PROOF. (i) Let u be an arbitrary vector in $P = v + S$. Then there exists $v_1 \in S$ such that

$$u = v + v_1. \Rightarrow v = u - v_1.$$

Let w be an arbitrary vector in P. Then,

$$w = v + v_2 \quad \text{for some } v_2 \in S.$$

$$\Rightarrow \quad w = (u - v_1) + v_2 \qquad [\because v = u - v_1]$$

$$\Rightarrow \quad w = u + (v_2 - v_1) \qquad [\because v_2 - v_1 \in S]$$

$$\Rightarrow \quad w \in u + S$$

Thus, $P \subset u + S$.

Now let $w' \in u + S$. Then,

$$w' = u + v_3 \quad \text{for some } v_3 \in S.$$

$$\Rightarrow \quad w' = (v + v_1) + v_3 \qquad [\because v = u - v_1]$$

$$\Rightarrow \quad w' = v + (v_1 + v_3)$$

$$\Rightarrow \quad w' \in v + S \qquad [\because v_1 + v_3 \in S]$$

Thus, $u + S \subset P$. Hence, $u + S = P$

Since u is an arbitrary vector in P. Therefore, $u + S = P$ for all $u \in P$.

(ii) Let v_1, v_2 be in the same parallel of S, say $u + S$. Then there exist $u_1, u_2 \in S$ such that

$$v_1 = v + u_1, v_2 = v + u_2$$

$$\Rightarrow \quad v_1 - v_2 = u_1 - u_2 \Rightarrow v_1 - v_2 \in S \qquad [\because u_1, u_2 \in S \Rightarrow u_1 - u_2 \in S]$$

Conversely, if $v_1 - v_2 \in S$, then there exists $u \in S$ such that

$$v_1 - v_2 = u$$

$$\Rightarrow \quad v_1 = v_2 + u$$

$$\Rightarrow \quad v_1 \in v_2 + S. \qquad [\because u \in S]$$

Also, $v_2 = v_2 + 0_V$ where $0_V \in S$

$\Rightarrow \quad v_2 \in v_2 + S$.

Hence, v_1, v_2 are in the same parallel of S.

Q.E.D.

EXERCISE 2.4

1. Let $V = R^3$, $S = \{(0,\ y,\ z) : y,\ z \in R\}$ and $T = \{(x,\ 0,\ z) : x,\ \ z \in R\}$
   Show that S and T are subspaces of V such that $V = S + T$. But, V is not the direct sum of S and T.
2. Let $V(R)$ be the vector space of all 2×2 matrices over R. Let S be the set of all symmetric matrices in V and T be the set of all skew-symmetric matrices in V. Show that S and T are subspaces of V such that $V = S \oplus T$.
3. Let $V = R^3$, $S = \{(x,\ y,\ 0) : x,\ y \in R\}$ and $T = \{(0, 0, z) : z \in R\}$.
   Show that S and T are subspaces of V such that $V = S \oplus T$.
4. Let $S_1,\ S_2,\ S_3$ be the following subspaces of vector space $V = R^3$:
   $$S_1 = \{(x,\ y,\ z) :\ x = z,\ x,\ y,\ z \in R\}$$
   $$S_2 = \{(x,\ y,\ z) :\ x + y + z = 0,\ x,\ y,\ z \in R\}$$
   $$S_3 = \{(0,\ 0,\ z) :\ z \in R\}$$
   Show that: (i) $V = S_1 + S_2$ (ii) $V = S_2 + S_3$ (iii) $V = S_1 + S_3$ When is the sum direct?
5. Let S and T be the two-dimensional subspaces of R^3. Show that $S \cap T \neq \{0\}$.
6. Let $S,\ S_1,\ S_2$ be the subspaces of a vector space $V(F)$. Show that
   $$(S \cap S_1) + (S \cap S_2) \subseteq S \cap (S_1 + S_2)$$
   Find the subspaces of R^2 for which equality does not hold.
7. Let V be the vector space of $n \times n$ square matrices. Let S be the subspace of upper triangular matrices, and let T be the subspaces of lower triangular matrices.
   Find (i) $S \cap T$ (ii) $S + T$.
8. Let F be a subfield of the field C of complex numbers and let V be the vector space of all 2×2 matrices over F. Let $S = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x,\ y,\ z \in F \right\}$ and $T = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x,\ y \in F \right\}$.
   Show that S and T are subspaces of V such that $V = S + T$ but $V \neq S \oplus T$.
   $\left[\text{Hint: } S \cap T = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in F \right\} \quad \therefore \quad S \cap T \neq \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \right]$
9. Let S and T be subspaces of a vector space $V(F)$ such that $V = S + T$ and $S \cap T = \{0_V\}$. Prove that each vector v in V there are unique vectors v_1 in S and v_2 in T such that $v = v_1 + v_2$.
10. Let U and W be vector spaces over a field F and let $V = U \times W = \{(u,\ w) : u \in U$ and $w \in W\}$. Then V is a vector space over field F with addition in V and scalar multiplication on V defined by
    $$(u_1,\ w_1) + (u_2,\ w_2) = (u_1 + u_2,\ w_1 + w_2)$$
    and, $$k(u,\ w) = (ku,\ kw).$$
    Let $S = \{(u,\ 0) : u \in U\}$ and $T = \{(0,\ w) : w \in W\}$. Show that

Since S is a subgroup of additive abelian group V. Therefore, for any $u+S, v+S \in V/S$

$$u+S = v+S \iff u-v \in S$$

Thus, we have defined vector addition, scalar multiplication on V/S and equality of any two elements of V/S. Now we proceed to prove that V/S is a vector space over field F under the above defined operations of addition and scalar multiplication.

THEOREM-1 *Let V be a vector space over a field F and let S be a subspace of V. Then, the set*

$$V/S = \{u+S : u \in V\}$$

is a vector space over field F for the vector addition and scalar multiplication on V/S defined as follows:

$$(u+S)+(v+S) = (u+v)+S$$

and, $\quad a(u+S) = au+S \quad$ *for all $u+S,\ v+S \in V/S$ and for all $a \in F$.*

PROOF. First we shall show that above rules for vector addition and scalar multiplication are well defined, i.e. are independent of the particular representative chosen to denote a coset.

Let $\quad u+S = u'+S,$ where $u, u' \in V$ and, $v+S = v'+S,$ where $v, v' \in V.$

Then,

$$u+S = u'+S \Rightarrow \ u-u' \in S \text{ and, } v+S = v'+S \Rightarrow \ v-v' \in S.$$

Now,

$$u-u' \in S, v-v' \in S$$

$\Rightarrow \quad (u-u')+(v-v') \in S \qquad [\because\ S$ is a subspace of $V]$

$\Rightarrow \quad (u+v)-(u'+v') \in S$

$\Rightarrow \quad (u+v)+S = (u'+v')+S$

$\Rightarrow \quad (u+S)+(v+S) = (u'+S)(v'+S)$

Thus, $\quad u+S = u'+S$ and $v+S = v'+S$

$\Rightarrow \ (u+S)+(v+S) = (u'+S)+(v'+S)$ for all $u,\ u',\ v,\ v' \in V.$

Therefore, vector addition on V/S is well defined.

Again,

$$\lambda \in F,\ u-u' \in S \ \Rightarrow\ \lambda(u-u') \in S \Rightarrow\ \lambda u - \lambda u' \in S \ \Rightarrow\ \lambda u + S = \lambda u' + S$$

Therefore, scalar multiplication on V/S is well defined.

$V-1$: *V/S is an abelian group under vector addition*:

Associativity: For any $u+S, v+S, w+S \in V/S$, we have

$$\begin{aligned}
[(u+S)+(v+S)]+(w+S) &= ((u+v)+S)+(w+S) \\
&= \{(u+v)+w\}+S \\
&= \{u+(v+w)\}+S \qquad \text{[By associativity of vector addition on } V\text{]} \\
&= (u+S)+((v+w)+S) \\
&= (u+S)+[(v+S)+(w+S)]
\end{aligned}$$

So, vector addition is associative on V/S.

Commutativity: For any $u+S, v+S \in V/S$, we have

$$\begin{aligned}
(u+S)+(v+S) &= (u+v)+S \\
&= (v+u)+S \qquad \text{[By commutativity of vector addition on } V\text{]} \\
&= (v+S)+(u+S)
\end{aligned}$$

So, vector addition is commutative on V/S.

Existence of additive identity: Since $0_V \in V$ therefore, $0_V+S=S \in V/S$.

Now,

$$(u+S)+(0_V+S) = (u+0_V)+S = (u+S) \qquad \text{for all } u+S \in V/S$$

$$\therefore \quad (u+S)+(0_V+S) = (u+S) = (0_V+S)+(u+S) \quad \text{for all } u+S \in V/S.$$

So $0_V+S=S$ is the identity element for vector addition on V/S.

Existence of additive inverse: Let $u+S$ be an arbitrary element of V/S. Then,

$$u \in V \Rightarrow \; -u \in V \Rightarrow \; (-u)+S \in V/S$$

Thus, for each $u+S \in V/S$ there exists $(-u)+S \in V/S$ such that

$$(u+S)+((-u)+S) = [u+(-u)]+S = 0_V+S = S$$

$$\Rightarrow (u+S)+((-u)+S) = 0_V+S = (-u)+S+(u+S) \; \text{[By commutativity of addition on } V/S\text{]}$$

So, each $u+S \in V/S$ has its additive inverse $(-u)+S \in V/S$.

Hence, V/S is an abelian group under vector addition.

$V-2$: *For any* $u+S,\ v+S \in V/S$ *and* $\lambda, \mu \in F$, *we have*

(i) $\lambda[(u+S)+(v+S)] = \lambda[(u+v)+S]$

$= [\lambda(u+v)]+S$

$= (\lambda u+\lambda v)+S$ [By V-2(i) in V]

$= (\lambda u+S)+(\lambda v+S)$ [By definition of addition on V/S]

$= \lambda(u+S)+\lambda(v+S)$ [By def. of scalar multiplication on V/S]

(ii) $(\lambda+\mu)(u+S) = (\lambda+\mu)u+S$ [By def. of scalar multiplication on V/S]

$= (\lambda u+\mu u)+S$ [By V-2(ii) in V]

$= (\lambda u+S)+(\mu u+S)$ [By definition of addition on V/S]

$= \lambda(u+S)+\mu(u+S)$

(iii) $(\lambda\mu)(u+S) = (\lambda\mu)u+S$

$= \lambda(\mu u)+S$ [By V-2(iii) in V]

$= \lambda\,(\mu u+S)$

$= \lambda[\mu(u+S)]$

(iv) $1(u+S) = 1u+S = u+S$ [By V-2(iv) in V]

Hence, V/S is a vector space over field F.

Q.E.D.

QUOTIENT SPACE. *Let V be a vector space over a field F and let S be a subspace of V. Then the set $V/S = \{u+S : u \in V\}$ is a vector space over field F for the vector addition and scalar multiplication defined as*

$$(u+S)+(v+S) = (u+v)+S$$
$$a(u+S) = au+S$$

For all $u+S, v+S \in V/S$ and for all $a \in F$.

This vector space is called a quotient space of V by S.

EXERCISE 2.5

1. If S is a subspace of a vector space $V(F)$, then show that there is one-one correspondence between the subspaces of V containing S and subspaces of the quotient space V/S.
2. Define quotient space.

2.6 LINEAR COMBINATIONS

LINEAR COMBINATION. *Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n$ be n vectors in V and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n scalars in F. Then the vector $\lambda_1 v_1 + \cdots + \lambda_n v_n$ (or $\sum_{i=1}^{n} \lambda_i v_i$) is called a linear combination of $v_1, v_2, \ldots, v_n$. It is also called a linear combination of the set $S = \{v_1, v_2, \ldots, v_n\}$. Since there are finite number of vectors in S, it is also called a finite linear combination of S.*

If S is an infinite subset of V, then a linear combination of a finite subset of S is called a finite linear combination of S.

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Express $v = (-2, 3)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (1, 1)$ and $v_2 = (1, 2)$.*

SOLUTION Let x, y be scalars such that

$$v = xv_1 + yv_2$$

$$\Rightarrow \quad (-2, 3) = x(1, 1) + y(1, 2)$$

$$\Rightarrow \quad (-2, 3) = (x + y, x + 2y)$$

$$\Rightarrow \quad x + y = -2 \text{ and } x + 2y = 3$$

$$\Rightarrow \quad x = -7, y = 5$$

Hence, $v = -7v_1 + 5v_2$.

EXAMPLE-2 *Express $v = (-2, 5)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (-1, 1)$ and $v_2 = (2, -2)$.*

SOLUTION Let x, y be scalars such that

$$v = xv_1 + yv_2$$

$$\Rightarrow \quad (-2, 5) = x(-1, 1) + y(2, -2)$$

$$\Rightarrow \quad (-2, 5) = (-x + 2y, x - 2y)$$

$$\Rightarrow \quad -x + 2y = -2 \text{ and } x - 2y = 5$$

This is an inconsistent system of equations and so it has no solution.

Hence, v cannot be written as the linear combination of v_1 and v_2.

EXAMPLE-3 *Express $v = (1, -2, 5)$ in R^3 as a linear combination of the following vectors:*

$$v_1 = (1, 1, 1), v_2 = (1, 2, 3), v_3 = (2, -1, 1)$$

This is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ Applying } R_2 \to R_2 + 3R_1, R_3 \to R_3 + (-2)R_1$$

$$\text{or, } \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix} \text{ Applying } R_3 \to R_3 + \frac{5}{2}R_2$$

This is an inconsistent system of equations and so has no solution. Hence, v cannot be written as a linear combination of v_1, v_2 and v_3.

EXAMPLE-5 *Express the polynomial $f(x) = x^2 + 4x - 3$ in the vector space V of all polynomials over R as a linear combination of the polynomials $g(x) = x^2 - 2x + 5, h(x) = 2x^2 - 3x$ and $\phi(x) = x + 3$.*

SOLUTION Let α, β, γ be scalars such that

$$f(x) = u\, g(x) + v\, h(x) + w\, \phi(x) \quad \text{for all } x \in R$$

$$\Rightarrow \quad x^2 + 4x - 3 = u(x^2 - 2x + 5) + v(2x^2 - 3x) + w(x + 3 \quad \text{for all } x \in R$$

$$\Rightarrow \quad x^2 + 4x - 3 = (u + 2v)x^2 + (-2u - 3v + w)x + (5u + 3w) \quad \text{for all } x \in R \qquad \text{(i)}$$

$$\Rightarrow \quad u + 2v = 1, -2u - 3v + w = 4, 5u + w = -3 \qquad \text{(ii)}$$

The matrix form of this system of equations is

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

This system of equations is equivalent to

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -10 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -8 \end{bmatrix} \text{ Applying } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 5R_1$$

$$\text{or, } \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 52 \end{bmatrix} \text{ Applying } R_3 \to R_3 + 10R_2$$

Thus, the system of equations obtained in (i) is consistent and is equivalent to

$$u + 2v = 1, v + w = 6 \text{ and } 13w = 52 \Rightarrow \quad u = -3, v = 2, w = 4$$

Hence, $$f(x) = -3g(x) + 2h(x) + 4\phi(x).$$

REMARK. *The equation (i) obtained in the above solution is an identity in x, that is it holds for any value of x. So, the values of u, v and w can be obtained by solving three equations which can be obtained by given any three values to variable x.*

SOLUTION Let x, y be scalars such that

$$u = xv_1 + yv_2$$
$$\Rightarrow \quad (1,k,4) = x(1,2,3) + y(2,3,1)$$
$$\Rightarrow \quad (1,k,4) = (x+2y, 2x+3y, 3x+y)$$
$$\Rightarrow \quad x+2y = 1, 2x+3y = k \text{ and } 3x+y = 4$$

Solving $x+2y=1$ and $3x+y=4$, we get $x = \frac{7}{5}$ and $y = -\frac{1}{5}$

Substituting these values in $2x+3y=k$, we get $k = \frac{11}{5}$

EXAMPLE-8 *Consider the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,1)$ in $R^3(R)$. Find conditions on a,b,c so that $u = (a,b,c)$ is a linear combination of v_1 and v_2.*

SOLUTION Let x, y be scalars in R such that

$$u = xv_1 + yv_2$$
$$\Rightarrow \quad (a,b,c) = x(1,2,3) + y(2,3,1)$$
$$\Rightarrow \quad (a,b,c) = (x+2y, 2x+3y, 3x+y)$$
$$\Rightarrow \quad x+2y = a, 2x+3y = b \text{ and } 3x+y = c$$

Solving first two equations, we get $x = -3a+2b$ and $y = 2a-b$.

Substituting these values in $3x+y=c$, we get

$3(-3a+2b)+(2a-b) = c$ or, $7a-5b+c=0$ as the required condition.

EXERCISE 2.6

1. Express $v = (3,-2)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (-1,1)$ and $v_2 = (2,1)$.
2. Express $v = (-2,5)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (2,-1)$ and $v_2 = (-4,2)$.
3. Express $v = (3,7,-4)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (1,2,3)$, $v_2 = (2,3,7)$ and $v_3 = (3,5,6)$.
4. Express $v = (2,-1,3)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (3,0,3)$, $v_2 = (-1,2,-5)$ and $v_3 = (-2,-1,0)$.
5. Let V be the vector space of all real polynomials over field R of all real numbers. Express the polynomial $p(x) = 3x^2+5x-5$ as a linear combination of the polynomials $f(x) = x^2+2x+1, g(x) = 2x^2+5x+4$ and $h(x) = x^2+3x+6$.
6. Let $V = P_2(t)$ be the vector space of all polynomials of degree less than or equal to 2 and t be the indeterminate. Write the polynomial $f(t) = at^2+bt+c$ as a linear combination of the polynomials $p_1(t) = (t-1)^2, p_2(t) = t-1$ and $p_3(t) = 1$.
7. Write the vectors $u = (1,3,8)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,1)$.

8. Write the vector $u = (2,4,5)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,1)$.
9. Consider the vector space $R^{2\times 2}$ of all 2×2 matrices over the field R of real numbers. Express the matrix $M = \begin{bmatrix} 5 & -6 \\ 7 & 8 \end{bmatrix}$ as a linear combination of the matrices

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

ANSWERS

1. $v = \dfrac{-7}{3}v_1 + \dfrac{1}{3}v_2$ 2. v is not expressible as a linear combination of v_1 and v_2.
3. $v = 2v_1 - 4v_2 + 3v_3$ 4. v is not expressible as a linear combination of v_1, v_2, v_3.
5. $p(x) = 3f(x) + g(x) - 2h(x)$ 6. $f(t) = ap_1(t) + (2a+b)p_2(t) + (a+b+c)p_3(t)$
7. $u = 3v_1 - v_2$ 8. Not Possible 9. $M = 5E_{11} - 6E_{12} + 7E_{21} + 8E_{22}$

2.7 LINEAR SPANS

LINEAR SPAN OF A SET. *Let V be a vector space over a field F and let S be a subset of V. Then the set of all finite linear combinations of S is called linear span of S and is denoted by $[S]$.*

Thus,

$$[S] = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_1, \lambda_2, \ldots, \lambda_n \in F,\ n \in N \text{ and } v_1, v_2, \ldots, v_n \in S \right\}.$$

If S is a finite set, say $S = \{v_1, v_2, \ldots, v_n\}$, then

$$[S] = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in F \right\}.$$

In this case $[S]$ is also written as $[v_1, v_2, \ldots, v_n]$.

If $S = \{v\}$, then $[S] = F_v$ as shown in Example 5 on page 136.

Consider a subset $S = \{(1,0,0),(0,1,0)\}$ of real vector space R^3. Any linear combination of a finite number of elements of S is of the form $\lambda(1,0,0) + \mu(0,1,0) = (\lambda, \mu, 0)$. By definition, the set of all such linear combinations is $[S]$. Thus, $[S] = \{(\lambda, \mu, 0) : \lambda, \mu \in R\}$. Clearly $[S]$ is xy-plane in three dimensional Euclidean space R^3 and is a subspace of R^3. In fact, this is true for every non-void subset of a vector space. The following theorem proves this result.

THEOREM-1 *Let S be a non-void subset of a vector space $V(F)$. Then $[S]$, the linear span of S, is a subspace of V.*

PROOF. Since $0_V = 0v_1 + 0v_2 + \cdots + 0v_n$ for every positive integer n. Therefore, 0_V is a finite linear combination of S and so it is in $[S]$.

Thus, $[S]$ is a non-void subset of V.

Let u and v be any two vectors in $[S]$. Then,

$u = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n$ for some scalars $\lambda_i \in F$, some $u_i^{'s} \in S$, and a positive integer n.
and, $v = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_m v_m$ for some $\mu_i^{'s} \in F$, some $v_i^{'s} \in S$, and a positive integer m.

If λ, μ are any two scalars in F, then

$$\lambda u + \mu v = \lambda(\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n) + \mu(\mu_1 v_1 + \cdots + \mu_m v_m)$$

$$\Rightarrow \quad \lambda u + \mu v = (\lambda\lambda_1) u_1 + \cdots + (\lambda\lambda_n) u_n + (\mu\mu_1) v_1 + \cdots + (\mu\mu_m) v_m$$

Clearly, $\lambda u + \mu v$ is a finite linear combination of vectors in S and so it is in $[S]$.

Thus, $\lambda u + \mu v \in [S]$ for all $u, v \in [S]$ and for all $\lambda, \mu \in F$.

Hence, $[S]$ is a subspace of V.

Q.E.D.

THEOREM-2 *Let S be a subspace of a vector space $V(F)$. Then $[S]$ is the smallest subspace of V containing S.*

PROOF. By Theorem 1, $[S]$ is a subspace of V. Let v be an arbitrary vector in S. Then,

$$v = 1v \qquad \text{[By } V\text{-2(iv)]}$$

$\Rightarrow \quad v$ is a finite linear combination of S.

$\Rightarrow \quad v \in [S]$.

Thus, $S \subset [S]$.

Now it remains to prove that $[S]$ is the smallest subspace of V containing S. For this, we shall show that if there exists another subspace T containing S, then it also contains $[S]$. So let T be a subspace of V containing S. Let u be an arbitrary vector in $[S]$. Then,

$u = \lambda_1 v_1 + \cdots + \lambda_n v_n$ for some $\lambda_i^{'s} \in F$, some $v_i^{'s} \in S$ and a positive integer n.

Since $S \subset T$ and $v_i \in S$. Therefore,

$$v_i \in T \quad \text{for each } i \in \underline{n}$$

$$\Rightarrow \quad \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \in T \qquad [\because \quad T \text{ is a subspace of } V]$$

$$\Rightarrow \quad u \in T$$

Since u is an arbitrary vector in S. Therefore,

$$u \in T \quad \text{for all } u \in [S].$$

$$\Rightarrow \quad [S] \subset T.$$

Hence, $[S]$ is the smallest subspace of V containing S.

Q.E.D.

REMARK-1 *Since the null space $\{0_V\}$ is the smallest subspace of V containing the void set φ, Therefore, by convention, we take $[\varphi] = \{0_V\}$.*

REMARK-2 *It is clear from the foregoing discussion that if S is a non-void subset of a vector space $V(F)$, then S itself is smaller than $[S]$. In fact, $[S] \neq \{0_V\}$ always contains an infinite number of vectors whatever be the number of vectors in S.*

THEOREM-3 *Let $S = \{v_1, v_2, \ldots, v_n\}$ be a finite subset of a vector space $V(F)$. Then for any $0 \neq a \in F$*

(i) $[v_1, v_2, \ldots, v_k, \ldots, v_n] = [v_1, v_2, \ldots, av_k, \ldots, v_n]$

(ii) $[v_1, v_2, \ldots, v_k, \ldots, v_n] = [v_1 + av_k, v_2, \ldots, v_k, \ldots, v_n]$

(iii) If $v = \sum\limits_{i=1}^{n} a_i v_i$, *where* $a_i \in F$ *for all* $i \in \underline{n}$, *then* $[v_1, v_2, \ldots, v_n] = [v_1, v_2, \ldots, v_n, v]$.

PROOF. (i) Let $u \in [v_1, v_2, \ldots, v_k, \ldots, v_n]$. Then u is a linear combination of vectors $v_1, v_2, \ldots, v_k, \ldots, v_n$.

Consequently, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ in F such that

$$u = \lambda_1 v_1 + \cdots + \lambda_k v_k + \cdots + \lambda_n v_n$$

$\Rightarrow$ $u = \lambda_1 v_1 + \cdots + \lambda_k a^{-1}(av_k) + \cdots + \lambda_n v_n$ $\quad [\because \quad 0 \neq a \in F \therefore \quad a^{-1} \in F]$

$\Rightarrow$ u is a linear combination of $v_1, v_2, \ldots, av_k, \ldots, v_n$

$\Rightarrow$ $u \in [v_1, v_2, \ldots, av_k, \ldots, v_n]$

Thus, $u \in [v_1, v_2, \ldots, v_k, \ldots, v_n] \Rightarrow u \in [v_1, v_2, \ldots, av_k, \ldots, v_n]$

$\therefore$ $\quad [v_1, v_2, \ldots, v_k, \ldots, v_n] \subset [v_1, v_2, \ldots, av_k, \ldots, v_n]$ (i)

Now let u be an arbitrary vector in $[v_1, v_2, \ldots, av_k, \ldots, v_n]$. Then there exist scalar $\mu_1, \mu_2, \ldots, \mu_k, \ldots, \mu_n$ in F such that

$$u = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_k(av_k) + \cdots + \mu_n v_n$$

$\Rightarrow$ $u = \mu_1 v_1 + \mu_2 v_2 + \cdots + (\mu_k a) v_k + \cdots + \mu_n v_n$

$\Rightarrow$ u is a linear combination of $v_1, v_2, \ldots, v_k, \ldots, v_n$

$\Rightarrow$ $u \in [v_1, v_2, \ldots, v_k, \ldots, v_n]$

Thus, $u \in [v_1, v_2, \ldots, av_k, \ldots, v_n] \Rightarrow u \in [v_1, v_2, \ldots, v_k, \ldots, v_n]$

$\therefore$ $\quad [v_1, v_2, \ldots, av_k, \ldots, v_n] \subset [v_1, v_2, \ldots, v_k, \ldots, v_n]$ (ii)

From (i) and (ii), we obtain

$$[v_1, v_2, \ldots, v_k, \ldots, v_n] = [v_1, v_2 \ldots, av_k, \ldots, v_n]$$

(ii) Let u be an arbitrary vector in $[v_1, v_2, \ldots, v_k, \ldots, v_n]$. Then, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that

$$u = \sum_{i=1}^{n} \lambda_i v_i$$

$\Rightarrow$ $u = \lambda_1(v_1 + av_k) + \lambda_2 v_2 + \cdots + (\lambda_k - a\lambda_1) v_k + \cdots + \lambda_n v_n$

$\Rightarrow$ u is a linear combination of $v_1 + av_k, v_2, \ldots, v_k, \ldots, v_n$

$\Rightarrow$ $u \in [v_1 + av_k, v_2, \ldots, v_k, \ldots, v_n]$

(iii) We know that $[S]$ is a subspace of V. Therefore, by (ii) it follows that $\big[[S]\big] = [S]$.

(iv) Let u be an arbitrary vector in $[S \cup T]$. Then,
$u = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m + \mu_1 v_1 + \cdots + \mu_n v_n$ for some $u_i^{'s} \in S$, $v_i^{'s} \in T$, $\lambda_i^{'s}$, $\mu_i^{'s} \in F$, and positive integers m and n.

Now, $u_i^{'s} \in S$ and $\lambda_i^{'s} \in F \Rightarrow \lambda_1 u_1 + \cdots + \lambda_m u_m \in [S]$

and, $v_i^{'s} \in T$ and $\mu_i^{'s} \in F \Rightarrow \mu_1 v_1 + \cdots + \mu_n v_n \in [T]$.

$\therefore$ $\lambda_1 u_1 + \cdots + \lambda_m u_m + \mu_1 v_1 + \cdots + \mu_n v_n \in [S] + [T]$

$\Rightarrow$ $u \in [S] + [T]$

Thus, $u \in [S \cup T] \Rightarrow u \in [S] + [T]$. Consequently,

$$[S \cup T] \subset [S] + [T].$$

Now, let v be an arbitrary vector in $[S] + [T]$. Then,

$$v = u + w \text{ for some } u \in [S],\ w \in [T].$$

Now, $u \in [S] \Rightarrow u$ is a finite linear combination of S.

and, $w \in [T] \Rightarrow w$ is a finite linear combination of T.

Therefore, $v = u + w$ is a finite linear combination of $S \cup T$ and so it is in $[S \cup T]$.

Thus, $v \in [S] + [T] \Rightarrow v \in [S \cup T]$

Consequently, $[S] + [T] \subset [S \cup T]$

Hence, $[S \cup T] = [S] + [T]$

Q.E.D.

2.7.1 ROW AND COLUMN SPACES OF A MATRIX

Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix over a field F. The rows of A are
$R_1 = (a_{11}, a_{12}, a_{13}, \ldots a_{1n})$, $R_2 = (a_{21}, a_{22}, a_{23}, \ldots a_{2n}), \ldots, R_m = (a_{m1}, a_{m2}, a_{m3}, \ldots, a_{mn})$
These rows may be viewed as vectors in F^n. So, they span a subspace of F^n, which is known as the row space of matrix A and is denoted by $rowsp(A)$. That is,

$$rowsp(A) = [R_1, R_2, \ldots, R_m]$$

Thus, row space of a matrix is the subspace of F^n spanned the rows of the matrix.

Similarly, the column space of A is the subspace of F^m spanned by the columns $C_1 = (a_{11}, a_{21}, \ldots, a_{m1})$, $C_2 = (a_{12}, a_{22}, \ldots, a_{m2}) \ldots$, $C_n = (a_{1n}, a_{2n}, \ldots, a_{mn})$ and is denoted by $colsp(A)$.

Clearly, $colsp(A) = rowsp(A^T)$.

THEOREM-1 *Row equivalent matrices have the same row space.*

PROOF. Let A be a matrix and B be the matrix obtained by applying one of the following row operations on matrix A:

(i) Interchange R_i and R_j (ii) Replace R_i by kR_i (iii) Replace R_j by $R_j + kR_i$

Then each row of B is a row of A or a linear combination of rows of A. So, the row space of B is contained in the row space of A. On the other hand, we can apply the inverse elementary row operation on B to obtain A. So, the row space of A is contained in the row space of B. Consequently, A and B have the same row space.

Q.E.D.

THEOREM-2 *Let A and B be row canonical matrices. Then A and B have the same row space if and only if they have the same non-zero rows.*

EXAMPLE *Let* $v_1 = (1,2,-1,3), v_2 = (2,4,1,-2), v_3 = (3,6,3,-7), v_4 = (1,2,-4,11)$ *and* $v_5 = (2,4,-5,14)$ *be vectors in* $R^4(R)$ *such that* $S = \{v_1, v_2, v_3\}$ *and* $T = \{v_4, v_5\}$. *Show that* $[S] = [T]$.

SOLUTION There are two ways to show that $[S] = [T]$. Show that each of v_1, v_2, v_3 is a linear combination of v_3 and v_4 and show that each of v_4 and v_5 is a linear combination of v_1, v_2, v_3. But, this method is not very convenient. So, let us discuss an alternative method. Let A be the matrix whose rows are v_1, v_2, v_3 and B be the matrix whose rows v_4 and v_5. That is,

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix} \qquad \text{Applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_3 \to R_3 - 2R_2$$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

$$\Rightarrow \quad B \sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \qquad \text{Applying } R_2 \to R_2 - 2R_1$$

$$\Rightarrow \quad B \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix} \qquad \text{Applying } R_1 \to R_1 + R_2$$

Clearly, non-zero rows of the matrices in row canonical form are identical. Therefore, $rowsp(A) = rowsp(B)$.

Hence, $[S] = [T]$

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Prove that any vector $v = (a, b)$ in $R^2(R)$ is a linear combination of $v_1 = (1, 1)$ and $v_2 = (-1, 1)$.*

SOLUTION Let $v = xv_1 + yv_2$ for some $x, y \in R$. Then,

$$\begin{aligned} & (a,b) = x(1,1) + y(-1,1) \\ \Rightarrow \quad & (a,b) = (x-y, x+y) \\ \Rightarrow \quad & x - y = a \text{ and } x + y = b \end{aligned}$$

Clearly, this system of equations is consistent with unique solution as the determinant of the coefficient matrix is non-zero. The values of x and y are $\dfrac{a+b}{2}$ and $\dfrac{a-b}{2}$ respectively.

Hence, every vector in R^2 is a linear combination of v_1 and v_2.

EXAMPLE-2 *Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, 2)$ span $R^2(R)$.*

SOLUTION In order to show that vectors v_1 and v_2 span $R^2(R)$, it is sufficient to show that any vector in R^2 is a linear combination of v_1 and v_2. Let $v = (a, b)$ be an arbitrary vector in $R^2(R)$. Further, let

$$\begin{aligned} & v = xv_1 + yv_2 \\ \Rightarrow \quad & (a,b) = x(1,1) + y(1,2) \\ \Rightarrow \quad & (a,b) = (x+y, x+2y) \\ \Rightarrow \quad & x + y = a \text{ and } x + 2y = b \end{aligned}$$

This is a consistent system of equations for all $a, b \in R$. So, every vector in $R^2(R)$ is a linear combination of v_1 and v_2.

EXAMPLE-3 *Let $V = P_n(x)$ be the vector space of all polynomials of degree less than or equal to n over the field R of all real numbers. Show that the polynomials $1, x, x^2, \ldots, x^{n-1}, x^n$ span V.*

SOLUTION Any polynomial $f(x)$ of degree less than or equal to n can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \text{ where } a_0, a_1, a_2, \ldots, a_n \in R$$

$\Rightarrow$ $f(x)$ is a linear combination of $1, x, x^2, \ldots, x^{n-1}, x^n$

Hence, polynomials $1, x, x^2, \ldots, x^{n-1}, x^n$ span V.

EXAMPLE-4 *Prove that the polynomials $1, 1 + x, (1 + x)^2$ span the vector space $V = P_2(x)$ of all polynomials of degree at most 2 over the field R of real numbers.*

SOLUTION Let $f(x) = a + bx + cx^2$ be an arbitrary polynomial in V, where $a, b, c \in R$. In order to prove that the polynomials $1, 1 + x, (1 + x)^2$ span V, it is sufficient to show that

This system of equations is inconsistent, because the rank of the coefficient matrix is not equal to the rank of the augmented matrix.

EXAMPLE-10 *Let* $v_1 = (-1,2,0), v_2 = (3,2,-1)$ *and* $v_3 = (1,6,-1)$ *be three vectors in* $R^3(R)$. *Then show that* $[v_1,v_2] = [v_1,v_2,v_3]$.

SOLUTION Clearly,

$$[v_1,v_2] = \{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in R\}$$

and, $[v_1,v_2,v_3] = \{\mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3 : \mu_1,\mu_2,\mu_3 \in R\}$.

Let $v_3 = (1,6,-1) = \alpha_1 v_1 + \alpha_2 v_2$. Then,

$$(1,6,-1) = (-\alpha_1 + 3\alpha_2, 2\alpha_1 + 2\alpha_2, -\alpha_2)$$

$$\Rightarrow \quad -\alpha_1 + 3\alpha_2 = 1, 2\alpha_1 + 2\alpha_2 = 6, -\alpha_2 = -1.$$

$$\Rightarrow \quad \alpha_1 = 2, \alpha_2 = 1$$

$$\therefore \quad (1,6,-1) = 2(-1,2,0) + 1(3,2,-1)$$

$$\Rightarrow \quad \mu_3(1,6,-1) = 2\mu_3(-1,2,0) + \mu_3(3,2,-1)$$

$$\Rightarrow \quad \mu_3 v_3 = 2\mu_3 v_1 + \mu_3 v_2$$

Now, $[v_1,v_2,v_3] = \{\mu_1 v_1 + \mu_2 v_2 + 2\mu_3 v_1 + \mu_3 v_2 : \mu_1,\mu_2,\mu_3 \in R\}$

$$\Rightarrow \quad [v_1,v_2,v_3] = \{(\mu_1 + 2\mu_3)v_1 + (\mu_2 + \mu_3)v_2 : \mu_1,\mu_2,\mu_3 \in R\}$$

$$\Rightarrow \quad [v_1,v_2,v_3] = \{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1,\lambda_2 \in R\}$$

$$\Rightarrow \quad [v_1,v_2,v_3] = [v_1,v_2].$$

EXAMPLE-11 *Let* $v_1 = (1,2,-1), v_2 = (2,-3,2), v_3 = (4,1,3)$ *and* $v_4 = (-3,1,2)$ *be four vectors in* $R^3(R)$. *Then show that* $[v_1,v_2] \neq [v_3,v_4]$.

SOLUTION Suppose $[v_1,v_2] = [v_3,v_4]$. Then for given $\lambda_3,\lambda_4 \in R$ there exist $\lambda_1,\lambda_2 \in R$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 = \lambda_3 v_3 + \lambda_4 v_4$$

$$\Rightarrow \quad \lambda_1(1,2,-1) + \lambda_2(2,-3,2) = \lambda_3(4,1,3) + \lambda_4(-3,1,2)$$

$$\Rightarrow \quad (\lambda_1, 2\lambda_1, -\lambda_1) + (2\lambda_2, -3\lambda_2, 2\lambda_2) = (4\lambda_3, \lambda_3, 3\lambda_3) + (-3\lambda_4, \lambda_4, 2\lambda_4)$$

$$\Rightarrow \quad (\lambda_1 + 2\lambda_2, 2\lambda_1 - 3\lambda_2, -\lambda_1 + 2\lambda_2) = (4\lambda_3 - 3\lambda_4, \lambda_3 + \lambda_4, 3\lambda_3 + 2\lambda_4)$$

$$\Rightarrow \quad \lambda_1 + 2\lambda_2 = 4\lambda_3 - 3\lambda_4 \qquad \text{(i)}$$

$$2\lambda_1 - 3\lambda_2 = \lambda_3 + \lambda_4 \qquad \text{(ii)}$$

$$-\lambda_1 + 2\lambda_2 = 3\lambda_3 + 2\lambda_4 \qquad \text{(iii)}$$

Solving (i) and (iii), we get $\lambda_1 = \frac{1}{2}(\lambda_3 - 5\lambda_4), \lambda_2 = \frac{1}{4}(7\lambda_3 - \lambda_4)$

These values of λ_1 and λ_2 should satisfy (ii) but they do not satisfy (ii).

Hence, $[v_1,v_2] \neq [v_3,v_4]$.

EXAMPLE-12 *Let $R^{2\times 2}$ be the vector space of all 2×2 matrices. Show that the matrices* $E_{11}=\begin{bmatrix}1&0\\0&0\end{bmatrix}, E_{12}=\begin{bmatrix}0&1\\0&0\end{bmatrix}, E_{21}=\begin{bmatrix}0&0\\1&0\end{bmatrix}, E_{22}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ *form a spanning set of $R^{2\times 2}$.*

SOLUTION Let $M=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ be any matrix in $R^{2\times 2}$, where $a,b,c,d\in R$.

Clearly, $M=aE_{11}+bE_{12}+cE_{21}+dE_{22}$

Thus, any matrix M in $R^{2\times 2}$is expressible as a linear combination of $E_{11},E_{12},E_{21},E_{22}$. Hence, $\{E_{11},E_{12},E_{21},E_{22}\}$ forms a spanning set of $R^{2\times 2}$.

EXAMPLE-13 *Find one vector in $R^3(R)$ that spans the intersection of subspaces S and T where S is the xy-plane, i.e. $S=\{(a,b,0):a,b\in R\}$ and T is the space spanned by the vectors $v_1=(1,1,1)$ and $v_2=(1,2,3)$.*

SOLUTION Since S and T are subspaces of $R^3(R)$. Therefore, $S\cap T$ is a subspace of $R^3(R)$.

Let $u=(a,b,c)$ be a vector in $S\cap T$. Then,

$$u\in S \text{ and } u\in T$$

$\Rightarrow$ $c=0$ and u is linear combination of v_1 and v_2 $\quad [\because \quad S=\{(a,b,0):a,b\in R\}]$

$\Rightarrow$ $u=xv_1+yv_2$ for some $x,y\in R$

$\Rightarrow$ $(a,b,0)=x(1,1,1)+y(1,2,3)$

$\Rightarrow$ $(a,b,0)=(x+y,\ x+2y,\ x+3y)$

$\Rightarrow$ $x+y=a,\ x+2y=b,\ x+3y=0$

Solving first two equations, we get $x=2a-b, y=b-a$

Substituting these values in $x+3y=0$, we get $a=2b$

$\therefore$ $v=(2b,b,0), b\in R$.

EXAMPLE-14 *Is the vector $v=(3,-1,0,-1)$ in the subspace of $R^4(R)$ spanned by the vectors $v_1=(2,-1,3,2), v_2=(-1,1,1,-3)$ and $v_3=(1,1,9,-5)$?*

SOLUTION If v is expressible as a linear combination of v_1,v_2 and v_3, then v is the subspace spanned by v_1,v_2 and v_3, otherwise not. So, let

$$v=xv_1+yv_2+zv_3 \text{ for some scalars } x,y,z.$$

$\Rightarrow$ $(3,-1,0,-1)=x(2,-1,3,2)+y(-1,1,1,-3)+z(1,1,9,-5)$

$\Rightarrow$ $(3,-1,0,-1)=(2x-y+z,-x+y+z,3x+y+9z,2x-3y-5z)$

$\Rightarrow$ $2x-y+z=3,\ -x+y+z=-1,\ 3x+y+9z=0,\ 2x-3y-5z=-1$

This system of equations can be expressed in matrix form as follows:

$$\begin{bmatrix}2&-1&1\\-1&1&1\\3&1&9\\2&-3&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-1\\0\\-1\end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & 4 & 12 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \\ -3 \end{bmatrix} \text{Applying } R_1 \to R_1 + 2R_2,\ R_3 \to R_3 + 3R_2,\ R_4 \to R_4 + 2R_2$$

or, $$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -15 \\ -3 \end{bmatrix} \text{Applying } R_1 \to R_1 + R_4,\ R_2 \to R_2 + R_4,\ R_3 \to R_3 + 4R_4$$

Clearly, this is an inconsistent system of equations. So, v is not expressible as a linear combination of v_1, v_2, v_3. Hence, $v \notin [v_1, v_2, v_3]$.

EXERCISE 2.7

1. Show that the vectors $e_1^{(3)} = (1,0,0), e_2^{(3)} = (0,1,0)$ and $e_3^{(3)} = (0,0,1)$ form a spanning set of $R^3(R)$.

2. Show that the vectors $v_1 = (1,1,1), v_2 = (1,1,0)$ and $v_3 = (1,0,0)$ form a spanning set of $R^3(R)$.

3. Consider the vector space $V = P_n(t)$ consisting of all polynomials of degree $\leq n$ in indeterminate t. Show that the set of polynomials $1, t, t^2, t^3, \ldots, t^n$ forms a spanning set of V.

4. Show that the vectors $v_1 = (1,2,3), v_2 = (1,3,5)$ and $v_3 = (1,5,9)$ do not spam $R^3(R)$. [Hint: (2,7,8) cannot be written as a linear combination of v_1, v_2, v_3].

5. If v_1, v_2 and v_3 are vectors in a vector space $V(F)$ such that $v_1 + v_2 + v_3 = 0_V$, then show that v_1 and v_2 span the same subspace of V as v_2 and v_3.

   [Hint: Let S_1 and S_2 be subspaces spanned by vectors v_1, v_2 and v_2, v_3 respectively. i.e, $S_1 = [\{v_1, v_2\}]$ and $S_2 = [\{v_2, v_3\}]$. We have to show that $S_1 = S_2$.

   Let v be an arbitrary vector in S_1. Then,

$$\begin{aligned} & v \in S_1 \\ \Rightarrow\quad & v = \lambda_1 v_1 + \lambda_2 v_2 \quad \text{for some } \lambda_1, \lambda_2 \in F \\ \Rightarrow\quad & v = \lambda_1(-v_2 - v_3) + \lambda_2 v_2 & [\because\ v_1 + v_2 + v_3 = 0_V] \\ \Rightarrow\quad & v = (\lambda_1 - \lambda_2)v_2 + (-\lambda)v_3 \\ \Rightarrow\quad & v \in S_2 \\ \therefore\quad & S_1 \subset S_2. \end{aligned}$$

REMARK-4 *Note that the set $\{0_V\}$ in a vector space V(F) is always l.d., because $\lambda 0_V = 0_V$ for all non-zero $\lambda \in F$. On the other hand, a singleton set containing a non-zero vector v is always l.i., because $\lambda v = 0_V \Rightarrow \lambda = 0$.*

THEOREM-1 *Any list of vectors containing the null vector is always l.d.*

SOLUTION Let $\underline{v} = (v_1, v_2, \dots, v_n)$ be a list of vectors in a vector space $V(F)$ such that $v_k = 0_V$. Then for any $0 \neq \lambda \in F$

$$0v_1 + 0v_2 + \cdots + \lambda v_k + 0v_{k+1} + \cdots + 0v_n = 0_V + 0_V + \cdots + 0_V = 0_V$$

This shows that a non-trivial linear combination of $v_1, v_2, \dots, v_n$ equals to the null vector. Hence, $\underline{v}$ is a l.d. list.

Q.E.D.

REMARK. *It follows from the foregoing discussion that the nature of the null vector is so strong that its presence in the set always makes it l.d.*

THEOREM-2 *A list $\underline{v} = (v_1, v_2, \dots, v_n)$ of non-zero vectors in a vector space $V(F)$ is l.d. iff some one vector v_k of the list is a linear combination (l.c.) of the previous vectors of the list. Also, if for some $j, 1 \leq j \leq n$, the list $(v_1, v_2, \dots, v_j)$ is a list of linearly independent vectors, then $j < k$.*

PROOF. First suppose that the list $(v_1, v_2, \dots, v_n)$ of vectors in $V(F)$ is linearly dependent. Then there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ not all zero such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0_V$$

Let k be the largest positive integer such that $\lambda_k \neq 0$, so that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = 0_V$$

We claim that $k > 1$. For, if $k = 1$

$\lambda_1 v_1 = 0_V$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$

$\Rightarrow \quad \lambda_1 = 0 = \lambda_2 = \cdots = \lambda_n \qquad [\because \quad v_1 \neq 0_V \quad \therefore \lambda_1 v_1 = 0_V \Rightarrow \lambda_1 = 0]$

which contradicts the fact that $(v_1, v_2, \dots, v_n)$ is a l.d. list.

Now, $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = 0_V$

$\Rightarrow \quad v_k = (-\lambda_1 \lambda_k^{-1}) v_1 + \cdots + (-\lambda_{k-1} \lambda_k^{-1}) v_{k-1} \qquad$ (i)

$\Rightarrow \quad v_k$ is a linear combination of the previous vectors of the list.

Hence, some one vector of the list is a linear combination of the previous vectors of the list.

Conversely, suppose that a vector v_k (say) of the list $(v_1, v_2, \dots, v_k, \dots, v_n)$ is a linear combination of the previous vectors $v_1, v_2, \dots, v_{k-1}$. Then there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$ such that

$v_k = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{k-1} v_{k-1}$

$\Rightarrow \quad \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{k-1} v_{k-1} + (-1) v_k = 0_V$

$\Rightarrow \quad \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{k-1} v_{k-1} + (-1) v_k + 0 v_{k+1} + \cdots + 0 v_n = 0_V$

$\Rightarrow \quad$ A non-trivial linear combination of the list $(v_1, v_2, \dots, v_k, \dots, v_n)$ is zero vector

Hence, the list is linearly dependent.

Q.E.D.

If possible, let the list $(v_1, v_2, \ldots, v_j)$ be a list of independent vectors and $j \geq k$. Then, the list of vectors $v_1, v_2, \ldots, v_k$ is linear independent. But,

$$v_k = (-\lambda_1 \lambda_k^{-1})v_1 + \cdots + (-\lambda_{k-1}\lambda_k^{-1})v_{k-1} \qquad \text{[From (i)]}$$

$\Rightarrow$ v_k is a linear combination of $v_1, v_2, \ldots, v_{k-1}$

$\Rightarrow$ $v_1, v_2, \ldots, v_{k-1}, v_k$ are linearly dependent.

This is a contradiction. Hence, $j < k$.

THEOREM-3 *Let $S = \{v_1, v_2, v_3, \ldots v_k\}$ be a linearly independent set of vectors in a vector space $V(F)$ and v be a non-zero vector in $V(F)$. Then, $S_1 = S \cup \{v\}$ is linearly independent iff $v \notin [S]$.*

PROOF. First, let S_1 be linearly independent. Suppose that $v \in [S]$. Then, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ in F such that $v = \sum\limits_{i=1}^{k} \lambda_i v_i$

$\Rightarrow$ $S_1 = S \cup \{v\}$ is linearly dependent.

This contradicts the hypothesis that S_1 is linearly independent. Hence, $v \notin [S]$.

Conversely, let $v \notin [S]$. Then, we have to prove that S_1 is linearly independent. If possible, let S_1 be linearly dependent. Then, there exists a finite non-empty subset B (say) of S_1 that is linearly dependent.

Now,

$B \subset S_1 = S \cup \{v\}$ and S is linearly independent

$\Rightarrow$ B contains v and some vectors in $S = \{v_1, v_2, \ldots, v_k\}$

$\Rightarrow$ v is a linear combination of some vectors in S

$\Rightarrow$ v is a linear combination of vectors in S

$\Rightarrow$ $v \in [S]$

This is a contradiction. Hence, S_1 is linearly independent.

Q.E.D.

EXAMPLE-2 *Show that the set $S = \{(1,2,4),\ (1,0,0),\ (0,1,0),\ (0,0,1)\}$ is a linearly dependent set of vectors in $R^3(R)$.*

SOLUTION Clearly,

$$(1,2,4) = 1(1,0,0) + 2(0,1,0) + 4(0,0,1)$$

$\Rightarrow$ $(1,2,4)$ is linear combination of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Hence, S is a linearly dependent set of vectors.

THEOREM-4 *Each subset of a linearly independent set is linearly independent.*

PROOF. Let $S = \{v_1, v_2, \ldots, v_m\}$ be a linearly independent set of vectors in a vector space $V(F)$. Let $S' = \{v_1, v_2, \ldots, v_n\}$ $n \leqq m$, be an arbitrary finite subset of S. Then,

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0_V$$

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n + 0 v_{n+1} + \cdots + 0 v_m = 0_V$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0 \qquad [\because \ S \text{ is l.i.}]$$

Hence, S' is a linearly independent set.

Q.E.D.

THEOREM-5 *Any super set of linearly dependent set of vectors in a vector space is linearly dependent.*

PROOF. Let $S = \{v_1, v_2, \ldots, v_m\}$ be a linearly dependent set of vectors in a vector space $V(F)$, and let $S' = \{v_1, v_2, \ldots, v_m, v_{m+1}\}$ be a super set of S.

Now,

S is a linearly dependent set

$\Rightarrow$ There exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ not all zero such that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0_V$

$\Rightarrow$ $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m + \lambda_{m+1} v_{m+1} = 0_V$, where $\lambda_{m+1} = 0$

$\Rightarrow$ A non-trivial linear combination of $v_1, v_2, \ldots, v_m, v_{m+1}$ is zero vector $\left[\because \text{ At least one } \lambda_i \neq 0\right]$

$\Rightarrow$ S' is a linearly dependent set.

Q.E.D.

THEOREM-6 *Let S be any linearly independent subset of a vector space $V(F)$ and let $S = S_1 \cup S_2$ such that $S_1 \cap S_2 = \phi$. Prove that (i) $[S] = [S_1] + [S_2]$ (ii) $[S_1] \cap [S_2] = \phi$ i.e. $[S]$ is the direct sum of $[S_1]$ and $[S_2]$.*

PROOF. (i) By Theorem 4(iv) on page 169, we have

$$[S_1 \cup S_2] = [S_1] + [S_2] \text{ i.e,} \quad [S] = [S_1] + [S_2]$$

(ii) Let $S_1 = \{v_1, v_2, \ldots, v_m\}$ and $S_2 = \{u_1, u_2, \ldots, u_n\}$.

Let v be an arbitrary vector in $[S_1] \cap [S_2]$. Then,

$$v \in [S_1] \cap [S_2]$$

$$\Rightarrow \quad v \in [S_1] \text{ and } v \in [S_2]$$

$$\Rightarrow \quad v = \sum_{i=1}^{m} \lambda_i v_i \text{ and } v = \sum_{i=1}^{n} \mu_j u_j \text{ for some } \lambda_i \in F \text{ and } \mu_j \in F$$

$$\Rightarrow \quad \sum_{i=1}^{m} \lambda_i v_i = \sum_{j=1}^{n} \mu_j u_j$$

$$\Rightarrow \quad \sum_{i=1}^{m} \lambda_i v_i + \sum_{j=1}^{n} (-\mu_j) u_j = 0_V$$

$$\Rightarrow \quad \lambda_i = 0 \text{ and } \mu_j = 0 \quad \text{for all } i \in \underline{m} \text{ and } j \in \underline{n} \qquad \left[\because \ S = S_1 \cup S_2 = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\} \text{ is } l.i.\right]$$

$$\therefore \quad v = 0_V$$

Since v is an arbitrary vector in $[S_1] \cap [S_2]$. Therefore, $[S_1] \cap [S_2] = \phi$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Let F be a field with unity 1. Show that the vectors $e_1^{(n)} = (1,0,0,\ldots,0)$, $e_2^{(n)} = (0,1,0,\ldots,0), e_3^{(n)} = (0,0,1,\ldots,0)\ldots, e_n^{(n)} = (0,0,0,\ldots,1)$ are linearly independent in the vector space $F^n(F)$.*

SOLUTION Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be scalars such that

$$\lambda_1 e_1^{(n)} + \lambda_2 e_2^{(n)} + \lambda_3 e_3^{(n)} + \cdots + \lambda_n e_n^{(n)} = 0$$

$$\Rightarrow \quad (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = (0,0,0,\ldots,0)$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$$

Hence, $e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)}$ are linearly independent.

Particular Cases:

(i) $e_1^{(2)} = (1,0), e_2^{(2)} = (0,1)$ are linearly independent vectors in $R^2(R)$.

(ii) $e_1^{(3)} = (1,0,0), e_2^{(3)} = (0,1,0), e_3^{(3)} = (0,0,1)$ are linearly independent vectors in $R^3(R)$.

EXAMPLE-2 *Show that the vectors $v_1 = (1,1,0), v_2 = (1,3,2), v_3 = (4,9,5)$ are linearly dependent in $R^3(R)$.*

SOLUTION Let x, y, z be scalars i.e., real numbers such that

$$xv_1 + yv_2 + zv_3 = 0$$

$$\Rightarrow \quad x(1,1,0) + y(1,3,2) + z(4,9,5) = (0,0,0)$$

$$\Rightarrow \quad (x+y+4z, x+3y+9z, 2y+5z) = (0,0,0)$$

$$\Rightarrow \quad x+y+4z = 0,\ x+3y+9z = 0,\ 0x+2y+5z = 0$$

This is a homogeneous system of linear equations. The determinant of the coefficient matrix A is

$$|A| = \begin{vmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{vmatrix} = 0$$

So, the system has non-trivial solutions. Hence, given vectors are linearly dependent in $R^3(R)$.

Aliter In order to check the linear independence or dependence of vectors, we may follow the following algorithm:

ALGORITHM

Step I *Form a matrix A whose columns are given vectors.*
Step II *Reduce the matrix in step-I to echelon form.*
Step III *See whether all columns have pivot elements or not. If all columns have pivot elements, then given vectors are linearly independent. If there is a column not having a pivot element, then the corresponding vector is a linear combination of the preceding vectors and hence linearly dependent.*

In Example 2, the matrix A whose columns are v_1, v_2, v_3 is

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\therefore \quad A \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 2 & 5 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} \textcircled{1} & 1 & 4 \\ 0 & \textcircled{2} & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - R_2$$

Pivots in the echelon form of matrix A have been encircled. We observe that the third column does not have a pivot. So, the third vector v_3 is a linear combination of the first two vectors v_1 and v_2. Thus, the vectors v_1, v_2, v_3 are linearly dependent.

EXAMPLE-3 *Show that the vectors* $v_1 = (1,2,3), v_2 = (2,5,7), v_3 = (1,3,5)$ *are linearly independent in* $R^3(R)$.

SOLUTION Let x, y, z be scalars such that

$$xv_1 + yv_2 + zv_3 = 0$$

$$\Rightarrow \quad x(1,2,3) + y(2,5,7) + z(1,3,5) = (0,0,0)$$

$$\Rightarrow \quad (x+2y+z, 2x+5y+3z, 3x+7y+5z) = (0,0,0)$$

$$\Rightarrow \quad x+2y+z = 0,\ 2x+5y+3z = 0,\ 3x+7y+5z = 0$$

The determinant of the coefficient matrix A of the above homogeneous system of equations is given by

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{vmatrix} = 1 \neq 0$$

So, the above system of equations has trivial solution only, i.e. $x = y = z = 0$.

Hence, the given vectors are linearly independent in $R^3(R)$.

EXAMPLE-4 *Show that the vectors* $v_1 = (1,1,2,4), v_2 = (2,-1,-5,2), v_3 = (1,-1,-4,0)$ *and* $v_4 = (2,1,1,6)$ *are linearly dependent in* $R^4(R)$.

SOLUTION Let x, y, z, t be scalars in R such that

$$xv_1 + yv_2 + zv_3 + tv_4 = 0$$

$$\Rightarrow \quad x(1,1,2,4) + y(2,-1,-5,2) + z(1,-1,-4,0) + t(2,1,1,6) = 0$$

$$\Rightarrow \quad (x + 2y + z + 2t, x - y - z + t, 2x - 5y - 4z + t, 4x + 2y + 0z + 6t) = (0,0,0,0)$$

$$\Rightarrow \quad x + 2y + z + 2t = 0,\ x - y - z + t = 0,\ 2x - 5y - 4z + t = 0,\ 4x + 2y + 0z + 6t = 0$$

The coefficient matrix A of the above homogeneous system of equations is

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - 4R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - 3R_2,\ R_4 \to R_4 - 2R_2$$

Clearly, rank of $A = 2 <$ Number of unknowns. So, the above system has non-trivial solutions. Hence, given vectors are linearly dependent in $R^4(R)$.

Aliter The matrix A whose columns are v_1, v_2, v_3, v_4 is

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

The echelon form of matrix A is

$$\begin{bmatrix} \textcircled{1} & 2 & 1 & 2 \\ 0 & \textcircled{-3} & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We observe that the third and fourth columns does not have pivots. So, third and fourth vectors are linear combinations of first two vectors v_1 and v_2. Thus, the vectors are linearly dependent.

EXAMPLE-5 *Determine whether the vectors $f(x) = 2x^3 + x^2 + x + 1, g(x) = x^3 + 3x^2 + x - 2$ and $h(x) = x^3 + 2x^2 - x + 3$ in the vector space $R[x]$ of all polynomials over the real number field are linearly independent or not.*

SOLUTION Let a, b, c be real numbers such that

$$af(x) + bg(x) + ch(x) = 0 \quad \text{for all } x$$

$$\Rightarrow \quad a(2x^3 + x^2 + x + 1) + b(x^3 + 3x^2 + x - 2) + c(x^3 + 2x^2 - x + 3) = 0 \quad \text{for all } x$$

$$\Rightarrow \quad (2a + b + c)x^3 + (a + 3b + 2c)x^2 + (a + b - c)x + (a - 2b + 3c) = 0 \quad \text{for all } x$$

$$\Rightarrow \quad 2a + b + c = 0,\ a + 3b + 2c = 0,\ a + b - c = 0,\ a - 2b + 3c = 0$$

The coefficient matrix A of the above system of equations is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix} \text{Applying } R_1 \leftrightarrow R_3$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & -3 & 4 \end{bmatrix} \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 9 \\ 0 & -1 & 3 \\ 0 & 0 & -5 \end{bmatrix} \text{Applying } R_2 \to R_2 + 2R_3,\ R_4 \to R_4 - 3R_3$$

$$\Rightarrow \qquad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 9 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{Applying } R_4 \to R_4 + \frac{5}{9}R_2$$

$$\Rightarrow \qquad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix} \text{Applying } R_2 \leftrightarrow R_3$$

Clearly, rank of A is equal to 3 which is equal to the number of unknowns a, b, c. So, the system has trivial solution only, i.e. $a = b = c = 0$.

So, $f(x), g(x), h(x)$ are linearly independent in $R[x]$.

EXAMPLE-6 *Let V be the vector space of functions from R into R. Show that the functions $f(t) = \sin t, g(t) = e^t, h(t) = t^2$ are linearly independent in V.*

SOLUTION Let x, y, z be scalars such that

$$x f(t) + y\, g(t) + z\, h(t) = 0 \text{ for all } t \in R$$

Putting $t = 0, \pi$ and $\frac{\pi}{2}$, respectively we get

$$x f(0) + y\, g(0) + z h(0) = 0$$
$$x f(\pi) + y\, g(\pi) + z\, h(\pi) = 0$$
$$x f\left(\frac{\pi}{2}\right) + y\, g\left(\frac{\pi}{2}\right) + z\, h\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \qquad y = 0,\ y\, e^{\pi} + z\, \pi^2 = 0, x + y\, e^{\frac{\pi}{2}} + h\, \frac{\pi^2}{4} = 0$$

$$\Rightarrow \qquad y = 0, z = 0, x = 0$$

Thus,

$$x f(t) + y g(t) + z h(t) = 0 \Rightarrow x = y = z = 0$$

Hence, $f(t), g(t), h(t)$ are linearly independent.

EXAMPLE-7 *Show that in the vector space $F[x]$ of all polynomials in an indeterminate x, the set $S = \{1, x, x^2, \dots\}$ is linearly independent.*

PROOF. Let $S' = \{x^{n_1}, x^{n_2}, \dots, x^{n_k}\}$ be a finite subset of S, where $n_1, n_2, \dots, n_k$ are distinct non-negative integers.

Let $\lambda_1, \lambda_2, \dots, \lambda_k \in F$ be such that

$$\lambda_1 x^{n_1} + \lambda_2 x^{n_2} + \cdots + \lambda_k x^{n_k} = 0(x) \qquad [0(x) \text{ is zero polynomial over } F]$$

$$\Rightarrow \qquad \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0 \qquad \text{[By the definition of equality of two polynomials]}$$

So, S' is a linearly independent set.

Since S' is an arbitrary finite subset of S. Therefore, every finite subset of S is linearly independent. Consequently, S is linearly independent.

Q.E.D.

EXAMPLE-8 *Show that the vectors (matrices)* $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *in* $R^{2\times 2}$ *are linearly independent.*

SOLUTION Let $\lambda_1, \lambda_2, \lambda_3 \in R$ be such that

$$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 = O_{2\times 2} \quad \text{(Null matrix).}$$

$$\Rightarrow \quad \lambda_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 & \lambda_1 + \lambda_3 \\ \lambda_1 & \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 0,\ \lambda_1 + \lambda_3 = 0,\ \lambda_1 = 0,\ \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Hence, A_1, A_2, A_3 are linearly independent vectors in $R^{2\times 2}$.

EXAMPLE-9 *If* v_1, v_2 *are vectors in a vector space* $V(F)$ *and* $\lambda_1, \lambda_2 \in F$, *show that the set* $\{v_1, v_2, \lambda_1 v_1 + \lambda_2 v_2\}$ *is linearly dependent set.*

SOLUTION Since the vector $\lambda_1 v_1 + \lambda_2 v_2$ in the set $\{v_1, v_2, \lambda_1 v_1 + \lambda_2 v_2\}$ is a linear combination of other two vectors. Therefore, by Theorem 2 on page 180 the set is linearly dependent.

Aliter. Clearly, $(-\lambda_1)v_1 + (-\lambda_2)v_2 + 1(\lambda_1 v_1 + \lambda_2 v_2) = 0_V$. Therefore, a non-trivial linear combination of $v_1, v_2, \lambda_1 v_1 + \lambda_2 v_2$ equals to the null vector. Hence, the given set is linearly dependent.

2.8.1 LINEAR DEPENDENCE AND ECHELON MATRICES

Consider the following echelon matrix A, whose pivots have been circled:

$$A = \begin{bmatrix} 0 & \textcircled{2} & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & \textcircled{4} & 3 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & \textcircled{7} & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{6} & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We observe that the rows R_2, R_3, R_4 have 0's in the second column below the non-zero pivot in R_1, and hence any linear combination of R_2, R_3 and R_4 must have 0 as its entry as the second component. Whereas R_1 has a non-zero entry 2 as the second component. Thus, R_1 cannot be a linear combination of the rows below it. Similarly, the rows R_3 and R_4 have 0's in the third column below the non-zero pivot in R_2, and hence R_2 cannot be a linear combination of the rows below it. Finally, R_3 cannot be a multiple of R_4 as R_4 has a 0 in the fifth column below the non-zero pivot in R_3. Thus, if we look at the rows from the bottom and move upward, we find that out of rows R_4, R_3, R_2, R_1 non row is a linear combination of the preceding rows. So, R_1, R_2, R_3, R_4 are linearly independent vectors $R^7(R)$.

The above discussion suggests the following theorem.

THEOREM-1 *The non-zero rows of a matrix in echelon form are linearly independent.*

EXAMPLE-1 *Let v_1, v_2, v_3 be vectors in a vector space $V(F)$, and let $\lambda_1, \lambda_2 \in F$. Show that the set $\{v_1, v_2, v_3\}$ is linearly dependent if the set $\{v_1 + \lambda_1 v_2 + \lambda_2 v_3, v_2, v_3\}$ is linearly dependent.*

SOLUTION If the set $\{v_1 + \lambda_1 v_2 + \lambda_2 v_3, v_2, v_3\}$ is linearly dependent, then there exist scalars $\lambda, \mu, \nu \in F$ (not all zero) such that

$$\lambda(v_1 + \lambda_1 v_2 + \lambda_2 v_3) + \mu v_2 + \nu v_3 = 0_V$$

$$\Rightarrow \quad \lambda v_1 + (\lambda\lambda_1 + \mu)v_2 + (\lambda\lambda_2 + \nu)v_3 = 0_V \qquad \text{(i)}$$

The set $\{v_1, v_2, v_3\}$ will be linearly dependent if in (i) at least one of the scalar coefficients is non-zero.

If $\lambda \neq 0$, then the set will be linearly dependent whatever may be the values of μ and ν. But, if $\lambda = 0$, then at least one of μ and ν should not be equal to zero and hence at least one of $\lambda\lambda_1 + \mu$ and $\lambda\lambda_2 + \nu$ will not be zero (since $\lambda = 0 \Rightarrow \lambda\lambda_1 + \mu = \mu$ and $\lambda\lambda_2 + \nu = \nu$).

Hence, from (i), we find that the scalars λ, $\lambda\lambda_1 + \mu$, $\lambda\lambda_2 + \nu$ are not all zero. Consequently, the set $\{v_1, v_2, v_3\}$ is linearly dependent.

EXAMPLE-2 *Suppose the vectors u, v, w are linearly independent vectors in a vector space $V(F)$. Show that the vectors $u + v, u - v, u - 2v + w$ are also linearly independent.*

SOLUTION Let $\lambda_1, \lambda_2, \lambda_3$ be scalars such that

$$\lambda_1(u+v) + \lambda_2(u-v) + \lambda_3(u - 2v + w) = 0_V$$

$$\Rightarrow \quad (\lambda_1 + \lambda_2 + \lambda_3)u + (\lambda_1 - \lambda_2 - 2\lambda_3)v + \lambda_3 w = 0_V$$

$$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 0,\ \lambda_1 - \lambda_2 - 2\lambda_3 = 0,\ 0\lambda_1 + 0\lambda_2 + \lambda_3 = 0 \qquad \left[\because\ u, v, w \text{ are linearly independent}\right]$$

The coefficient matrix A of the above system of equations is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |A| = -2 \neq 0$$

So, the above system of equations has only trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Thus, $u + v$, $u - v$, $u - 2v + w$ are linearly independent vectors.

EXERCISES 2.8

1. Mark each of the following true or false.
   - (i) For any subset S of a vector space $V(F)$, $[S+S]=[S]$
   - (ii) If S and T are subsets of a vector space $V(F)$, then $S \neq T \Rightarrow [S] \neq [T]$.
   - (iii) If S and T are subsets of a vector space $V(F)$, then $S \subset T \Rightarrow [S] \subset [T]$
   - (iv) Every subset of a linearly independent set is linearly independent.
   - (v) Every superset of a linearly independent set is linearly independent.
   - (vi) Every subset of a linearly dependent set is linearly dependent.
   - (vii) Every superset of a linearly dependent set is linearly dependent.
   - (viii) A subset of a linearly dependent set can never be linearly dependent.
   - (ix) The void set is linearly dependent.
   - (x) If a non-void subset of a vector space is not linearly dependent, then it is linearly independent.
   - (xi) Any set of vectors containing the null vector is linearly dependent.
   - (xii) Every singleton set consisting of a non-zero vector is linearly independent.
   - (xiii) Intersection of two linearly independent subsets of a vector space is linearly independent.
   - (xiv) Union of two linearly independent subsets of a vector space is linearly independent.
2. Which of the following subsets S of R^3 are linearly independent?
   - (i) $S=\{(1,2,1),(-1,1,0),(5,-1,2)\}$
   - (ii) $S=\{(1,1,0),(0,0,1),(1,5,2)\}$
   - (iii) $S=\{(1,0,0),(1,1,1),(0,0,0)\}$
   - (iv) $S=\{(1,3,2),(1,-7,8),(2,1,1)\}$
   - (v) $S=\{(1,5,2),(1,0,0),(0,1,0)\}$
3. Which of the following subsets S of $R[x]$ are linearly independent?
   - (i) $S=\{1,x,x^2,x^3,x^4\}$
   - (ii) $S=\{1,x-x^2,x+x^2,3x\}$
   - (iii) $S=\{x^2-1,\ x+1,\ x-1\}$
   - (iv) $S=\{x,\ x-x^3,\ x^2+x^4,\ x+x^2+x^4+\frac{1}{2}\}$
   - (v) $S=\{1,\ 1+x,\ 1+x+x^2,\ x^4\}$
   - (vi) $S=\{1,\ x,\ x(1-x)\}$
   - (vii) $S=\left\{1,\ x,\ 1+x+x^2\right\}$

4. Let $f(x),\ g(x),\ h(x),\ k(x) \in R[x]$ be such that $f(x) = 1,\ g(x) = x,\ h(x) = x^2$ and $k(x) = 1 + x + x^2$ for all $x \in R$. Show that $f(x),\ g(x),\ h(x),\ k(x)$ are linearly dependent but any three of them are linearly independent.
5. Which of the following subsets of the space of all continuous functions on R are linearly independent?
   (i) $S = \{\sin x,\ \cos x,\ \sin(x+1)\}$
   (ii) $S = \{xe^x,\ x^2e^x,\ (x^2 + x - 1)e^x\}$
   (iii) $S = \{\sin^2 x,\ \cos 2x,\ 1\}$
   (iv) $S = \{x,\ \sin x,\ \cos x\}$
   (v) $S = \{x,\ x^2,\ e^{2x}\}$
6. Prove that the set $S = \{1, \mathrm{i}\}$ is linearly independent in the vector space C of all complex numbers over the field R of real numbers while it is linearly dependent in the vector space C of all complex numbers over the field of complex numbers.
7. If the set $\{v_1,\ v_2,\ v_3\}$ is linearly independent in a vector space $V(F)$, then prove that the set $\{v_1 + v_2,\ v_2 + v_3,\ v_3 + v_1\}$ is also linearly independent.
8. Prove that the vectors $v_1 = (1,\ 1,\ 0,\ 0)$, $v_2 = (0,\ 0,\ 0,\ 3)$, and $v_3 = (0,\ 1,\ -1,\ 0)$ in $F^4(F)$ are linearly independent if F is a field of characteristic zero and are linearly dependent if F is of characteristic 3.
9. If $\{v_1,\ v_2, \ldots, v_n\}$ is a linearly independent set of vectors in a vector space $V(F)$ and $\{v_1,\ v_2, \ldots, v_n,\ u\}$ is a linearly dependent set in V. Then show that u is a linear combination of $v_1,\ v_2, \ldots, v_n$.
10. If the set $S = \{v_1,\ v_2, \ldots, v_n\}$ is a linearly independent set of vectors in a vector space $V(F)$ and $u \in [S]$, then show that representation of u as a linear combination of $v_1,\ v_2, \ldots, v_n$ is unique.
11. Show that the vectors $v_1 = (1 + i,\ 2i)$ and $v_2 = (1,\ 1 + i)$ are linearly dependent in $C^2(C)$ but linearly independent in $C^2(R)$.
12. If $v_1,\ v_2,\ v_3$ are linearly dependent vectors of $V(F)$ where F is any subfield of the field of complex numbers, then so also are $v_1 + v_2,\ v_2 + v_3,\ v_3 + v_1$.
13. Show that the functions $f(t) = \sin t,\ g(t) = \cos t,\ h(t) = t$ from R to R are linearly independent.
14. Find a maximal linearly independent subsystem of the system of vectors:

$$v_1 = (2, -2, -4),\ v_2 = (1, 9, 3),\ v_3 = (-2, -4, 1),\ v_4 = (3, 7, -1).$$

ANSWERS

1. (i) F (ii) F (iii) T (iv) T (v) F (vi) F (vii) T (viii) F (ix) F (x) T (xi) T (xii) T (xiii) T (xiv) F
2. (i), (ii), (iv), (v) 3. (i), (iii), (iv), (v), (vi), (vii) 5. (ii), (iv), (v)

14. $\{v_1,\ v_2\}$

2.9 BASIS AND DIMENSION

BASIS. *A non-void subset B of a vector space $V(F)$ is said to be a basis for V, if*

(i) B spans V, i.e. $[B] = V$.
and, (ii) B is linearly independent (l.i.).

In other words, a non-void subset B of a vector space V is a linearly independent set of vectors in V that spans V.

FINITE DIMENSIONAL VECTOR SPACE. *A vector space $V(F)$ is said to be a finite dimensional vector space if there exists a finite subset of V that spans it.*

A vector space which is not finite dimensional may be called an infinite dimensional vector space.

REMARK. *Note that the null vector cannot be an element of a basis, because any set containing the null vector is always linearly dependent.*

For any field F and a positive integer n, the set $B = \{e_1^{(n)},\ e_2^{(n)}, \ldots, e_n^{(n)}\}$ spans the vector space $F^n(F)$ and is linearly independent. Hence, it is a basis for F^n. This basis is called standard basis for F^n.

REMARK. *Since the void set φ is linearly independent and spans the null space $\{0_V\}$. Therefore, the void set φ is the only basis for the null space $\{0_V\}$.*

Consider the subset $B = \{e_1^{(3)},\ e_2^{(3)},\ e_3^{(3)}\}$, where $e_1^{(3)} = (1,\ 0,\ 0)$, $e_2^{(3)} = (0,\ 1,\ 0)$, $e_3^{(3)} = (0,\ 0,\ 1)$, of the real vector space R^3. The set B spans R^3, because any vector $(a,\ b,\ c)$ in R^3 can be written as a linear combination of $e_1^{(3)}$, $e_2^{(3)}$ and $e_3^{(3)}$, namely,

$$(a,\ b,\ c) = ae_1^{(3)} + be_2^3 + ce_3^{(3)}$$

Also, B is linearly independent, because

$$\lambda\ e_1^{(3)} + \mu\ e_2^{(3)} + \nu e_3^{(3)} = (0,\ 0,\ 0)$$
$$\Rightarrow \quad \lambda(1,\ 0,\ 0) + \mu(0,\ 1,\ 0) + \nu(0,\ 0,\ 1) = (0,\ 0,\ 0)$$
$$\Rightarrow \quad (\lambda,\ \mu,\ \nu) = (0,\ 0,\ 0) \Rightarrow \lambda = \mu = \nu = 0$$

Hence, B is a basis for the real vector space R^3.

Now consider the subset $B_1 = \{(1,\ 1,\ 0),\ (1,\ 0,\ 1),\ (0,\ 1,\ 1)\}$ of R^3.
This set is also a basis for R^3, because any vector $(a,b,c) \in R^3$ can be written as

$$(a,\ b,\ c) = \left(\frac{a+b-c}{2}\right)(1,\ 1,\ 0) + \left(\frac{a_1+c-b}{2}\right)(1,\ 0,\ 1) + \left(\frac{b+c-a}{2}\right)(0,\ 1,\ 1)$$

and, $\lambda(1,\ 1,\ 0) + \mu(1,\ 0,\ 1) + \nu(0,\ 1,\ 1) \Rightarrow \lambda = \mu = \nu = 0$.

REMARK. *It follows from the above discussion that a basis for a vector space need not be unique. In fact, we shall show that corresponding to every non-zero vector in a vector space V one can obtain a basis for V.*

EXAMPLE-1 *Show that the set $B = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space of all real polynomials of degree not exceeding n.*

SOLUTION Let $\lambda_0, \lambda_1, \ldots, \lambda_n \in R$ be such that

$$\lambda_0 \cdot 1 + \lambda_1 \cdot x + \cdots + \lambda_n x^n = 0(x) \qquad \text{(zero polynomial)}$$

Then,

$$\lambda_0 \cdot 1 + \lambda_1 \cdot x + \cdots + \lambda_n x^n = 0(x) \quad \text{(zero polynomial)}$$

$$\Rightarrow \quad \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n = 0 + 0x + 0x^2 + \cdots + 0x^n + \ldots$$

$$\Rightarrow \quad \lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$$

Therefore, the set B is linearly independent.
Also, the set B spans the vector space $P_n(x)$ of all real polynomials of degree not exceeding n, because every polynomial of degree less than or equal n is a linear combination of B.
Hence, B is a basis for the vector space $P_n(x)$ of all real polynomials of degree not exceeding n.

EXAMPLE-2 *Show that the infinite set $\{1, x, x^2, \ldots\}$ is a basis for the vector space $R[x]$ of all polynomials over the field R of real numbers.*

SOLUTION The set $B = \{1, x, x^2, \ldots\}$ is linearly independent. Also, the set B spans $R[x]$, because every real polynomial can be expressed as a linear combination of B.

Hence, B is a basis for $R[x]$. Since B is an infinite set. Therefore, $R[x]$ is an infinite dimensional vector space over R.

EXAMPLE-3 *Let $a, b \in R$ such that $a < b$. Then the vector space $C[a, b]$ of all real valued continuous functions on $[a, b]$ is an infinite dimensional vector space.*

EXAMPLE-4 *The field R of real numbers is an infinite dimensional vector space over its subfield Q of rational numbers. But, it is a finite dimensional vector space over itself.*

SOLUTION Since π is a transcendental number, that is, π is not a root of any polynomial over Q. Thus, for any $n \in N$ and $a_i \in Q (i = 0, 1, 2, \ldots, n)$

$$a_0 + a_1\pi + a_2\pi^2 + \cdots + a_n\pi^n \neq 0$$

$\Rightarrow 1, \pi, \pi^2, \pi^3, \ldots, \pi^n, \ldots$ are linearly independent over Q.

Hence, R is an infinite dimensional vector space over Q.

EXAMPLE-5 *The set $\{1, i\}$ is a basis for the vector space C of all complex numbers over the field R of real numbers.*

We have defined basis of a vector space and we have seen that basis of a vector space need not be unique. Now a natural question arises, does it always exist?. The answer is in the affirmative as shown in the following theorem.

THEOREM–1 *Every finite dimensional vector space has a basis.*

PROOF. Let V be a finite dimensional vector space over a field F. If V is the null space, then the void set φ is its basis, we are done. So, let V be a non-null space. Since V is finite dimensional. So, there exists a finite subset $S = \{v_1, v_2, \ldots, v_n\}$ of V that spans V. If S is a linearly independent set, we are done. If S is not linearly independent, then by Theorem 2 on page 180, there exists a vector $v_k \in S$ which is linear combination of the previous vectors. Remove v_k from S and let $S_1 = S - \{v_k\}$. By Theorem 3 on page 167, S_1 spans V.

If S_1 is a linearly independent set, the theorem is proved. If not we repeat the above process on S_1 and omit one more vector to obtain $S_2 \subset S_1$. Continuing in this manner, we obtain successively $S_1 \supset S_2 \supset S_3, \ldots$, where each S_i spans V.

Since S is a finite set and each S_i contains one vector less than S_{i-1}. Therefore, we ultimately arrive at a linearly independent set that spans V. Note that this process terminates before we exhaust all vectors in S, because if not earlier, then after $(n-1)$ steps, we shall be left with a singleton set containing a non-zero vector that spans V. This singleton set will form a basis for V because each singleton set containing a non-zero vector in V forms a linearly independent set. Hence, V has a basis.

Q.E.D.

REMARK. *This theorem remains valid if the word finite is omitted. In that case, the proof requires the use of Zorn's lemma. This lemma involves several concepts which are beyond the scope of this book.*

NOTE. *In future unless otherwise mentioned a vector space will mean a finite dimensional vector space.*

THEOREM-2 *The representation of a vector in terms of the basis vectors is unique.*

PROOF. Let V be a vector space over a field F, and let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for V.

Let v be an arbitrary vector in V. We have to prove that v has a unique representation in terms of vectors in B.

Let $v = \lambda_1 b_1 + \cdots + \lambda_n b_n$ and $v = \mu_1 b_1 + \cdots + \mu_n b_n$ be two representations for $v \in V$ as a linear combination of basis vectors. Then,

$$\lambda_1 b_1 + \cdots + \lambda_n b_n = \mu_1 b_1 + \cdots + \mu_n b_n$$

$$\Rightarrow \quad (\lambda_1 - \mu_1) b_1 + \cdots + (\lambda_n - \mu_n) b_n = 0_V$$

$$\Rightarrow \quad \lambda_1 - \mu_1 = 0, \ldots, \lambda_n - \mu_n = 0 \qquad [\because B \text{ is a linearly independent set}]$$

$$\Rightarrow \quad \lambda_1 = \mu_1,\ \lambda_2 = \mu_2, \ldots, \lambda_n = \mu_n$$

Hence, v has a unique representation in terms of vectors in B. Q.E.D.

THEOREM-3 *Let S be a subset of a vector space $V(F)$ such that S spans V. Then S contains a basis for V.*

PROOF. By Theorem 1, the basis for V is obtained by removing those vectors from S, which are linear combinations of previous vectors in S. Hence, S contains a basis for V.

Q.E.D.

THEOREM-4 *A subset B of a vector space $V(F)$ is a basis for V if every vector in V has a unique representation as a linear combination of vectors of B.*

PROOF. First suppose that B is a basis for V. Then B spans V and so every vector in V is a linearly combination of vectors of B. To prove the uniqueness, we consider the following two cases:

Case I When B is finite, say $B = \{b_1, b_2, \ldots, b_n\}$.

Let v be an arbitrary vector in V and suppose that

$$v = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n \quad \text{and, also} \quad v = \mu_1 b_1 + \mu_2 b_2 + \cdots + \mu_n b_n,$$

where $\lambda_i,\ \mu_i \in F$ for all $i \in \underline{n}$. Then,

$$\lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n = \mu_1 b_1 + \mu_2 b_2 + \cdots + \mu_n b_n$$

$$\Rightarrow \quad (\lambda_1 - \mu_1) b_1 + \cdots + (\lambda_n - \mu_n) b_n = 0_V$$

$$\Rightarrow \quad \lambda_1 - \mu_1 = 0, \ldots, \lambda_n - \mu_n = 0 \qquad [\because B \text{ is linearly independent}]$$

$$\Rightarrow \quad \lambda_1 = \mu_1, \lambda_2 = \mu_2, \ldots, \lambda_n = \mu_n.$$

Thus, $v \in V$ has a unique representation as a linear combination of vectors of B.

Case II When B is infinite.

Let $v \in V$ and suppose that $v = \sum\limits_{b \in B} \lambda_b\, b$ and, also $v = \sum\limits_{b \in B} \mu_b\, b$

where $\lambda_b, \mu_b \in F$ and each sum contains only finite number of terms (i.e. λ_b, μ_b are zero in all except a finite number of terms). Then,

$$\sum_{b \in B} \lambda_b\, b = \sum_{b \in B} \mu_b\, b$$

$$\Rightarrow \quad \sum_{b \in B} (\lambda_b - \mu_b) b = 0_V$$

$$\Rightarrow \quad \lambda_b - \mu_b = 0 \quad \text{for all } b \in B \qquad [\because B \text{ is linearly independent}]$$

$$\Rightarrow \quad \lambda_b = \mu_b \quad \text{for all } b \in B$$

This proves that $v \in V$ has a unique representation as a linear combination of vectors of B.

Conversely, suppose that every vector in V has a unique representation as a linear combination of vectors of B. Then, B spans V. Further suppose that

$$\lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n = 0_V \quad \text{for some } \lambda_1, \lambda_2, \ldots, \lambda_n \in F,\ b_1, b_2, \ldots, b_n \in B.$$

Then,

$$\lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n = 0b_1 + 0b_2 + \cdots + 0b_n$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \quad \left[\begin{array}{l}\because \text{ Every vector in } V \text{ has a unique representation} \\ \text{as a linear combination of vectors in } B\end{array}\right]$$

Thus, the only linear combination of vectors of B that equals to the null vector is the trivial linear combination. Hence, B is a basis for V.

Q.E.D.

The following theorem proves an extremely important result that one cannot have more linearly independent vectors than the number of vectors in a spanning set.

THEOREM-5 *Let V be a vector space over a field F. If V is spanned by the set $\{v_1, v_2, \ldots, v_n\}$ of n vectors in V and if $\{w_1, w_2, \ldots, w_m\}$ is a linearly independent set of vectors in V, then $m \leqq n$. Moreover, V can be spanned by a set of n vectors containing the set $\{w_1, w_2, \ldots, w_m\}$.*

PROOF. We shall prove both the results together by induction on m.

First we shall prove the theorem for $m = 1$.

Let $\{w_1\}$ be a linearly independent set. Then $w_1 \neq 0_V$.

Now,

$w_1 \in V$

$\Rightarrow$ w_1 is a linear combination of $v_1, v_2, \ldots, v_n$ [$\because$ $\{v_1, v_2, \ldots, v_n\}$ spans V]

$\Rightarrow$ $\{w_1, v_1, v_2, \ldots, v_n\}$ is linearly dependent set [By Theorem 2 on page 180]

$\Rightarrow$ There exist $v_k \in \{w_1, v_1, \ldots, v_n\}$ such that v_k is a linear combination of the preceeding vectors

Let us rearrange the vectors $w_1, v_1, \ldots, v_n$ in such a way that v_n is a linear combination of the previous vectors. Removing v_n from the set $\{w_1, v_1, \ldots, v_n\}$ we obtain the set $\{w_1, v_1, \ldots, v_{n-1}\}$ of n vectors containing w_1 such that it spans V and $n - 1 \geqq 0 \Rightarrow n \geqq 1 = m$.

Note that the vector removed is one of v's because the set of w's is linearly independent.

Hence, the theorem holds for $m = 1$.

Now suppose that the theorem is true for m. This means that for a given set $\{w_1, \ldots, w_m\}$ of m linearly independent vectors in V, we have

(i) $m \leqq n$

and, (ii) there exists a set of n vectors $w_1, \ldots, w_m, v_1, \ldots, v_{n-m}$ in V that spans V.

To prove the theorem for $m + 1$, we have to show that for a given set of $(m + 1)$ linearly independent vectors $w_1, w_2, \ldots, w_m, w_{m+1}$ in V, we have

(i) $m + 1 \leqq n$

and, (ii) there exists a set of n vectors containing $w_1, \ldots, w_m, w_{m+1}$ that spans V.

we repeat the above process. Continuing in this manner we remove, one by one, every vector which is a linear combination of the preceding vectors, till we obtain a linearly independent set spanning V. Since S is a linearly independent set, v_i is a linear combination of the preceding vectors. Therefore, the vectors removed are some of b_i's. Hence, the reduced linearly independent set consists of all v_i's and some of b_i's. This reduced set $\{v_1, v_2, \ldots, v_m, b_1, b_2, \ldots, b_{n-m}\}$ is a basis for V containing S.

Q.E.D.

This theorem can also be stated as

"Any l.i. set of vectors of a finite dimensional vector space is a part of its basis."

COROLLARY-1 *In a vector space $V(F)$ of dimension n*

(i) any set of n linearly independent vectors is a basis,

and, (ii) any set of n vectors that spans V is a basis.

PROOF. (i) Let $B = \{b_1, b_2, \ldots, b_n\}$ be a set of n linearly independent vectors in vector space V. Then by Theorem 7, B is a part of the basis. But, the basis cannot have more than n vectors. Hence, B itself is a basis for V.

(ii) Let $B = \{b_1, b_2, \ldots, b_n\}$ be a set of n vectors in V that spans V. If B is not linearly independent, then there exists a vector in B which is linear combination of the preceding vectors and by removing it from B we will obtain a set of $(n-1)$ vectors in V that will also span V. But, this is a contradiction to the fact that a set of $(n-1)$ vectors cannot span V. Hence, B is a linearly independent set. Consequently, it is a basis for V.

Q.E.D.

COROLLARY-2 *Let $V(F)$ be a vector space of dimension n, and let S be a subspace of V. Then,*

(i) every basis of S is a part of a basis of V.

(ii) $\dim S \leqq \dim V$.

(iii) $\dim S = \dim V \Leftrightarrow S = V$.

and, (iv) $\dim S < \dim V$*, if S is a proper subspace of V.*

PROOF. (i) Let $B' = \{b_1, b_2, \ldots, b_m\}$ be a basis for S. Since S is a subspace of V. Therefore, B' is a linearly independent set of vectors in V and hence it is a part of a basis for V.

(ii) Since basis for S is a part of the basis for V. Therefore, $\dim S \leqq \dim V$.

(iii) Let $\dim S = \dim V = n$. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for V so that $[B] = S$.

Since B is a linearly independent set of n vectors in V and $\dim V = n$. Therefore, B spans V, i.e. $[B] = V$.
Thus, $S = [B] = V \Rightarrow \; S = V$.
Conversely, let $S = V$. Then,

$$
\begin{aligned}
& S = V \\
\Rightarrow \quad & S \text{ is a subspace of } V \text{ and } V \text{ is a subspace of } S \\
\Rightarrow \quad & \dim S \leqq \dim V \text{ and } \dim V \leqq \dim S \qquad \text{[By (ii)]} \\
\Rightarrow \quad & \dim S = \dim V.
\end{aligned}
$$

(iv) Let S be a proper subspace of V. Then there exists a vector $v \in V$ such that $v \notin S$. Therefore, v cannot be expressed as a linear combination of vectors in B', the basis for S. That is, $v \notin [S]$. Consequently, the set $\{b_1, b_2, \ldots, b_m, v\}$ forms a linearly independent subset of V. Therefore, the basis for V will contain more than m vectors.

Hence, $\dim S < \dim V$.

Q.E.D.

THEOREM-8 *Let $V(F)$ be a finite dimensional vector space, and let m be a positive integer. Then the function space $V^m = \{f : f : \underline{m} \to V\}$ is also finite dimensional and*

$$\dim(V^m) = m.\dim V$$

PROOF. Let $\dim V = n$, and let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for V. By Example 11 on page 126, the function space V^m is a vector space over field F. For any $i \in \underline{m}$, $j \in \underline{n}$ define a mapping

$$
\begin{aligned}
& \varphi_{ij} : \underline{m} \to V && \text{given by} \\
& \varphi_{ij}(k) = \delta_{ik} b_j && \text{for all } k \in \underline{m}.
\end{aligned}
$$

Here δ_{ik} is Kronecker delta

Let $B' = \left\{\varphi_{ij} : i \in \underline{m}, j \in \underline{n}\right\}$. Clearly $O(B') = mn$.

We shall now show that B' is a basis for the function space V^m.

$B'$is l.i.: Let $a_{ij} \in F, i \in \underline{m}, j \in \underline{n}$ be such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \varphi_{ij} = \widehat{0} \quad \text{(zero function)}$$

Then,

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\varphi_{ij}\right)(k) = \widehat{0}(k) \quad \text{for all } k \in \underline{m}$$

$$\Rightarrow \quad \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\varphi_{ij}(k) = 0_V$$

$$\Rightarrow \quad \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\delta_{ik}b_j = 0_V$$

$$\Rightarrow \quad \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\delta_{ik}\right)b_j = 0_V$$

$$\Rightarrow \quad \sum_{j=1}^{n} a_{kj}b_j = 0_V \qquad [\because \quad \delta_{ik} = 1 \text{ for } i = k]$$

$$\Rightarrow \quad a_{kj} = 0 \quad \text{for all } j \in \underline{n} \qquad [\because \quad B = \{b_1, \ldots, b_n\} \text{ is a basis for } V]$$

Thus, $a_{kj} = 0$ for all $k \in \underline{m}$ and for all $j \in \underline{n}$.

So, $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\varphi_{ij} = \widehat{0} \Rightarrow a_{ij} = 0$ for all $i \in \underline{m}$ and for all $j \in \underline{n}$.

Therefore, B' is a linearly independent set.

B' spans V^m: Let $f : \underline{m} \to V$ be an arbitrary mapping in V^m. Then for each $k \in \underline{m}$, $f(k) \in V$.

Since B is a basis for V. Therefore, $f(k) \in V$ is expressible as a linear combination of vectors in B. Let

$$f(k) = \sum_{j=1}^{n} \lambda_{kj}b_j \qquad \text{(i)}$$

For any $k \in \underline{m}$, we have

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\varphi_{ij}\right)(k) = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\varphi_{ij}(k) = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\delta_{ik}b_j$$

$$\Rightarrow \quad \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\varphi_{ij}\right)(k) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} \lambda_{ij}\delta_{ik}\right)b_j$$

$$\Rightarrow \quad \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\varphi_{ij}\right)(k) = \sum_{j=1}^{n} \lambda_{kj}b_j = f(k) \qquad \text{[From (i)]}$$

$$\therefore \quad f = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\varphi_{ij}$$

Hence, B_1 is a basis for $S+T$. Consequently, we have
$\dim S+T =$ number of vectors in $B_1 = p+q-r = \dim S + \dim T - \dim(S\cap T)$.

Q.E.D.

COROLLARY. *If S and T are subspaces of a finite-dimensional vector space $V(F)$ such that $S\cap T = \{0_V\}$, then*

$$\dim(S+T) = \dim S + \dim T.$$

PROOF. We have,

$$\begin{aligned} & S\cap T = \{0_V\} \\ \Rightarrow\quad & \dim(S\cap T) = 0 \\ \text{Thus,}\quad & \dim(S+T) = \dim S + \dim T - \dim(S\cap T) \\ \Rightarrow\quad & \dim(S+T) = \dim S + \dim T. \end{aligned}$$

Q.E.D.

In three-dimensional Euclidean space R^3, let $S =$ the xy-plane and $T =$ the yz-plane. Clearly, S and T are subspaces of R^3 and $\dim S = 2$, $\dim T = 2$. Clearly, $S\cap T = y$-axis, whose dimension is 1, and $S+T = R^3$. Therefore, $\dim S+T = \dim R^3 = 3$ and $\dim S + \dim T - \dim(S\cap T) = 2+2-1 = 3$. So, $\dim(S+T) = \dim S + \dim T - \dim(S\cap T)$. This verifies the result of Theorem 9. On the other hand, if $S =$ the xy-plane and $T = z$-axis, then $S\cap T = \{0_V\}$ and $S+T = R^3$. Also, $\dim R^3 = \dim(S+T) = 3 = 2+1 = \dim S + \dim T$. This verifies the result of corollary to Theorem 9.

THEOREM-10 *If a finite dimensional vector space $V(F)$ is direct sum of its two subspaces S and T, then*

$$\dim V = \dim S + \dim T.$$

PROOF. Since V is direct sum of S and T, therefore, $V = S+T$ and $S\cap T = \{0_V\}$.

Now, $S\cap T = \{0_V\} \quad\Rightarrow\quad \dim(S\cap T) = 0$
By Theorem 9, we have

$$\begin{aligned} & \dim(S+T) = \dim S + \dim T - \dim(S\cap T) \\ \Rightarrow\quad & \dim(S+T) = \dim S + \dim T. \end{aligned}$$

Q.E.D.

THEOREM-11 *Every subspace of a finite dimensional vector space has a complement.*

PROOF. Let V be a finite dimensional vector space over a field F, and let S be a subspace of V. Further, let $\dim V = n$ and $\dim S = m$.

Let $B_1 = \{b_1, b_2, \ldots, b_m\}$ be a basis for S. Then B is a linearly independent set of vectors in B. So it can be extended to a basis for V. Let $B = \{b_1, b_2, \ldots, b_m, v_1, v_2, \ldots, v_{n-m}\}$ be extended basis for V. Let T be a subspace of V spanned by the set $B_2 = \{v_1, v_2, \ldots, v_{n-m}\}$

We shall now show that T is the complement of S.

Let v be an arbitrary vector in V. Then,

$$v = \sum_{i=1}^{m} \lambda_i b_i + \sum_{i=1}^{n-m} \mu_i v_i \text{ for some } \lambda_i, u_i \in F \qquad [\because \quad B \text{ is a basis for } V]$$

Since B_1 and B_2 are bases for S and T respectively. Therefore,

$$\sum_{i=1}^{m} \lambda_i b_i \in S \text{ and, } \sum_{i=1}^{n-m} \mu_i v_i \in T$$

$$\Rightarrow \qquad v = \sum_{i=1}^{m} \lambda_i b_i + \sum_{i=1}^{n-m} \mu_i v_i \in S + T$$

Thus, $v \in V \Rightarrow \; v \in S + T$

$\therefore \quad V \subset S + T$

Also, $S + T \subset V$.

Hence, $V = S + T$.

Let $u \in S \cap T$. Then, $u \in S$ and $u \in T$.

Now,

$$u \in S \Rightarrow \text{ There exists } \lambda_1, \ldots, \lambda_m \in F \text{ such that } u = \sum_{i=1}^{m} \lambda_i b_i \qquad [\because \quad B_1 \text{ is a basis for } S]$$

and,

$$u \in T \Rightarrow \text{ There exists } \mu_1, \ldots, \mu_{n-m} \in F \text{ such that } u = \sum_{i=1}^{n-m} \mu_i v_i \qquad [\because \quad B_2 \text{ is a basis for } T]$$

$$\therefore \quad \sum_{i=1}^{m} \lambda_i b_i = \sum_{i=1}^{n-m} \mu_i v_i$$

$$\Rightarrow \qquad \lambda_1 b_1 +, \ldots, + \lambda_m b_m + (-\mu_1) v_1 + \cdots + (-\mu_{n-m}) v_{n-m} = 0_V$$

$$\Rightarrow \qquad \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0 = \mu_1 = \cdots = \mu_{n-m} \qquad [\because \quad B \text{ is a basis for } V]$$

$$\Rightarrow \qquad u = 0 b_1 + \cdots + 0 b_m = 0_V$$

Thus, $u \in S \cap T \Rightarrow \; u = 0_V$

$\therefore \quad S \cap T = \{0_V\}$

Hence, $V = S \oplus T$. Consequently, T is complement of S.

Q.E.D.

THEOREM-12 *Let $V(F)$ be a finite dimensional vector space, and let S be a subspace of V. Then,* $\dim(V/S) = \dim V - \dim S$.

PROOF. Let $\dim V = n, \dim S = m$. Then $n \geqq m$. Let $B_1 = \{v_1, v_2, \ldots, v_m\}$ be a basis for S. Then, B_1 is a linearly independent set of vectors in V. So it can be extended to a basis for V, say $B_2 = \{v_1, \ldots, v_m, b_1, \ldots, b_{n-m}\}$.

We shall now show that the set $B=\{b_1+S, b_2+S, \ldots, b_{n-m}+S\}$ is basis for V/S.

To prove that B is a basis for V/S, we have to show that B is a linearly independent set and B spans V/S.

B is a l.i set in V/S: Let $\lambda_1, \ldots, \lambda_{n-m}$ be scalars in F such that

$$\lambda_1(b_1+S)+\cdots+\lambda_{n-m}(b_{n-m}+S)=S \quad \text{(zero of } V/S)$$

Then,

$$(\lambda_1 b_1+S)+\cdots+(\lambda_{n-m}b_{n-m}+S)=S \qquad \text{[By def. of scalar multiplication on } V/S]$$

$$\Rightarrow \quad (\lambda_1 b_1+\cdots+\lambda_{n-m}b_{n-m})+S=S$$

$$\Rightarrow \quad \lambda_1 b_1+\cdots+\lambda_{n-m}b_{n-m}\in S \qquad [\because\ S \text{ is additive subgroup of } (V,+)\ \therefore\ S+u=S\Leftrightarrow u\in S]$$

$$\Rightarrow \quad \lambda_1 b_1+\cdots+\lambda_{n-m}b_{n-m} \text{ is a linear combination of } B_1=\{v_1,\ldots,v_m\}\ [\because\ B_1 \text{ is a basis for } S]$$

$$\Rightarrow \quad \lambda_1 b_1+\cdots+\lambda_{n-m}b_{n-m}=\mu_1 v_1+\cdots+\mu_m v_m \quad \text{for some } \mu_i\in F$$

$$\Rightarrow \quad \lambda_1 b_1+\cdots+\lambda_{n-m}b_{n-m}+(-\mu_1)v_1+\cdots+(-\mu_m)v_m=0_V$$

$$\Rightarrow \quad \lambda_1=\lambda_2=\cdots=\lambda_{n-m}=0=\mu_1=\cdots=\mu_m \qquad [\because\ B_2 \text{ is linearly independent}]$$

Thus, only the trivial linear combination of vectors of B equals to the null vector of V/S.

Hence, B is a linearly independent set in V/S.

B spans V/S: Let $u+S$ be an arbitrary vector in V/S. Then,

$$u\in V$$

$$\Rightarrow \quad u \text{ is a linear combination of } B_2 \qquad [\because\ B_2 \text{ is a basis for } V]$$

$$\Rightarrow \quad \text{there exist } \lambda_1,\ldots,\lambda_m,\ \mu_1,\ldots,\mu_{n-m} \text{ in } F \text{ such that}$$

$$u=\lambda_1 v_1+\cdots+\lambda_m v_m+\mu_1 b_1+\cdots+\mu_{n-m}b_{n-m}$$

$$\Rightarrow u=v+\mu_1 b_1+\cdots+\mu_{n-m}b_{n-m}, \qquad \text{where } v=\lambda_1 v_1+\cdots+\lambda_m v_m\in S\ [\because\ B_1 \text{is a basis for } S]$$

$$\Rightarrow u=(\mu_1 b_1+\cdots+\mu_{n-m}b_{n-m})+v \qquad [\because\ \text{vector addition is commutative on } V]$$

$$\Rightarrow u+S=(\mu_1 b_1+\cdots+\mu_{n-m}b_{n-m})+v+S$$

$$\Rightarrow u+S=(\mu_1 b_1+\cdots+\mu_{n-m}b_{n-m})+S \qquad [\because\ v\in S\ \therefore\ v+S=S]$$

$$\Rightarrow u+S=(\mu_1 b_1+S)+\cdots+(\mu_{n-m}b_{n-m}+S)$$

$$\Rightarrow u+S=\mu_1(b_1+S)+\cdots+\mu_{n-m}(b_{n-m}+S)$$

$\Rightarrow u+S$ is a linear combination of B.

Since $u+S$ is an arbitrary vector in V/S. Therefore, every vector in V/S is a linear combination of B. Consequently, B spans V/S, Hence, B is a basis for V/S.

$\therefore$ $\dim(V/S) =$ Number of vectors in $B = n - m = \dim V - \dim\ S$.

Q.E.D.

CO-DIMENSION. *Let $V(F)$ be a finite dimensional vector space and let S be a subspace of V. Then codimension of S is defined to be the dimension of the quotient space V/S.*

Clearly, Codimension $S = \dim V - \dim S$. *[By Theorem 12]*

MAXIMAL LINEARLY INDEPENDENT SET. *Let $V(F)$ be a vector space. Then a linearly independent set of vectors in V is said to be a maximal linearly independent set if every set of vectors in V containing it as a proper subset is a linearly dependent set.*

THEOREM-13 *If S is a maximal linearly independent set of vectors in a finite dimensional vector space $V(F)$, then S spans V, i.e. $[S] = V$.*

PROOF. Since V is a finite dimensional vector space. Therefore, S is a finite set. Let $S = \{v_1, \dots, v_n\}$, and let $v = \sum_{i=1}^{n} \lambda_i v_i$ be an arbitrary vector in V. Since S is a maximal linearly independent set. Therefore, the larger set $S' = \{v_1, \dots, v_n, v\}$ must be linearly dependent. Hence, we can find scalars $\mu_1, \mu_2, \dots, \mu_n, \mu$ not all zero such that

$$\mu_1 v_1 + \cdots + \mu_n v_n + \mu v = 0_V$$

We claim that $\mu \neq 0$, because

$\mu = 0$

$\Rightarrow$ $\mu_1 v_1 + \cdots + \mu_n v_n = 0_V$ where at least one $\mu_i \neq 0$.

$\Rightarrow$ S is a linearly dependent

$\Rightarrow$ a contradiction.

Now, $\mu_1 v_1 + \cdots + \mu_n v_n + \mu v = 0_V$

$\Rightarrow$ $\mu v = -\sum_{i=1}^{n} \mu_i v_i$

$\Rightarrow$ $v = \sum_{i=1}^{n} (\mu_i \mu^{-1}) v_i$ $[\because \mu \neq 0 \ \therefore \ \mu^{-1} \in F]$

$\Rightarrow$ v is a linear combination of S

Hence, S spans V.

Q.E.D.

THEOREM-14 *Let $V(F)$ be a finite dimensional vector space. Then a set of n linearly independent vectors in V is maximal iff $n = \dim V$.*

PROOF. Let $S = \{v_1, \dots, v_n\}$ be a set of n linearly independent vectors in V.

Hence, $B_1 \cup B_2$ is a basis for V.

Now we shall show that $V = S_1 \oplus S_2$. For this, it is sufficient to show that $S_1 \cap S_2 = \{0_V\}$.

Let $v \in S_1 \cap S_2$. Then, $v \in S_1$ and $v \in S_2$.

Since $B_1 = \{v_1, \ldots, v_m\}$ is a basis for S_1. Therefore, there exist scalars $\lambda_1, \ldots, \lambda_m \in F$ such that

$$v = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Since $B_2 = \{u_1, \ldots, u_n\}$ is a basis for S_2 and $v \in S_2$. Therefore, there exist scalars $\mu_1, \mu_2, \ldots, \mu_n \in F$ such that

$$v = \mu_1 u_1 + \cdots + \mu_n u_n$$

$$\therefore \quad \lambda_1 v_1 + \cdots + \lambda_m v_m = \mu_1 u_1 + \cdots + \mu_n u_n$$

$$\Rightarrow \quad \lambda_1 v_1 + \cdots + \lambda_m v_m + (-\mu_1) u_1 + \cdots + (-\mu_n) u_n = 0_V$$

$$\Rightarrow \quad \lambda_1 = \cdots = \lambda_m = 0 = \mu_1 = \cdots = \mu_n \qquad [\because \ B_1 \cup B_2 \text{ is a basis for } V]$$

$$\Rightarrow \quad v = 0v_1 + \cdots + 0v_m = 0_V$$

Thus, $v \in S_1 \cap S_2 \Rightarrow v = 0_V$

$\therefore \quad S_1 \cap S_2 = \{0_V\}$

Hence, $V = S_1 \oplus S_2$

Q.E.D

THEOREM-17 *Let S and T be subspaces of a finite dimensional vector space V such that $V = S \oplus T$. Let B_1 and B_2 be bases for S and T respectively. Then $B_1 \cup B_2$ is basis for V and* $\dim V = \dim S + \dim T$.

PROOF. Since V is a finite dimensional vector space over a field F, therefore, S and T are also finite dimensional vector spaces over field F. Let $\dim S = m$ and $\dim T = n$. Further let $B_1 = \{b_1, b_2, \ldots, b_m\}$ be a basis for S and $B_2 = \{b_{m+1}, b_{m+2}, \ldots, b_{m+n}\}$ be a basis for T. Let $B = B_1 \cup B_2 = \{b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_{m+n}\}$.

We shall now establish that B is a basis for V.

B is linearly independent: Let $\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n}$ be scalars in F such that

$$\lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_m b_m + \lambda_{m+1} b_{m+1} + \cdots + \lambda_{m+n} b_{m+n} = 0_V$$

$$\Rightarrow \quad \lambda_1 b_1 + \cdots + \lambda_m b_m = -(\lambda_{m+1} b_{m+1} + \cdots + \lambda_{m+n} b_{m+n}) \qquad \text{(i)}$$

$$\Rightarrow \quad \lambda_1 b_1 + \cdots + \lambda_m b_m \in T \qquad \left[\because \ -\sum_{i=1}^{n} \lambda_{m+i} b_{m+i} \in T\right]$$

THEOREM-20 *A vector space $V(F)$ is the direct sum of its $n(\geq 2)$ subspaces $S_1, S_2, \dots, S_n$ i.e. $V = S_1 \oplus S_2 \oplus S_3 \oplus \cdots \oplus S_n$ iff*

$$\text{(i) } V = S_1 + S_2 + \cdots + S_n \quad \text{(ii) } S_i \cap \sum_{\substack{j \neq i \\ j=i}}^{n} S_j = \{0_V\} \quad \text{for each } i = 1, 2, \dots, n$$

PROOF. First suppose that $V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$. Then, $V = S_1 + S_2 + \cdots + S_n$

Let v be an arbitrary vector in $S_i \cap \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} S_j$. Then,

$$v \in S_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j \Rightarrow v \in S_i \text{ and } v \in \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j$$

Now, $v \in S_i$

$\Rightarrow \quad v = v_i \quad$ for some $v_i \in S_i$

$\Rightarrow \quad v = 0_V + 0_V + \cdots + v_i + 0_V + \cdots + 0_V \qquad [\because \quad 0_V \in S_j$ for all $j]$

and, $\quad v \in \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} S_j$

$\Rightarrow$ There exist vectors $v_j \in S_j,\ j = 1, 2, \dots, n$ and $j \neq i$ such that

$$v = v_1 + v_2 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_n$$

But, $V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$. So, each vector in V has unique representation as the sum of vectors in $S_1, S_2, \dots, S_n$.

$\therefore \quad v_1 = 0_V = v_2 = \cdots = v_{i-1} = v_i = v_{i+1} = \cdots = v_n$

$\Rightarrow \quad v = 0_V$

Since v is an arbitrary vector in $S_i \cap \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} S_j$.

$$\therefore \quad S_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j = \{0_V\}$$

Conversely, let $S_1, S_2, \dots, S_n$ be subspaces of $V(F)$ such that (i) and (ii) hold. We have to prove that V is the direct sum of its subspaces $S_1, S_2, \dots, S_n$.

Let v be an arbitrary vector in V. Then,

$v \in S_1 + S_2 + \cdots + S_n \qquad [\because \quad V = S_1 + S_2 + \cdots + S_n]$

$\Rightarrow \quad v = v_1 + v_2 + \cdots + v_n, \quad$ where $v_i \in S_i;\ i = 1, 2, \dots, n$

B spans V: Let v be an arbitrary vector in V. Since V is the direct sum of $S_1,\ldots,S_k$. Therefore, there exist unique $v_1 \in S_1,\ldots,v_k \in S_k$ such that

$$v = v_1 + \cdots + v_k.$$

Since for each $i \in \underline{k}$, B_i is a basis for S_i. Therefore, each v_i can be expressed as a linear combination of vectors in B_i. Consequently, v is expressible as a linear combination of vectors in $\bigcup\limits_{i=1}^{k} B_i = B$. Hence, B spans V.

B is a l.i. set in V: To show that B is a linearly independent set, let $\lambda_1^1,\ldots,\lambda_{n_1}^1,\lambda_1^2,\ldots,\lambda_{n_2}^2,\ldots,$ $\lambda_1^i,\ldots,\lambda_{n_i}^i,\ldots,\lambda_k^k,\ldots,\lambda_{n_k}^k$ be scalars in F such that

$$\begin{aligned} &\left(\lambda_1^1 b_1^1 + \cdots + \lambda_{n_1}^1 b_{n_1}^1\right) + \cdots + \left(\lambda_1^i b_1^i + \cdots + \lambda_{n_i}^i b_{n_i}^i\right) + \cdots + \left(\lambda_k^k b_1^k + \cdots + \lambda_{n_k}^k b_{n_k}^k\right) = 0_V \\ \Rightarrow\quad &\left(\lambda_1^1 b_1^1 + \cdots + \lambda_{n_1}^1 b_{n_1}^1\right) + \cdots + \left(\lambda_k^k b_1^k + \cdots + \lambda_{n_k}^k b_{n_k}^k\right) = \underbrace{0_V + 0_V + \cdots + 0_V}_{k-\text{terms}} \qquad \text{(i)} \end{aligned}$$

Since V is the direct sum of $S_1,\ldots,S_k$. Therefore, each vector in V is uniquely expressible as a sum of vectors one in each S_i. Consequently, (i) implies that

$$\begin{aligned} &\lambda_1^i b_1^i + \cdots + \lambda_{n_i}^i b_{n_i}^i = 0_V \quad \text{for all } i = 1,2,\ldots,k \\ \Rightarrow\quad &\lambda_1^i = \cdots = \lambda_{n_i}^i = 0 \qquad \text{for all } i = 1,2,\ldots,k \qquad [\because\ B_i \text{ is a basis for } S_i] \end{aligned}$$

Hence, B is a linearly independent set in V.

Thus, B is a basis for V.

(ii) $\Rightarrow$ (i).

Let $B = \bigcup\limits_{i=1}^{k} B_i$ be a basis for V. Then we have to show that V is the direct sum of $S_1,S_2,\ldots,S_k$.

Let v be an arbitrary vector in V. Then,

$$\begin{aligned} &v = \sum_{i=1}^{k}\left(\lambda_1^i b_1^i + \cdots + \lambda_{n_i}^i b_{n_i}^i\right) \qquad \left[\because\ B = \bigcup_{i=1}^{k} B_i \text{ is a basis for } V\right] \\ \Rightarrow\quad &v = \sum_{i=1}^{k} v_i \text{ , where } v_i = \lambda_1^i b_1^i + \cdots + \lambda_{n_i}^i b_{n_i}^i \in S_i \qquad [\because\ B_i \text{ is a basis for } S_i] \end{aligned}$$

Thus, v is expressible as a sum of vectors one in each S_i. Since v is an arbitrary vector in V. Therefore, each vector in V is expressible as a sum of vectors one in each S_i. Consequently, $V = S_1 + S_2 + \cdots + \cdots + S_k$.

Now we shall show that the expression of each vector in V as a sum of vectors one in each S_i is unique. If possible, let $v \in V$ be such that

$$v = \sum_{i=1}^{k}\left(\lambda_1^i b_1^i + \cdots + \lambda_{n_i}^i b_{n_i}^i\right) \text{ and } v = \sum_{i=1}^{k}\left(\mu_1^i b_1^i + \cdots + \mu_{n_i}^i b_{n_i}^i\right).$$

Then, $$\sum_{i=1}^{k}\left(\lambda_1^i b_1^i+\cdots+\lambda_{n_i}^i b_{n_i}^i\right)=\sum_{i=1}^{k}\left(\mu_1^i b_1^i+\cdots+\mu_{n_i}^i b_{n_i}^i\right)$$

$$\Rightarrow \quad \sum_{i=1}^{k}\left\{(\lambda_1^i-\mu_1^i)b_1^i+(\lambda_2^i-\mu_2^i)b_2^i+\cdots+(\lambda_{n_i}^i-\mu_{n_i}^i)b_{n_i}^i\right\}=0_V$$

$$\Rightarrow \quad \lambda_1^i-\mu_1^i=0, \lambda_2^i-\mu_2^i=0,\ldots,\lambda_{n_i}^i-\mu_{n_i}^i=0 \quad \text{for } i=1,2,\ldots,k \quad \left[\because \quad B=\bigcup_{i=1}^{k} B_i \text{ is l.i.}\right]$$

$$\Rightarrow \quad \lambda_1^i=\mu_1^i,\ldots,\lambda_{n_i}^i=\mu_{n_i}^i \quad \text{for } i=1,2,\ldots,k$$

Thus, v is uniquely expressible as a sum of vectors one in each S_i. Therefore, every vector in V is expressible as a sum of vectors one in each S_i.

Hence, V is the direct sum of $S_1, S_2, \ldots, S_k$.

Q.E.D.

THEOREM-22 *Let S and T be two subspaces of a vector space $V(F)$ such that $S\cap T=\{0_V\}$. If B_1 and B_2 be bases of S and T respectively, then show that $B=B_1\cup B_2$ is a basis of $S+T$ and $B_1\cap B_2=\varphi$.*

PROOF. Let $B_1=\{b_1,b_2,\ldots,b_m\}$ and $B_2=\{b_1',b_2',\ldots,b_n'\}$ be bases of S and T respectively. Then, B_1 and B_2 are linearly independent subsets of S and T respectively. Therefore, $B_1\cap B_2$ is also linearly independent and $B_1\cap B_2\subseteq S\cap T$.

$$\therefore \quad B_1\cap B_2\subseteq\{0_V\}$$

$$\Rightarrow \quad B_1\cap B_2=\varphi \qquad [\because \quad B_1 \text{ and } B_2 \text{ are l.i. and any l.i. set cannot have null vector in it}]$$

We shall now show that $B=B_1\cup B_2$ is a basis for $S+T$.

$B=B_1\cup B_2$ is a l.i. subset of $S+T$: Clearly, S and T are subsets of $S+T$. So, $S\cup T\subseteq S+T$.

$$\therefore \quad B_1\cup B_2\subseteq S\cup T\subseteq S+T.$$

Let $\lambda_i (i=1,2,\ldots,m)$ and $\mu_j (j=1,2,\ldots,n)$ be scalars in F such that

$$\sum_{i=1}^{m}\lambda_i b_i+\sum_{j=1}^{n}\mu_j b_j'=0_V$$

$$\Rightarrow \quad \sum_{i=1}^{m}\lambda_i b_i=-\sum_{j=1}^{n}\mu_j b_j'=u \text{ (say)}$$

Clearly, $\sum_{i=1}^{m}\lambda_i b_i\in S$ and $-\sum_{j=1}^{n}\mu_j b_j'\in T$

$$\Rightarrow \quad u\in S \text{ and } u\in T$$

$$\Rightarrow \quad u\in S\cap T$$

$$\Rightarrow \quad u=0_V \qquad [\because \quad S\cap T=\{0_V\}]$$

$$\therefore \quad \sum_{i=1}^{m}\lambda_i b_i=0_V \text{ and, } -\sum_{j=1}^{n}\mu_j b_j'=0_V$$

$$\Rightarrow \quad \lambda_i=0 (i=1,2,\ldots,m), \mu_j=0 (j=1,2,\ldots,n) \; [\because \quad B_1 \text{ and } B_2 \text{ are linearly independent}]$$

SOLUTION (i) Since $\dim R^3 = 3$. So, a basis of $R^3(R)$ must contain exactly 3 elements. Hence, B_1 is not a basis of $R^3(R)$.

(ii) Since $\dim R^3 = 3$. So, B_2 will form a basis of $R^3(R)$ if and only if vectors in B_2 are linearly independent. To check this, let us form the matrix A whose rows are the given vectors as given below.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

Since non-zero rows of a matrix in echelon form are linearly independent. So, let us reduce A to echelon form.

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 + 3R_2$$

Clearly, the echelon form of A has no zero rows. Hence, the three vectors are linearly independent and so they form a basis of R^3.

(iii) Since $(n+1)$ or more vectors in a vector space of dimension n are linearly dependent. So, B_3 is a linearly dependent set of vectors in $R^3(R)$. Hence, it cannot be a basis of R^3.

(iv) The matrix A whose rows are the vectors in B_4 is given by

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

Since non-zero rows of a matrix in echelon form are linearly independent. So, let us reduce A to echelon form.

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 5R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 + 2R_2$$

The echelon form of A has a zero row, hence the given vectors are linearly dependent and so B_4 does not form a basis of R^3.

EXAMPLE-2 *Let V be the vector space of all 2×2 matrices over a field F. Prove that V has dimension 4 by finding a basis for V.*

SOLUTION Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be four matrices in V, where 1 is the unity element in F.

Let $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.

We shall now show that B forms a basis of V.

B is l.i.: Let x, y, z, t be scalars in F such that

$$xE_{11} + yE_{12} + zE_{21} + tE_{22} = 0$$

$$\Rightarrow \quad \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x = y = z = t = 0$$

So, B is l.i.

B spans v: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary matrix in V. Then,

$$A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$$

$\Rightarrow$ A is expressible as a linear combination of matrices in B.

So, B spans V. Thus, B is a basis of V.

Hence, $\dim V = 4$.

EXAMPLE-3 *Let $v_1 = (1, i, 0), v_2 = (2i, 1, 1), v_3 = (0, 1+i, 1-i)$ be three vectors in $C^3(C)$. Show that the set $B = \{v_1, v_2, v_3\}$ is a basis of $C^3(C)$.*

SOLUTION We know that $\dim C^3(C) = 3$. Therefore, B will be a basis of $C^3(C)$ if B is a linearly independent set. Let $x, y, z \in C$ such that

$$xv_1 + yv_2 + zv_3 = 0$$

$$\Rightarrow \quad \big(x + 2iy, xi + y + z(1+i), y + z(1-i)\big) = (0, 0, 0)$$

$$\Rightarrow \quad x + 2iy = 0, xi + y + z(1+i) = 0, y + z(1-i) = 0$$

$$\Rightarrow \quad x = y = z = 0.$$

Hence, B is a basis of $C^3(C)$.

EXAMPLE-4 *Determine whether $(1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5)$ form a basis of $R^4(R)$. If not, find the dimension of the subspace they span.*

SOLUTION Given four vectors can form a basis of $R^4(R)$ iff they are linearly independent as the dimension of R^4 is 4.

The matrix A having given vectors as its rows is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix}$$

Since non-zero rows of a matrix in echelon form are linearly independent. So, let us reduce A to echelon form

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_2, R_4 \to R_4 - 2R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - 3R_2, R_4 \to R_4 - 4R_3$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_4 \to R_4 - R_3$$

The echelon matrix has a zero row. So, given vectors are linearly independent and do not form a basis of R^4. Since the echelon matrix has three non-zero rows, so the four vectors span a subspace of dimension 3.

TYPE II **ON EXTENDING A GIVEN SET TO FORM A BASIS OF A GIVEN VECTOR SPACE**

EXAMPLE-5 *Extend the set* $\{u_1 = (1,1,1,1), u_2 = (2,2,3,4)\}$ *to a basis of* R^4.

SOLUTION Let us first form a matrix A with rows u_1 and u_2, and reduce it to echelon form:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We observe that the vectors $v_1 = (1,1,1,1)$ and $v_2 = (0,0,1,2)$ span the same space as spanned by the given vectors u_1 and u_2. In order to extend the given set of vectors to a basis of $R^4(R)$, we need two more vectors u_3 and u_4 such that the set of four vectors v_1, v_2, u_3, u_4 is linearly independent. For this, we chose u_3 and u_4 in such a way that the matrix having v_1, v_2, u_3, u_4 as its rows is in echelon form. Thus, if we chose $u_3 = (0,a,0,0)$ and $u_4 = (0,0,0,b)$, where a, b are non-zero real numbers, then v_1, u_3, v_2, u_4 in the same order form a matrix in echelon form. Thus, they are linearly independent, and they form a basis of R^4. Hence, u_1, u_2, u_3, u_4 also form a basis of R^4.

EXAMPLE-6 *Let* $v_1 = (-1,1,0), v_2 = (0,1,0)$ *be two vectors in* $R^3(R)$ *and let* $S = \{v_1, v_2\}$. *Extend set* S *to a basis of* $R^3(R)$.

SOLUTION Let A be the matrix having v_1 and v_2 as its two rows. Then,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Clearly, A is in echelon form. In order to form a basis of $R^3(R)$, we need one more vector such that the matrix having that vector as third row and v_1, v_2 as first and second rows is in echelon form. If we take $v_3 = (0, 0, a)$, where $a(\neq 0) \in R$, then matrix having its three rows as v_1, v_2, v_3 is in echelon form.Thus, v_1, v_2, v_3 are linearly independent and they form a basis of $R^3(R)$.

REMARK. *Sometimes, we are given a list of vectors in the vector space $R^n(R)$ and we want to find a basis for the subspace S of R^n spanned by the given vectors, that is, a basis of $[S]$. The following two algorithms help us for finding such a basis of $[S]$.*

ALGORITHM 1 *(Row space algorithm)*

Step I *Form the matrix A whose rows are the given vectors*

Step II *Reduce A to echelon form by elementary row operations.*

Step III *Take the non-zero rows of the echelon form. These rows form a basis of the subspace spanned by the given set of vectors.*

In order to find a basis consisting of vectors from the original list of vectors, we use the following algorithm.

ALGORITHM 2 *(Casting-out algorithm)*

Step I *Form the matrix A whose columns are the given vectors.*

Step II *Reduce A to echelon form by elementary row operations.*

Step III *Delete (cast out) those vectors from the given list which correspond to columns without pivots and select the remaining vectors in S which correspond to columns with pivots. Vectors so selected form a basis of $[S]$.*

TYPE III **ON FINDING THE DIMENSION OF SUBSPACE SPANNED BY A GIVEN SET OF VECTORS**

EXAMPLE-7 *Let S be the set consisting of the following vectors in R^5:*

$v_1 = (1, 2, 1, 3, 2), v_2 = (1, 3, 3, 5, 3), v_3 = (3, 8, 7, 13, 8), v_4 = (1, 4, 6, 9, 7), v_5 = (5, 13, 13, 25, 19)$

Find a basis of $[S]$ (i.e the subspace spanned by S) consisting of the original given vectors. Also, find the dimension of $[S]$.

SOLUTION Let A be the matrix whose columns are the given vectors.Then,

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 2 & 3 & 8 & 4 & 13 \\ 1 & 3 & 7 & 6 & 13 \\ 3 & 5 & 13 & 9 & 25 \\ 2 & 3 & 8 & 7 & 19 \end{bmatrix}$$

Let us now reduce A to echelon form by using elementary row operations.

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 5 & 8 \\ 0 & 2 & 4 & 6 & 10 \\ 0 & 1 & 2 & 5 & 9 \end{bmatrix} \qquad \begin{array}{l}\text{Applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 3R_1 \text{ and } R_5 \to R_5 - R_1\end{array}$$

$$\Rightarrow \qquad A \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \qquad \text{Applying } R_3 \to R_3 - 2R_2, R_4 \to R_4 - 2R_2, R_5 \to R_5 - R_1$$

$$\Rightarrow \qquad A \sim \begin{bmatrix} \textcircled{1} & 1 & 3 & 1 & 5 \\ 0 & \textcircled{1} & 2 & 2 & 3 \\ 0 & 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_4 \to R_4 - 2R_3, R_5 \to R_5 - 2R_3$$

We observe that pivots (encircled entries) in the echelon form of A appear in the columns C_1, C_2, C_4. So, we "cast out" the vectors u_3 and u_5 from set S and the remaining vectors u_1, u_2, u_4, which correspond to the columns in the echelon matrix with pivots, form a basis of $[S]$. Hence, $\dim[S] = 3$.

EXAMPLE-8 *Let S be the set consisting of following vectors in* R^4*:*

$$v_1 = (1, -2, 5, -3), v_2 = (2, 3, 1, -4), v_3 = (3, 8, -3, -5).$$

(i) Find a basis and dimension of the subspace spanned by S, i.e. $[S]$.

(ii) Extend the basis of $[S]$ *to a basis of* R^4.

(iii) Find a basis of $[S]$ *consisting of the original given vectors.*

SOLUTION (i) Let A be the matrix whose rows are the given vectors. Then,

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

Let us now reduce A to echelon form. The row reduced echelon form of A is as given below.

$$A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let B denote the matrix whose rows are v_1, v_2, v_3. Then, $B = \begin{bmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{bmatrix}$

$$\Rightarrow \quad B = \begin{bmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{bmatrix}$$

$$\Rightarrow \quad B \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \qquad \text{Applying } R_2 \to R_2 - 3R_1, R_3 \to R_3 - 3R_1$$

$$\Rightarrow \quad B \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_3 \to R_3 - 3R_2$$

$$\Rightarrow \quad B \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_1 \to R_1 + 2R_2$$

Clearly, A and B have the same row canonical form. So, row spaces of A and B are equal. Hence, $[S] = [T]$.

EXERCISE 2.9

1. Mark each of the following true or false.
   - (i) The vectors in a basis of a vector space are linearly dependent.
   - (ii) The null (zero) vector may be a part of a basis.
   - (iii) Every vector space has a basis.
   - (iv) Every vector space has a finite basis.
   - (v) A basis cannot have the null vector.
   - (vi) If two bases of a vector space have one common vector, then the two bases are the same.
   - (vii) A basis for $R^3(R)$ can be extended to a basis for $R^4(R)$.
   - (viii) Any two bases of a finite dimensional vector space have the same number of vectors.
   - (ix) Every set of $n+1$ vectors in an n-dimensional vector space is linearly dependent.
   - (x) Every set of $n+1$ vectors in an n-dimensional vector space is linearly independent.
   - (xi) An n-dimensional vector space can be spanned by a set $n-1$ vectors in it.
   - (xii) Every set of n linearly independent vectors in an n-dimensional vector space is a basis.

(xiii) A spanning set of n vectors in an n-dimensional vector space is a basis.

(xiv) $[v]_B$ is independent of B.

(xv) If S and T are subspaces of a vector space $V(F)$, then
$\dim S < \dim T \Rightarrow S \subsetneqq T$.

2. Find three bases for the vector space $R^2(R)$ such that no two of which have a vector in common.
3. Determine a basis for each of the following vector spaces:

    (i) $R^2(R)$

    (ii) $R(\sqrt{2}) = \{a + b\sqrt{2} : a,\ b \in R\}$ over R

    (iii) $Q(\sqrt{2}) = \{a + b\sqrt{2} : a,\ b \in Q\}$ over Q

    (iv) C over R

    (v) C over itself

    (vi) $Q(i) = \{a + ib :\ a,\ b \in Q\}$ over Q

4. Which of the following subsets B form a basis for $R^3(R)$?

    (i) $B = \{(1,0,0), (1,1,0), (1,1,1)\}$

    (ii) $B = \{(0,0,1), (1,0,1)(1,-1,1), (3,0,1)\}$

    (iii) $B = \{(1,2,3), (3,1,0), (-2,1,3)\}$

    (iv) $B = \{(1,1,1), (1,2,3), (-1,0,1)\}$

    (v) $B = \{(1,1,2), (1,2,5), (5,3,4)\}$

5. Which of the following subsets B form a basis for the given vector space V?

    (i) $B = \{(1,0), (i,0), (0,1), (0,i)\}, V = C^2(R)$.

    (ii) $B = \{(1,i,1+i), (1,i,1-i), (i,-i,1)\}, V = C^3(C)$.

    (iii) $B = \{(1,1,1,1), (2,6,4,5), (1,2,1,2), (0,3,2,3)\}, V = R^4(R)$.

    (iv) $B = \{1,\ \sin x,\ \sin^2 x,\ \cos^2 x\}, V = C[-\pi, \pi]$

    (v) $B = \{x-1,\ x^2+x-1,\ x^2-x+1\}, V =$ Space of all real polynomials of degree at most two.

6. Extend the set $S = \{(1,1,1,1), (1,2,1,2)\}$ to a basis for $R^4(R)$.
7. Extend the set $S = \{(3,-1,2)\}$ to two different bases for $R^3(R)$.
8. Consider the set S of vectors in $R^3(R)$ consisting of $v_1 = (1,1,0), v_2 = (0,2,3)$, $v_3 = (1,2,3), v_4 = (1,-2,3)$.

    (i) Prove that S is linearly dependent.

    (ii) Find a subset B of S such that B is a basis of $[S]$.

    (iii) Find $\dim[S]$.

9. Let S be the subspace of R^5 spanned by the vectors. $v_1=(1,2,-1,3,4), v_2=(2,4,-2,6,8)$, $v_3=(1,3,2,2,6), v_4=(1,4,5,1,8), v_5=(2,7,3,3,9)$. Find a subset of these vectors that form a basis of S.
10. Prove that the set $S=\{a+2ai : a\in R\}$ is a subspace of the vector space $C(R)$. Find a basis and dimension of S. Let $T=\{a-2ai : a\in R\}$. Prove that $C(R)=S\oplus T$.
11. Extend each of the following subsets of R^3 to a basis of $R^3(R)$:
    (i) $B_1=\{(1,2,0)\}$ (ii) $B_2=\{(1,-2,0),(0,1,1)\}$
12. Find a basis of the vector space $R^4(R)$ that contains the vectors $(1,1,0,0),(0,1,0,-1)$.
13. Find four matrices A_1,A_2,A_3,A_4 in $R^{2\times 2}(R)$ such that $A_i^2=A_i$ for $1\le i\le 4$ and $\{A_1,A_2,A_3,A_4\}$ is a basis of $R^{2\times 2}$.
14. Let S be the set of all matrices A in $R^{2\times 2}$ which commute with the matrix $\begin{bmatrix}1&1\\2&2\end{bmatrix}$. Prove that S is a subspace of $R^{2\times 2}$. Find $\dim S$.
15. Let $V=R^{3\times 3}, S$ be the set of symmetric matrices and T be the set of skew-symmetric matrices in V. Prove that S and T are subspaces of V such that $V=S\oplus T$. Find dimensions of S and T.
16. Express $R^5(R)$ as a direct sum of two subspaces S and T with $\dim S=2, \dim T=3$.
17. If S and T be two subspaces of a finite dimensional vector space $V(F)$ such that

$$\dim V=\dim S+\dim T \text{ and } S\cap T=\{0_V\}.$$

    Prove that $V=S\oplus T$.
18. Let $V(F)$ be a finite dimensional vector space, and let $S_1,S_2,\dots,S_k$ be subspaces of V such that

$$V=S_1+S_2+\cdots+S_k \text{ and } \dim V=\dim S_1+\dim S_2+\cdots+\dim S_k.$$

    Prove that $V=S_1\oplus S_2\oplus\cdots\oplus S_k$.
19. Show by means of an example that under certain circumstances, it is possible to find three subspaces S_1,S_2,S_3 of a finite dimensional vector space $V(F)$ such that $V=S_1\oplus S_2= S_2\oplus S_3=S_3\oplus S_1$. What does this imply about the dimension of V?
20. Give an example of an infinite dimensional vector space $V(F)$ with a subspace S such that V/S is a finite dimensional vector space.
21. Construct three subspaces S_1,S_2,S_3 of a vector space V such that $V=S_1\oplus S_2=S_1\oplus S_3$ but $S_2\neq S_3$
22. Give an example of a vector space $V(R)$ having three different non-zero subspaces S_1,S_2,S_3 such that

$$V=S_1\oplus S_2=S_2\oplus S_3=S_3\oplus S_1.$$

23. Let $S=\{(a,b,0) : a,b\in R\}$ be a subspace of R^3. Find its two different complements in R^3.

24. Determine a basis of the subspace spanned by the vectors:
$v_1 = (1,\ 2,\ 3),\ v_2 = (2,\ 1,\ -1),\ v_3 = (1,\ -1,\ -4),\ v_4 = (4,\ 2,\ -2)$

25. Show that the vectors $v_1 = (1,\ 0,\ -1),\ v_2 = (1,\ 2,\ 1),\ v_3 = (0,\ -3,\ 2)$ form a basis for R^3. Express each of the standard basis vectors $e_1^{(3)},\ e_2^{(3)},\ e_3^{(3)}$ as a linear combination of $v_1,\ v_2,\ v_3$.

ANSWERS

1. (i) F (ii) F (iii) T (iv) F (v) T (vi) F (vii) F (viii) T (ix) T (x) F (xi) F (xii) T (xiii) T (xiv) F (xv) F

2. $B_1 = \{(1,0),(0,1)\}, B_2 = \{(1,-1),(2,3)\}, B_3 = \{(-2,1),(2,-3)\}$.
Vectors (a_1,b_1) and (a_2,b_2) form a basis of $R^2(R)$ iff $a_1b_2 \neq a_2b_1$.

3. (i) $B = \{(1,0),(0,1)\}$ (ii) $B = \{1,\sqrt{2}\}$ (iii) $B = \{1,\sqrt{2}\}$ (iv) $B = \{1,i\}$
(v) $B = \{1,i\}$ (vi) $B = \{1,i\}$

4. (i)

5. (ii), (iii)

6. $\{(1,1,1,1),(1,2,1,2),(0,0,1,0),(0,0,0,1)\}$

7. $\{(3,-1,2),(0,1,0),(0,0,1)\},\{(3,-1,2),(0,-1,0),(0,0,2)\}$

8. (ii) $B = \{(1,1,0),(0,2,3)\}$ (ii) $\dim[S] = 2$

9. $\{v_1,v_3,v_5\}$

10. $\{1+2i\}$, 1

11. (i) $\{(1,2,0),(0,1,0),(0,0,1)\}$ (ii) $\{(1,-2,0),(0,1,1),(0,0,1)\}$

12. $\{(1,1,0,0),(0,1,0,-1),(0,0,1,0),(0,0,0,1)\}$

13. $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

15. $\dim S = \dim T = 3$

16. $S = \{(a,b,0,0,0) : a,b \in R\}$, $T = \{(0,0,a,b,c) : a,b,c \in R\}$

19. $\dim V$ is an even integer.

21. $V = R^2(R), S_1 = \{(a,0) : a \in R\}, S_2 = \{(0,a) : a \in R\}$ and $S_3 = \{(a,a) : a \in R\}$

23. (i) $S_1 = \{(0,0,a) : a \in R\}$ (ii) $\{(a,a,a) : a \in R\}$

24. $\{v_1,\ v_2\}$.

25. $e_1^{(3)} = \frac{7}{10}v_1 + \frac{3}{10}v_2 + \frac{1}{5}v_3,\ e_2^{(3)} = -\frac{1}{5}v_1 + \frac{1}{5}v_2 - \frac{1}{5}v_3,\ e_3^{(3)} = \frac{-3}{10}v_1 + \frac{3}{10}v_2 + \frac{1}{5}v_3$

(ii) If $v=(a,b)$, then

$$
\begin{aligned}
& v = xv_1 + yv_2 \\
\Rightarrow\quad & (a,b) = (x+2y, x+3y) \\
\Rightarrow\quad & x+2y = a, x+3y = b \\
\Rightarrow\quad & x = 3a-2b, y = -a+b
\end{aligned}
$$

Hence, the coordinate matrix $[v]_B$ of v relative to the basis B is given by $[v_B] = \begin{bmatrix} 3a-2b \\ -a+b \end{bmatrix}$

EXAMPLE-2 *Find the coordinate vector of $v=(1,1,1)$ relative to the basis $B=\{v_1=(1,2,3), v_2=(-4,5,6),\ v_3=(7,-8,9)\}$ of vector space R^3.*

SOLUTION Let $x,y,z \in R$ be such that

$$
\begin{aligned}
& v = xv_1 + yv_2 + zv_3 \\
\Rightarrow\quad & (1,1,1) = x(1,2,3) + y(-4,5,6) + z(7,-8,9) \\
\Rightarrow\quad & (1,1,1) = (x-4y+7z,\ 2x+5y-8z,\ 3x+6y+9z) \\
\Rightarrow\quad & x-4y+7z = 1,\ 2x+5y-8z = 1,\ 3x+6y+9z = 1 \\
\Rightarrow\quad & x = \frac{7}{10},\ y = \frac{-2}{15},\ z = \frac{-1}{30}
\end{aligned}
$$

Hence $[v]_B = \begin{bmatrix} \frac{7}{10} \\ \frac{-2}{15} \\ \frac{-1}{30} \end{bmatrix}$ is the coordinate vector of v relative to basis B.

EXAMPLE-3 *Find the coordinates of the vector (a,b,c) in the real vector space R^3 relative to the ordered basis (b_1,b_2,b_3), where $b_1=(1,0,-1)$, $b_2=(1,1,1)$, $b_3=(1,0,0)$.*

SOLUTION Let $\lambda,\mu,\gamma \in R$ be such that

$$
\begin{aligned}
& (a,b,c) = \lambda(1,0,-1) + \mu(1,1,1) + \gamma(1,0,0). \\
\Rightarrow\quad & (a,b,c) = (\lambda+\mu+\gamma, \mu, -\lambda+\mu) \\
\Rightarrow\quad & \lambda+\mu+\gamma = a,\ \mu = b,\ -\lambda+\mu = c \\
\Rightarrow\quad & \lambda = b-c,\ \mu = b,\ \gamma = a-2b+c.
\end{aligned}
$$

Hence, the coordinates of vector $(a,b,c) \in R^3$ relative to the given ordered basis are $(b-c,\ b,\ a-2b+c)$.

EXAMPLE-4 *Let V be the vector space of all real polynomials of degree less than or equal to two. For a fixed $c \in R$, let $f_1(x) = 1, f_2(x) = x+c, f_3(x) = (x+c)^2$. Obtain the coordinates of $c_0+c_1x+c_2x^2$ relative to the ordered basis (f_1, f_2, f_3) of V.*

SOLUTION Let $\lambda, \mu, \gamma \in R$ be such that

$$c_0+c_1x+c_2x^2 = \lambda f_1(x) + \mu f_2(x) + \gamma f_3(x).$$

Then,

$$\begin{aligned} & c_0+c_1x+c_2x^2 = (\lambda+\mu c+\gamma c^2) + (\mu+2\gamma c)x + \gamma x^2 \\ \Rightarrow\quad & \lambda+\mu c+\gamma c^2 = c_0,\ \mu+2\gamma c = c_1,\ \gamma = c_2 \qquad \text{[Using equality of two polynomials]} \\ \Rightarrow\quad & \lambda = c_0 - c_1c + c_2c^2,\ \mu = c_1 - 2c_2c, \gamma = c_2. \end{aligned}$$

Hence, the coordinates of $c_0+c_1x+c_2x^2$ relative to the ordered basis (f_1, f_2, f_3) are $(c_0 - c_1c + c_2c^2, c_1 - 2c_2c,\ c_2)$.

EXAMPLE-5 *Consider the vector space $P_3[t]$ of polynomials of degree less than or equal to 3.*

(i) Show that $B = \{(t-1)^3, (t-1)^2, (t-1), 1\}$ is a basis of $P_3[t]$.

(ii) Find the coordinate matrix of $f(t) = 3t^3 - 4t^2 + 2t - 5$ relative to basis B.

SOLUTION (i) Consider the polynomials in B in the following order:

$$(t-1)^3, (t-1)^2, (t-1), 1$$

We see that no polynomial is a linear combination of preceding polynomials. So, the polynomials are linearly independent, and, since $\dim R_3[t] = 4$. Therefore, B is a basis of $P_3[t]$.

(ii) Let x, y, z, s be scalars such that

$$\begin{aligned} & f(t) = x(t-1)^3 + y(t-1)^2 + z(t-1) + s(1) \\ \Rightarrow\quad & f(t) = xt^3 + (-3x+y)t^2 + (3x-2y+z)t + (-x+y-z+s) \\ \Rightarrow\quad & 3t^3 - 4t^2 + 2t - 5 = xt^3 + (-3x+y)t^2 + (3x-2y+z)t + (-x+y-z+s) \\ \Rightarrow\quad & x = 3, -3x+y = -4, 3x-2y+z = 2, -x+y-z+s = -5 \\ \Rightarrow\quad & x = 3, y = 13, z = 19, s = 4 \end{aligned}$$

Hence, the coordinate matrix of $f(t)$ relative to the given basis is $[f(t)]_B = \begin{bmatrix} 3 \\ 13 \\ 19 \\ 4 \end{bmatrix}$.

EXAMPLE-6 *Let F be a field and n be a positive integer. Find the coordinates of the vector $(a_1, a_2, \ldots, a_n) \in F^n$ relative to the standard basis.*

$$\lambda u + \mu v = \lambda(0,\ b_1,\ c_1) + \mu(0,\ b_2,\ c_2)$$

$$\Rightarrow \qquad \lambda u + \mu v = (0,\ \lambda b_1 + \mu b_2,\ \lambda c_1 + \mu c_2) \in S.$$

Hence, S is a subspace of R^3.

REMARK. *Geometrically-R^3 is three dimensional Euclidean plane and S is yz-plane which is itself a vector space. Hence, S is a sub-space of R^3. Similarly, $\{(a,\ b,\ 0) : a,\ b \in R\}$, i.e. xy-plane, $\{(a, 0, c) : a, c \in R\}$, i.e. xz plane are subspaces of R^3. In fact, every plane through the origin is a subspace of R^3. Also, the coordinate axes $\{(a, 0, 0) : a \in R\}$, i.e. x-axis, $\{(0, a, 0) : a \in R\}$, i.e. y-axis and $\{(0, 0, a) : a \in R\}$, i.e. z-axis are subspaces of R^3.*

EXAMPLE-2 *Let V denote the vector space R^3, i.e. $V = \{(x, y, z) : x, y, z \in R\}$ and S consists of all vectors in R^3 whose components are equal, i.e. $S = \{(x, x, x) : x \in R\}$. Show that S is a subspace of V.*

SOLUTION Clearly, S is a non-empty subset of R^3 as $(0, 0, 0) \in S$.

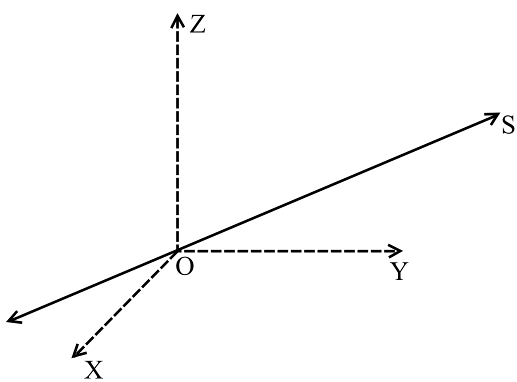
Let $u = (x, x, x)$ and $v = (y, y, y) \in S$ and $a, b \in R$. Then,

$$au + bv = a(x, x, x) + b(y, y, y)$$

$$\Rightarrow \qquad au + av = (ax + by, ax + by, ax + by) \in S.$$

Thus, $au + bv \in S$ for all $u,\ v \in S$ and $a, b \in R$.
Hence, S is *a* subspace of V.

REMARK. *Geometrically S, in the above example, represents the line passing through the origin O and having direction ratios proportional to 1,1,1 as shown in the following figure.*



EXAMPLE-3 *Let a_1, a_2, a_3 be fixed elements of a field F. Then the set S of all triads (x_1, x_2, x_3) of elements of F, such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$, is a subspace of F^3.*

SOLUTION Let $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3)$ be any two elements of S. Then $x_1, x_2, x_3, y_1, y_2, y_3 \in F$ are such that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \quad \text{(i)}$$

and,
$$a_1y_1 + a_2y_2 + a_3y_3 = 0 \quad \text{(ii)}$$

Let a, b be any two elements of F. Then,

$$au + bv = a(x_1, x_2, x_3) + b(y_1, y_2, y_3) = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3).$$

Now,

$$\begin{aligned} & a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3) \\ &= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3) \\ &= a0 + b0 = 0 \quad \text{[From (i) and (ii)]} \end{aligned}$$

$\therefore \quad au + bv = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \in S.$

Thus, $au + bv \in S$ for all $u, v \in S$ and all $a, b \in F$. Hence, S is a subspace of F^3.

EXAMPLE-4 *Show that the set S of all $n \times n$ symmetric matrices over a field F is a subspace of the vector space $F^{n \times n}$ of all $n \times n$ matrices over field F.*

SOLUTION Note that a square matrix A is symmetric, if $A^T = A$.

Obviously, S is a non-void subset of $F^{n \times n}$, because the null matrix $0_{n \times n}$ is symmetric.

Let $A, B \in S$ and $\lambda, \mu \in F$. Then,

$$\begin{aligned} & (\lambda A + \mu B)^T = \lambda A^T + \mu B^T \\ \Rightarrow \quad & (\lambda A + \mu B)^T = \lambda A + \mu B \quad [\because \ A, B \in S \ \therefore \ A^T = A, B^T = B] \\ \Rightarrow \quad & \lambda A + \mu B \in S. \end{aligned}$$

Thus, S is a non-void subset of $F^{n \times n}$ such that $\lambda A + \mu B \in S$ for all $A, B \in S$ and for all $\lambda, \mu \in F$.

Hence, S is a subspace of $F^{n \times n}$.

EXAMPLE-5 *Let V be a vector space over a field F and let $v \in V$. Then $F_v = \{av : a \in F\}$ is a subspace of V.*

SOLUTION Since $1 \in F$, therefore $v = 1\, v \in F_v$. Thus, F_v is a non-void subset of V.

Let $\alpha, \beta \in F_v$. Then, $\alpha = a_1 v$, $\beta = a_2 v$ for some $a_1, a_2 \in F$.

$$\Rightarrow \quad \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} x+y+z+t & x-y-z \\ x+y & x \end{bmatrix}$$

$$\Rightarrow \quad x+z+t = 2, x-y-z = 3, x+y = 4, x = -7$$

$$\Rightarrow \quad x = -7, y = 11, z = -21, t = 30$$

Hence, the coordinate matrix of A relative to the basis B_2 is $[A]_{B_2} = \begin{bmatrix} 7 \\ 11 \\ -21 \\ 30 \end{bmatrix}$.

EXERCISE 2.10

1. Find the coordinate matrices of vectors $u = (5,3)$ and $v = (a,b)$ relative to the ordered basis $B = \{(1,-2),(4,-7)\}$ of vector space $R^2(R)$.

2. The vectors $v_1 = (1,2,0), v_2 = (1,3,2), v_3 = (0,1,3)$ form a basis of $R^3(R)$. Find the coordinate matrix of vector v relative to the ordered basis $B = \{v_1, v_2, v_3\}$, where

   (i) $v = (2,7,-4)$ (ii) $v = (a,b,c)$

3. $B = \{t^3+t^2,\ t^2+t,\ t+1, 1\}$ is an ordered basis of vector space $P_3[t]$. Find the coordinate matrix $f(t)$ relative to B where:

   (i) $f(t) = 2t^3+t^2-4t+2$ (ii) $f(t) = at^3+bt^2+ct+d$

4. $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is an ordered basis of vector space $R^{2\times 2}(R)$.
   Find the coordinate matrix of matrix A relative to B where:

   (i) $A = \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$ (ii) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

5. Consider the vector space $R_2(t)$ of all polynomials of degree less than or equal to 2. Show that the polynomials $f_1(t) = t-1, f_2(t) = t-1, f_3(t) = (t-1)^2$ form a basis of $P_2[t]$. Find the coordinate matrix of polynomial $f(t) = 2t^2-5t+9$ relative to the ordered basis $B = \{f_1(t), f_2(t), f_3(t)\}$.

6. Find the coordinate vector of $v = (a,b,c)$ in $R^3(R)$ relative to the ordered basis $B = \{v_1, v_2, v_3\}$, where $v_1 = (1,1,1), v_2 = (1,1,0), v_3 = (1,0,0)$.

ANSWERS

1. $[u]_B = \begin{bmatrix} -41 \\ 11 \end{bmatrix}, [v]_B = \begin{bmatrix} -7a-4b \\ 2a+b \end{bmatrix}$

2. (i) $[v]_B = \begin{bmatrix} -11 \\ 13 \\ -10 \end{bmatrix}$ (ii) $[v]_B = \begin{bmatrix} 7a-3b+c \\ -6a+3b-c \\ 4a-2b+c \end{bmatrix}$

3. (i) $[f(t)]_B = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 2 \end{bmatrix}$ (ii) $[f(t)]_B = \begin{bmatrix} a \\ b-c \\ a-b+c \\ -a+b-c+d \end{bmatrix}$

4. (i) $[A]_B = \begin{bmatrix} d \\ c-d \\ b+c-2d \\ a-b-2c+2d \end{bmatrix}$ (ii) $[A]_B = \begin{bmatrix} 7 \\ -1 \\ -13 \\ 10 \end{bmatrix}$

5. $[f(t)]_B = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$ 6. $[v]_B = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} x+y+z+t & x-y-z \\ x+y & x \end{bmatrix}$$

$$\Rightarrow \quad x+z+t=2, x-y-z=3, x+y=4, x=-7$$

$$\Rightarrow \quad x=-7, y=11, z=-21, t=30$$

Hence, the coordinate matrix of A relative to the basis B_2 is $[A]_{B_2} = \begin{bmatrix} 7 \\ 11 \\ -21 \\ 30 \end{bmatrix}$.

EXERCISE 2.10

1. Find the coordinate matrices of vectors $u = (5,3)$ and $v = (a,b)$ relative to the ordered basis $B = \{(1,-2),(4,-7)\}$ of vector space $R^2(R)$.

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   (i) $v = (2,7,-4)$ (ii) $v = (a,b,c)$

3. $B = \{t^3+t^2,\ t^2+t,\ t+1, 1\}$ is an ordered basis of vector space $P_3[t]$. Find the coordinate matrix $f(t)$ relative to B where:

   (i) $f(t) = 2t^3+t^2-4t+2$ (ii) $f(t) = at^3+bt^2+ct+d$

4. $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is an ordered basis of vector space $R^{2\times 2}(R)$. Find the coordinate matrix of matrix A relative to B where:

   (i) $A = \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$ (ii) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

5. Consider the vector space $R_2(t)$ of all polynomials of degree less than or equal to 2. Show that the polynomials $f_1(t) = t-1, f_2(t) = t-1, f_3(t) = (t-1)^2$ form a basis of $P_2[t]$. Find the coordinate matrix of polynomial $f(t) = 2t^2-5t+9$ relative to the ordered basis $B = \{f_1(t), f_2(t), f_3(t)\}$.

6. Find the coordinate vector of $v = (a,b,c)$ in $R^3(R)$ relative to the ordered basis $B = \{v_1, v_2, v_3\}$, where $v_1 = (1,1,1), v_2 = (1,1,0), v_3 = (1,0,0)$.

ANSWERS

1. $[u]_B = \begin{bmatrix} -41 \\ 11 \end{bmatrix}, [v]_B = \begin{bmatrix} -7a-4b \\ 2a+b \end{bmatrix}$

2. (i) $[v]_B = \begin{bmatrix} -11 \\ 13 \\ -10 \end{bmatrix}$ (ii) $[v]_B = \begin{bmatrix} 7a-3b+c \\ -6a+3b-c \\ 4a-2b+c \end{bmatrix}$

3. (i) $[f(t)]_B = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 2 \end{bmatrix}$ (ii) $[f(t)]_B = \begin{bmatrix} a \\ b-c \\ a-b+c \\ -a+b-c+d \end{bmatrix}$

4. (i) $[A]_B = \begin{bmatrix} d \\ c-d \\ b+c-2d \\ a-b-2c+2d \end{bmatrix}$ (ii) $[A]_B = \begin{bmatrix} 7 \\ -1 \\ -13 \\ 10 \end{bmatrix}$

5. $[f(t)]_B = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$

6. $[v]_B = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$