Chapter 2

Foundation on Compressible Fluid Flow

2.1 COMPRESSIBLE FLUIDS

In everyday life one recognizes three states of matter, namely, solid, liquid and gaseous. Solids, liquids and gases are all comprised of molecules in continuous motion. However, arrangement of these molecules and the space between them, differ, giving rise to the characteristic properties of the three different states of matter. In solids, the molecules are densely and regularly packed and the movement loose is restrained by its neighbour molecules. In liquids, the structure is comparatively looser and the molecules are restrained to some degree by the surrounding molecules which can break away from the restraint causing a change of structure. In gaseous space there is no formal structure, the space between molecules is large and the molecules can move freely. A liquid is difficult to compress and it is regards as incompressible. A given mass of liquid occupies a fixed volume, irrespective of the size or shape of its container and a free surface is formed if the volume of the container is greater than that of the liquid. A gas is comparably easy to compress, change of volume with pressures is large. A gas completely fills any vessel in which it is placed and therefore does not form a free surface.

Fluid, liquid or gas flows under the action of force and deform continuously for a long as force is applied. Deformation is caused by shearing force. There can be no shear stress in a fluid at rest. Shear stress is developed when the fluid is in motion. The force per unit area F/A is the shear stress, τ and the deformation is measured by an angle, ϕ called shear strain proportional to shear stress. In solids ϕ is a fixed quantity for a given value of τ, since solid can resist shear stress permanently. In fluid, the shear strain ϕ continues to increase with time and fluids flows. It is found experimentally that the rate of shear strain is directly proportional to the shear stress. That is, $\tau \propto \frac{V}{y}$ the term V/y is the change in velocity with y and may be written in the differential form dV/dy. The constant of proportionality is known as dynamic viscosity, μ. Hence, $\tau = \mu \frac{dV}{dy}$, which is known as Newton's law of viscosity. Figure 2.1 provides an insight into fluids. Solids do not deform continuously under the application of a shear stress and hence denoted along the stress axis. On the other hand, viscous or ideal fluid is one in which the viscosity is neglected. Hence, it is represented as a line of zero shear stress.

Consider a streamline in which flow is taking place in x direction as shown in Figure 2.2 and consider cross-sectional area dA and length dx. The resultant force on the fluid element in the direction of x must be equal to the mass of fluid element multiplied by acceleration.

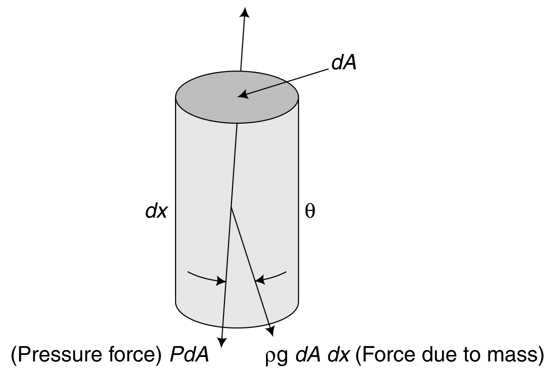


Fig. 2.2 Forces in a fluid element.

$$PdA - \left(P + \frac{dP}{dx}\,dx\right) dA - \rho g\ dA\ dx \cos\theta = \rho\ dA\ dx\ a_x$$

where, a_x is acceleration in x direction

Since $\dfrac{dx}{dt} = V$, velocity and $\dfrac{dV}{dt} = a_x$,

$$a_x = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial t} = V\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t}$$

If the flow is steady, $\dfrac{\partial V}{\partial t} = 0$, hence $a_x = V\dfrac{\partial V}{\partial x}$

Substituting the value $a_x = V\dfrac{\partial V}{\partial x}$ in equilibrium force equation:

$$-\frac{dP}{dx}\,dx\ dA - \rho g\ dA\ dx \cos\theta = \rho\ dA\ dx\ V\frac{dV}{dx}$$

and dividing by $\rho dA dx$

$$\frac{\partial P}{\rho dx} + g\cos\theta + V\frac{dV}{dx} = 0$$

Since $\cos\theta = \dfrac{dz}{dx}$

$$\frac{dP}{\rho} + gdz + VdV = 0$$

The above equation is known as Euler equation of motion.

If the flow is incompressible, ρ is a constant and integrating the Euler equation gives:

$$\int \frac{dP}{\rho} + \int g dz + \int V dV = \text{Constant}$$

$$\frac{P}{\rho} + gz + \frac{V^2}{2} = \text{Constant}$$

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{Constant}$$

The above equation is the Bernoulli equation where $\frac{P}{\rho g}$ corresponds to static pressure head, $\frac{V^2}{2g}$ is the kinetic head and z is potential energy per unit weight.

When the fluid is compressible in the integral $\int \frac{dP}{\rho}$, both P and ρ changes depend upon the type of process. The value of ρ is obtained in terms of P and substituted for integration. For an isothermal process $\frac{P}{\rho}$ is constant, hence, $\rho = \frac{P}{C_1}$.

Hence,
$$\int \frac{dP}{\rho} = \int \frac{dP}{C_1} = C_1 \int \frac{dP}{P} = C_1 \log_e P = \frac{P}{\rho} \log_e P$$

The Euler equation becomes, $\frac{P}{\rho} \log_e P + \frac{V^2}{2} + gz = \text{Constant}$

$$\frac{P}{\rho g} \log_e P + \frac{V^2}{2g} + z = \text{Constant}$$

When the process is adiabatic, the relation between pressure, P, and the density, ρ, is: $\frac{P}{\rho^\gamma} = \text{Constant}$. Hence, $\rho \left(\frac{P}{C_2}\right)^{\frac{1}{\gamma}}$

$$\int \frac{dP}{\rho} = \int \frac{dP}{\left(\frac{P}{C_2}\right)^{\frac{1}{\gamma}}} = \int \frac{C_2^{\frac{1}{\gamma}}}{P^{\frac{1}{\gamma}}} dp = C_2^{\frac{1}{\gamma}} \int \frac{1}{P^{\frac{1}{\gamma}}} dP$$

$$\int \frac{dP}{\rho} = \frac{C_2^{\frac{1}{\gamma}} P^{\frac{\gamma-1}{\gamma}}}{\frac{\gamma - 1}{\gamma}} = \frac{\gamma}{\gamma - 1} C_2^{\frac{1}{\gamma}} P^{\frac{\gamma-1}{\gamma}} = \frac{\gamma}{\gamma - 1} \left(\frac{P}{\rho^\gamma}\right)^{\frac{1}{\gamma}} P^{\frac{\gamma-1}{\gamma}}$$

Since
$$C_2^{\frac{1}{\gamma}} = \frac{P}{\rho^\gamma}$$

$$\int \frac{dP}{\rho} = \frac{\gamma}{\gamma - 1} \frac{P}{\rho}$$

Hence, the Euler equation becomes:

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$$

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P}{\gamma\rho} + \frac{V^2}{2g} + z = \text{Constant}$$

2.2.1 Stagnation Properties

Algebraic sum of the potential energy and kinetic energy considering the elevation of the flow is zero is called stagnation properties. The value of pressure, density and temperature at stagnation condition are called stagnation pressure, stagnation density and stagnation temperature respectively. They are denoted by p_0, ρ_0 and T_0. The stagnation point can occur when a fluid flows fast an immersed body or in the reservoir conditions.

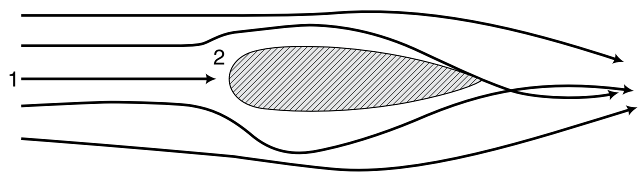


Fig. 2.3 Fluid flows fast an immersed body.

Consider a compressible fluid flowing past an immersed body under frictionless adiabatic condition. Consider two points one and two. Let P_1, V_1 and ρ_1 are the pressure, velocity and density at point 1 and P_2, V_2 and ρ_2 are pressure, velocity and density at point 2. Applying Euler equation

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{P_1 g} + \frac{V_1^2}{2g} + z_1 = \left(\frac{\gamma}{\gamma - 1}\right)\frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2$$

Since $z_1 = bz_2$, $\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left(\frac{\gamma}{\gamma - 1}\right)\frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g}$

At stagnation point (2), $V_2 = 0$, $P_2 = P_0$ and $\rho_2 = \rho_0$

$$\left(\frac{\gamma}{\gamma - 1}\right)\left[\frac{P_1}{\rho_1} - \frac{P_0}{\rho_0}\right] = \frac{V_1^2}{2g}$$

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1}\left[1 - \frac{P_0}{P_1} - \frac{\rho_1}{\rho_0}\right] = -\frac{V_1^2}{2g}$$

For the adiabatic process, $\frac{P_1}{\rho_1^\gamma}\frac{P_0}{\rho_0^\gamma}$ = a constant, $\frac{\rho_1}{\rho_0} = \left(\frac{P_1}{P_0}\right)^{\frac{1}{\gamma}}$

Hence, $\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1}\left[1 - \frac{P_0}{P_1}\left(\frac{P_1}{P_0}\right)^{\frac{1}{\gamma}}\right] = -\frac{V_1^2}{2g}$

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1}\left[1 - \left(\frac{P_0}{P_1}\right)^{1-\frac{1}{\gamma}}\right] = -\frac{V_1^2}{2g}$$

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1}\left[1 - \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = -\frac{V_1^2}{2g}$$

$$\left[1 - \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = -\frac{V_1^2}{2g}\left(\frac{\gamma - 1}{\gamma}\right)\frac{P_1}{\rho_1}$$

$$1 + \frac{V_1^2}{2g}\left(\frac{\gamma - 1}{\gamma}\right)\frac{P_1}{\rho_1} = \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

For the adiabatic process, the velocity of the sound,

$$c = \sqrt{\gamma RT} = \sqrt{\gamma\frac{P}{\rho}}. \text{ Hence, } c_1^2 = \gamma\frac{P_1}{\rho_1}$$

Hence, $$1 + \frac{V_1^2}{2g}\gamma\left(\frac{\gamma - 1}{\gamma}\right)\frac{1}{c_1^2} = \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$$1 + \frac{M_1^2}{2g}(\gamma - 1) = \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

where, $\frac{V_1^2}{c_1^2} = M_1^2$ and M_1 is the Mach number of the flow.

$$\frac{P_0}{P_1} = \left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}}$$

The above equation is the expression for stagnation pressure. For adiabatic flow, $\frac{\rho_0}{\rho_1} = \left[\frac{P_0}{P_1}\right]^{\frac{1}{\gamma}}$. Substituting in the above equation:

$$\frac{P_0}{P_1} = \left[\left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}}\right]^{\frac{1}{\gamma}} = \left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]^{\frac{1}{\gamma-1}}$$

The above equation gives the expression for stagnation density. From the equation of state $\frac{P_0}{\rho_0} = RT_0$, $T_0 = \frac{1}{R}\frac{P_0}{\rho_0}$ substituting the values for P_0 and ρ_0

$$T_0 = \frac{1}{R}\,\frac{P_1\left[1 + \frac{(\gamma - 1)}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}}}{\rho_1\left[1 + \frac{(\gamma - 1)}{2} M_1^2\right]^{\frac{1}{\gamma-1}}}$$

$$T_0 = \frac{P_1}{\rho_1 R}\left[1 + \frac{(\gamma - 1)}{2} M_1^2\right]$$

Since $\frac{P_1}{\rho_1} = RT_1$, $\quad \frac{T_0}{T_1} = \left[1 + \frac{\gamma - 1}{2} M_1^2\right]$

Fluid flow through an orifice or a nozzle as shown in Figure 2.4 can be taken as adiabatic and Euler equation will apply. If the reservoir is large $V_2 = 0$.

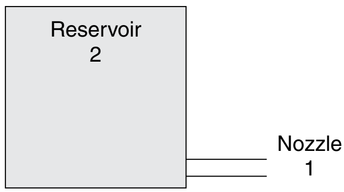


Fig. 2.4 Reservoir with a nozzle.

Using the equation for adiabatic condition, $\frac{\rho_2}{\rho_1} = \left[\frac{P_1}{P_2}\right]^{\frac{1}{\gamma}}$

$$\frac{V_1^2}{2} = \left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1}\left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

$$V_1 = \sqrt{2\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1}\left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

where suffix 2 indicates stagnation conditions.

Mass flow rate, $\quad \dot{m} = A\rho_1 V_1 = A\rho_1\sqrt{2\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_0}{\rho_1}\left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

2.2.2 Gas Flow in Nozzle and Ducts

From the continuity equation, mass flow rate is given by $\dot{m} = A\rho V$

Taking logarithm and differentiating

$\ln \rho + \ln A + \ln V =$ a constant

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$dV = -V\left[\frac{d\rho}{\rho} + \frac{dA}{A}\right]$$

From Euler equation $\dfrac{P}{\rho g} + \dfrac{V^2}{2g} + z$ a constant

For adiabatic energy transformation process stagnation properties remain constant

Hence, $$\frac{dP}{\rho} + VdV = 0 \text{ and } dP = -\rho VdV = \rho V^2\left[\frac{d\rho}{\rho} + \frac{dA}{A}\right]$$

$$\rho V^2 \frac{dA}{A} = dP - \rho V^2 \frac{d\rho}{\rho}$$

$$\frac{dA}{A} = \frac{dP}{\rho V^2} - \frac{d\rho}{\rho}$$

$$\frac{dA}{A} = \frac{dP}{\rho V^2}\left[1 - \frac{d\rho}{dP} V^2\right]$$

For an isentropic process, $\dfrac{dP}{\rho} = c^2$ where c is the sonic velocity.

Hence, $$\frac{dA}{A} = \frac{dP}{\rho V^2}\left[1 - \frac{V^2}{c^2}\right] \text{ and } \frac{dA}{A} = \frac{dP}{\rho V^2}[1 - M^2]$$

The above equation can be considered for both acceleration and deceleration processes for various values of Mach number.

Gases and vapour are expanded in nozzles by providing a pressure ratio across them. As per the above equation the shape of the passage depends on the local Mach number. Since the purpose of the nozzle is to accelerate the flow, pressure drop is always negative. Following three possible conditions can be considered for a nozzle for $M < 1$, dA is negative which indicates the area of nozzle decreases from $M = 0$ to $M = 1$ giving a convergent nozzle.

For $M = 1$, $dA = 0$, which indicates there is no change of passage at the point where the Mach number is unity. This section represents the throat of the passage.

For $M > 1$, dA is positive. This shows the area of nozzle increases, continuously giving a divergent passage.

For a deceleration flow or a diffuser, throat conditions remain same. However, for $M < 1$, dA is positive, a divergent passage acts as diffuser for subsonic flow and the convergent passage acts as diffuser for supersonic flow. Figure. 2.5 illustrates a nozzle with convergent and divergent parts. Throat condition is indicated by suffices 'Th' and input condition to the nozzle is the stagnation point. From the stagnation properties one can obtain

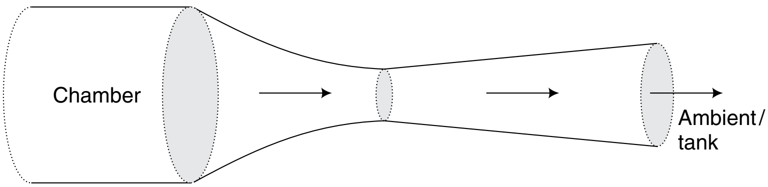


Fig. 2.5 Nozzle with convergent-divergent part.

$$\frac{P_0}{P_{Th}} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho_{Th}} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{1}{\gamma-1}}$$

$$\frac{T_0}{T_{Th}} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]$$

Substituting in the throat condition $M = 1$

$$\frac{T_{Th}}{T_0} = \left(\frac{2}{\gamma + 1}\right) = 0.833$$

$$\frac{P_{Th}}{P_0} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\lambda-1}} = 0.528$$

$$\frac{\rho_{th}}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}} = 0.634$$

Substituting the above equations in general properties of stagnation,

$$\frac{T_{Th}}{T} = \left(\frac{2}{\gamma + 1}\right) + \left(\frac{\gamma - 1}{\lambda + 1}\right)M^2$$

$$\frac{P_{Th}}{P} = \left[\left(\frac{2}{\gamma + 1}\right) + \left(\frac{\gamma - 1}{\lambda + 1}\right)M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_{Th}}{\rho} = \left[\left(\frac{2}{\gamma + 1}\right) + \left(\frac{\gamma - 1}{\lambda + 1}\right)M^2\right]^{\frac{1}{\gamma-1}}$$

Like temperature, pressure and density ratio, area ratio at a given section of passage is also an useful quantity. The ratio of area A to throat area can be obtained as follows. From the continuity equation

$$\rho_{AV} = \rho_{Th}\, A_{Th}\, V_{Th} \text{ and hence } \frac{A}{A_{Th}} = \frac{\rho_{Th}}{\rho}\,\frac{V_{Th}}{V}$$

Since
$$\frac{V_{Th}}{V} = \frac{c_{Th}}{V} = \frac{1}{M_{Th}} = \left[\frac{1 + \dfrac{\gamma - 1}{2}M^2}{\dfrac{\gamma + 1}{2}M^2}\right]^{\frac{1}{2}}$$

$$= \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1}M^2\right]^{\frac{1}{2}}$$

Hence,
$$\frac{A}{A_{Th}} = \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Area ratio for different Mach numbers is shown in Figure 2.6.

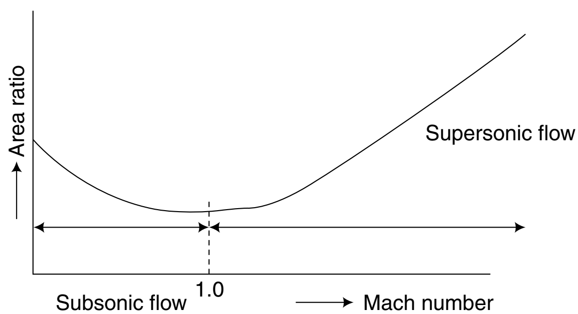


Fig. 2.6 Area ratio for different Mach numbers.

Another relation frequently used in compressible flow calculations is the function

$$\frac{A}{A_{Th}}\, x\, \frac{P}{P_{Th}} = \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

By simplification

$$\frac{A}{A_{Th}}\, x\, \frac{P}{P_{Th}} = \frac{\dfrac{1}{M}\left[\dfrac{2}{\gamma + 1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}{\left[1 + \dfrac{\gamma - 1}{2}M^2\right]^{\frac{1}{2}}}$$

2.2.3 Flow Analysis in Convergent-Divergent Nozzles

2.2.3.1 Convergent-Divergent Nozzle

A nozzle is a mechanical device designed to control the characteristics of a fluid flow as it exits from an enclosed chamber into some medium. A nozzle is often a pipe or tube of varying cross-sectional area and it can be used to direct or modify the flow of a fluid (liquid or gas). Nozzles are frequently used to control the rate of flow, speed, direction, mass, and/or the pressure of the stream that emerges from them. Frequently the purpose is to increase the kinetic energy of the flowing medium at the expense of its pressure energy and/or internal energy. Nozzles can be described as convergent (narrowing down from a wide diameter to a smaller diameter in the direction of the flow) or divergent (expanding from a smaller diameter to a larger one). A De Laval nozzle has a convergent section followed by a divergent section and is often called a convergent-divergent nozzle. Convergent nozzles accelerate fluids. If the nozzle pressure ratio is high enough the flow will reach sonic velocity at the narrowest point (nozzle throat). In this situation, nozzle is said to be choked. Increasing the nozzle pressure ratio further will not increase the throat Mach number beyond unity. Downstream (external to the nozzle) the flow is free to expand to supersonic velocities. Divergent nozzles slow fluids, if the flow is subsonic, but accelerate sonic or supersonic fluids.

Convergent-divergent nozzles can therefore accelerate fluids that have choked in the convergent section to supersonic speed. This convergent divergent process is more efficient than allowing a convergent nozzle to expand supersonically externally. Since exhaust velocity has to exceed air speed, supersonic aircraft also very typically use convergent-divergent nozzle despite the weight and cost penalties. Supersonic jet engines, like those employed in fighter and supersonic transport aircraft (Conecorde) indeed have relatively high nozzle pressure ratios. Because subsonic jet engines require low exhaust velocities, they require only subsonic exhaust and thus have modest nozzle pressure ratios and employ simple convergent nozzles.

Rocket motors use convergent-divergent nozzles, to maximize thrust and exhaust velocity and thus extremely high nozzle pressure ratios are employed. It is noted that Mach number one can be a very high speed for a hot gas, since heat significantly raises the speed of sound. Thus, the absolute speed reached at nozzle throat can be far higher than speed of sound at sea level. This fact is used extensively in rocketry where hypersonic flows are required.

The rocket nozzle can surely be described as the epitome of elegant simplicity. The primary function of a nozzle is to channel and accelerate the combustion products produced by the burning propellant in such a way as to maximize the velocity of the exhaust at the exit, to supersonic velocity. The familiar rocket nozzles also known as a convergent divergent or De Laval nozzle accomplish these remarkable features by simple geometry. In other words, it does this by varying the cross-sectional area (or diameter) for an exacting form. The analysis of a rocket nozzle involves the concept of steady, one-dimensional compressible fluid flow of an ideal gas. Briefly this means that:

- The flow of the fluid (gases + condensed particles) is constant and does not change over time during the burn.

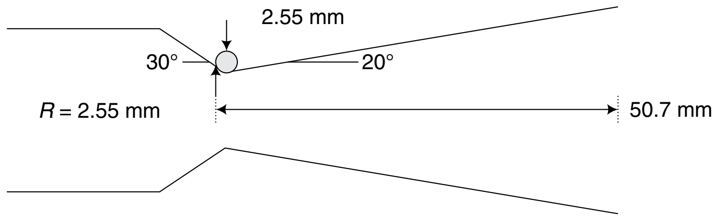


Fig. 2.7 Rothe nozzle configuration.

so that an expansion ratio that maximizes performance over a range of ambient pressures can be selected. The shape of the nozzle also affects how efficiently the expansion of the exhaust gases is converted into linear motion. The simplest nozzle shape is 12 degree internal angle cone, which is about 97% efficient. Smaller angles give slightly higher efficiency, larger angles give lower efficiency.

More complex shapes of revolution are frequently used such as parabolic shapes. This gives perhaps 1% higher efficiency than the cone nozzle and is shorter and lighter. These shapes are widely used on launch vehicles and rockets where weight is a criteria. They are of course, harder to fabricate, so are more costly. Nevertheless, other factors must also be considered that tend to alter the design from this expansion ratio based optimum. Some of the issues designers must deal with are the weight, length, manufacturability, cooling and aerodynamic characteristics of the nozzles.

The usual configuration for converging-diverging nozzle is shown in Figure 2.8. Gas flows through the nozzle from a region of high pressure (usually referred to as the chamber) to one of low pressure (referred to as the ambient or tank). The chamber is usually big enough so that any flow velocities here are negligible. The pressure here is denoted by the symbol P_0. Gas flows from the chamber into the converging portion of the nozzle, past the throat, through the diverging portion and then exhausts into the ambient as a jet. The pressure of the ambient is referred to as the back pressure, p_b.

To get a basic feel for the behaviour of the nozzle, imagine performing the simple experiment as shown in Figure 2.8. Here, a converging-diverging nozzle is used to connect two air cylinders. Cylinder *A* contains air at high pressure, and takes the place of the chamber. The convergent-divergent nozzle exhausts this air into cylinder *B* which takes the place of a tank. Imagine the controlling of the pressure in cylinder *B* and measuring the resulting mass flow rate through the nozzle. Experiment shows that the lower the pressure in cylinder *B*, the more is the mass flow rate through the nozzle. This is true, but up to a point. If lower back pressure is enough where the flow rate suddenly stops increasing altogether and it does not matter how much lower one makes the back pressure (even if you make it vacuum) one cannot get any more mass flow rate out of the nozzle. Then one says that the nozzle has choked. One could delay this behaviour by making the nozzle throat bigger but eventually the same thing would happen. The nozzle will become choked even if you eliminated the throat altogether and just had a converging nozzle.

The reason for this behaviour has to do with the way the flows behave at Mach number one. When the flow speed reaches the speed of sound, in a steady internal flow the Mach number

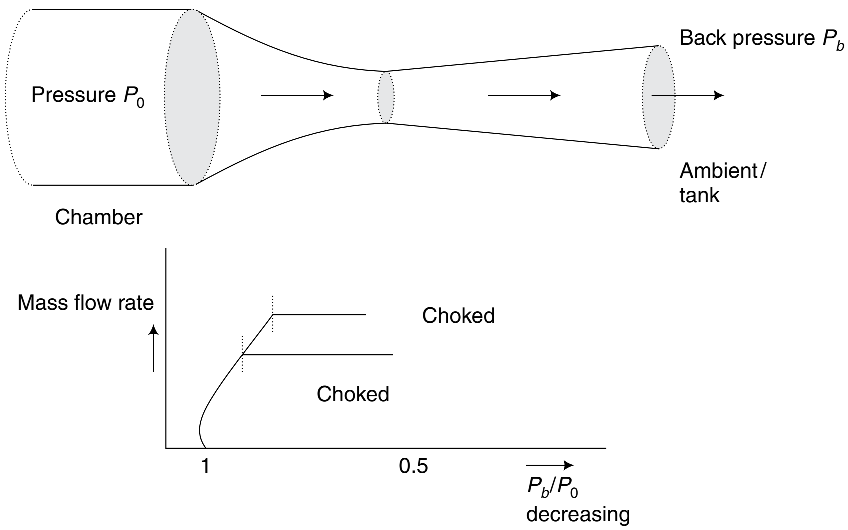


Fig. 2.8 Converging-diverging nozzle configuration.

can only reach one at a minimum in the cross-sectional area. When the nozzle is not choked, the flow through it is entirely subsonic. If one lowers the back pressure a little, the flow goes faster and the flow rate increases. As one lowers the back pressure further the flow speed at the throat eventually reaches the szzpeed of sound. Any further lowering of the back pressure cannot accelerate the flow through the nozzle any more because that would entail moving the point where $M = 1$ away from the throat where the area is minimum and so the flow gets stuck. The flow pattern downstream of the nozzle can still change if you lower the back pressure further, but the mass flow rate is now fixed because the flow in the throat and for that matter the entire converging section is now fixed too. The changes in the flow pattern after the nozzle has choked are not very important in the experiment because they do not change the mass flow rate. They are, however, very important. If one were using this nozzle to accelerate the flow out of the jet engine or rocket and create propulsion or if one just want to understand how high speed flows work.

Figure 2.9 (a) shows the flow through nozzle when it is completely subsonic, that is, the nozzle is not choked. The flow accelerates out of the chamber through the converging section, reaching the maximum (subsonic) speed at the throat. The flow, then, decelerates through the diverging section and exhaust into ambient as subsonic jet. Lowering the back pressure in this state increases the flow speed everywhere in the nozzle.

By lowering the back pressure far enough one eventually gets the situation as shown in Figure 2.9 (b). The flow pattern is exactly the same as in subsonic flow, except the flow speed at the throat has just reached Mach number one. Flow through the nozzle is now choked since further reduction in the back pressure cannot move the point of Mach number one away from the throat. However, the diverging section does not change as one lowers the back pressure.

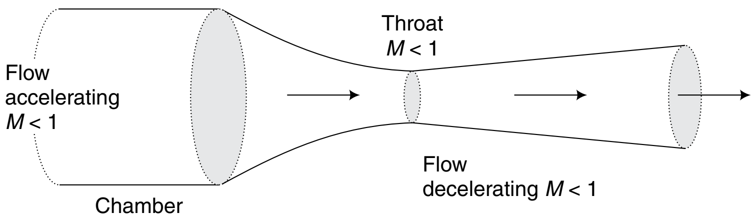
As p_b is lowered below that needed to just choke the flow a region of supersonic flow forms just downstream of the throat. Unlike a subsonic flow, the supersonic flow accelerates as the area gets bigger. This region of supersonic acceleration is terminated by a normal shock wave. The shock wave produces a near-instantaneous deceleration of the flow to subsonic speed. This subsonic flow then decelerates through the remainder of the diverging section and exhausts as a subsonic jet. In this regime if you lower or raise the back pressure it will increase or decrease the length of supersonic flow in the diverging section before the shock wave.

If one lowers p_b enough one can extend the supersonic region all the way down the nozzle until the shock is sitting at the nozzle exit (see Figure 2.9 (d)). Because there is a very long region of acceleration (entire nozzle length), the flow speed just before the shock will be very large in this case. However, after the shock the flow in the jet will still be subsonic.

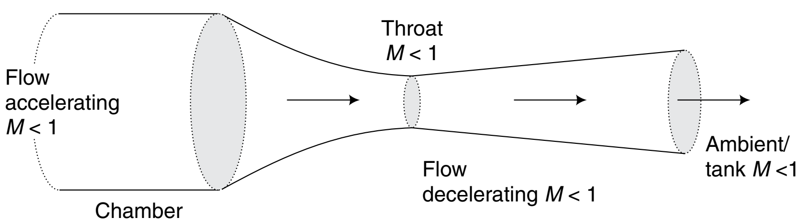
Lowering the back pressure further causes the shock to bend out into the jet (see Figure 2.9 (e)) and a complex pattern of shocks and reflections is set up in the jet which will now involve a mixture of subsonic and supersonic flows or (if the back pressure is low enough) just supersonic flow. Because the shock is no longer perpendicular to the flow near the nozzle walls, it deflects it upwards as it leaves the exit producing an initially contracting jet. One refers to this as overexpanding flow because in this case the pressure at the nozzle exit is lower than that in the ambient (back pressure), the flow has been expanded by the nozzle.

A further lowering of the back pressure changes and weaken the waves pattern in the jet. Eventually one will have lowered the back pressure enough so that it is now equal to the pressure at the nozzle exit. In this case, the waves in the jet disappear altogether (see Figure 2.9 (f)) and the jet will be uniformly supersonic. This situation, since it is often desirable, is referred to as design conditions.

Finally, if one lowers the back pressure even further one will create a new imbalance between the exit and back pressure (exit pressure greater than back pressure) (see Figure 2.9 (g)). In this situation (called underexpanded) what one calls expansion waves that produce gradual turning and acceleration in the jet form at the nozzle exit, initially turning in the flow at the jet edges outward in a plume and setting up a different type of complex wave pattern,



(a) Subsonic flow



(b) Flow just choked

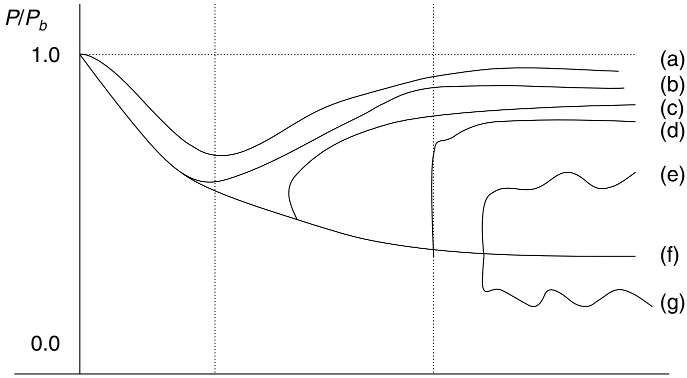


Fig. 2.9 Convergent-divergent nozzle at different operating conditions.

2.3 SHOCK WAVES, FANNO LINE AND RAYLEIGH LINE

In some situations shocks are undesirable because they interfere with the normal flow behaviour. For example, in turbomachines, if the area and properties of flow passage are incorrectly designed shock may occur on account of supersonic flow developed due to local acceleration. These shocks may be normal or inclined to the direction of local flow. They may cause boundary layer separation and deviation of flow from its desired direction.

Other undesirable forms of the shock waves are the sonic boom created by supersonic aircraft and the blast waves generated by an explosion. These waves have a damaging effect on human life and buildings. Figure 2.10 shows normal shock wave in constant area frictionless duct.

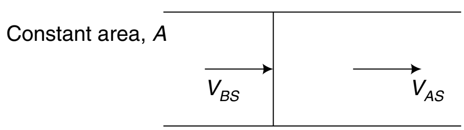


Fig. 2.10 Normal shock.

Conservation of mass equation for shock waves gives the flow rate,

$$\dot{m} = \rho_{BS} A_{BS} V_{BS} = \rho_{AS} A_{AS} V_{AS}$$

where suffixes '*BS*' indicate before shock and "*AS*" indicate after shock.

Since $$A_{BS} = A_{AS} = A,$$

$$\frac{\dot{m}}{A} = \rho_{BS} V_{BS} = \rho_{AS} V_{AS}.$$

From momentum equation

$$(\rho_{BS} - \rho_{AS}) A = \dot{m}(V_{BS} - V_{AS})$$

$$(\rho_{BS} - \rho_{AS}) = \frac{\dot{m}}{A} (V_{BS} - V_{AS})$$

Fanno line describes an adiabatic flow process in a constant area duct with friction. The stagnation enthalpy and the flow rate per unit area remain constant. On account of friction the process is irreversible. The process is governed by the equation of continuity, energy and state, $\frac{\dot{m}}{A} = \rho_{BS} V_{BS} = \rho_{AS} V_{AS}$, $H_{BS} + \frac{1}{2} V_{BSs}^2 = H_{AS} + \frac{1}{2} V_{AS}^2$ and $H = f(\phi\ \rho)$ and $\phi = f(P\rho)$ respectively, where H is the enthalpy ϕ is the entropy,

$$\rho_{AS} = \frac{\dot{m}}{A} \frac{1}{V_{AS}}$$

Enthalpy after the shock $H_{AS} = H_0 - \frac{1}{2} V_{AS}^2$, where H_0 is the stagnation enthalpy. Entropy $\phi_{AS} = f(P_{AS}\ \rho_{AS})$. For different values of V_{AS}, the Fanno line on the enthalpy-entropy diagram is shown in Figure 2.11. The constant pressure lines have also been shown for studying changes in pressure in a given Fanno process. The maximum entropy point on Fanno line is F. For small changes in the process may be considered reversible. Entropy at the point $F = 0$, then

$$H + \frac{1}{2}V^2 = \text{constant and } dH = -VdV$$

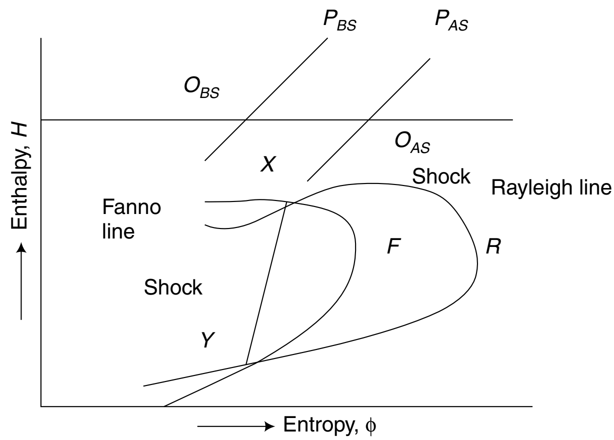


Fig. 2.11 Normal shock on Fanno and Rayleigh line.

Since ρV is constant, $\rho dV + V d\rho = 0$ and $dV = -\frac{V}{\rho} d\rho$

For an isentropic process, $dH = \frac{1}{\rho} dP$, Hence, $V^2 = c^2 = \frac{dP}{d\rho} = \left(\frac{\partial P}{\partial \rho}\right)_\phi$

The above equation demonstrates that the gas velocity at the maximum entropy part F on the Fanno line is sonic ($M = 1$). The upper branch of the curve is subsonic flow and the lower branch supersonic flow.

Rayleigh line describes a frictionless flow process in a constant area duct with heat transfer. Here also, the flow rate per unit area is constant.

Combining the above equation gives:

$$\frac{\gamma-1}{2}\frac{c_{Th}^2}{V_{BS}} - \frac{\gamma-1}{2}V_{BS} - \frac{\gamma-1}{2}\frac{c_{Tht}^2}{V_{AS}} - \frac{\gamma-1}{2}V_{AS} = \gamma(V_{BS} - V_{AS})$$

$$\frac{\gamma+1}{2}c_{Th}^2\left[\frac{1}{V_{BS}} - \frac{1}{V_{AS}}\right] + \frac{\gamma-1}{2}(V_{BS} - V_{AS}) = \gamma(V_{BS} - V_{AS})$$

$$\frac{\gamma+1}{2}c_{Th}^2\left[\frac{V_{AS} - V_{BS}}{V_{BS}V_{ASs}}\right] + \frac{\gamma-1}{2}(V_{BS} - V_{AS}) = \gamma(V_{BS} - V_{AS})$$

$$\frac{\gamma+1}{2}c_{Th}^2 + \frac{\gamma-1}{2}V_{BS}V_{AS} = \gamma V_{BS}\,V_{AS}$$

Rearranging the expression yields:

$$V_{BS}\,V_{AS} = a_{Th}^2$$

The above equation is known as Prandtl-Meyer relation

$$\frac{V_{BS}}{a_{Th}}\frac{V_{AS}}{a_{Th}} = 1$$

Since $T_{BSTh} = \dfrac{2}{\gamma+1}T_{0BS} = T_{ASTh} = \dfrac{2}{\gamma+1}T_{0AS}$, $c_{BSTh} = \sqrt{\gamma RT_{BDTh}}$ and $c_{ASTh} = \sqrt{\gamma RT_{ASTh}}$

Hence $c_{Th} = c_{BSTh} = c_{ASTh}$ and $\dfrac{V_{BS}}{a_{BSTh}} = M_{BSTh}$ and $\dfrac{V_{AS}}{C_{ASTh}} = M_{ASTh}$

$$M_{BSTh}M_{ASTh} = 1$$

where suffixes BSTh is the before shock at throat, *ASTh* is the after shock at throat, 0*AS* and 0*BS* are the stagnation points after shock and before shock respectively.

This is another useful form of the Prandile-Meyer Relation.

2.5 OBLIQUE SHOCKS

An oblique shock may be defined as a plane shock whose normal is inclined at an angle to the direction of flow. Figure 2.12 shows a supersonic flow in which the oblique shock is oriented at an angle α with the free stream. Let V_{BS} the velocity before shock and $(V_{BS})_1$ and $(V_{BS})_2$ be its respective components normal and parallel to the shock. In passing through shock parallel component undergoes no change while the normal component is affected. It must be noted that V_{BS} must be greater than the velocity of sound since normal shock can occur only in supersonic flow. Since V_{BS} must be in subsonic direction V_{AS} must change the direction towards shock. Also since V_{BS} is affected by shock for very weak shocks, flow after shock may be supersonic.

Note also the manner of turning of stream lines shows the vast difference in between subsonic and supersonic flows.

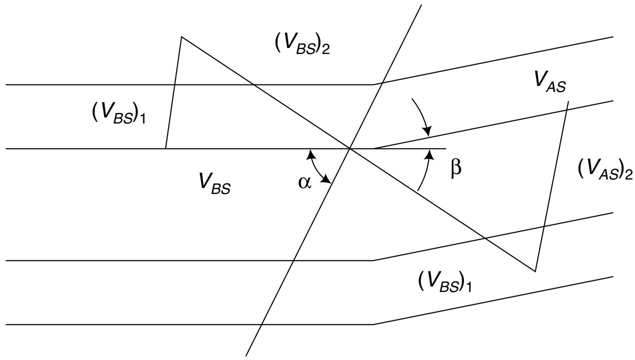


Fig. 2.12 Oblique shock waves.

For symmetrical three-dimensional supersonic flows, shock surfaces in the form of cones are generated instead of oblique plane shock surfaces. There are usually called conical shocks. Stream lines undergo change in direction and it is possible to have supersonic flow behind the shock depending upon its strength.

2.6 MACH WAVES

A simple elastic wave depends upon the elastic properties and the density of the fluid. A sound wave is an elastic wave and it travels with speed c, it may be noted that elastic waves of different frequencies and magnitude can travel through fluids and its entropy is separated from each other. These waves involve very small pressure variations unlike shock waves where comparatively large pressure variation occurs over a very narrow point and it moves with velocity $c = \sqrt{\gamma RT}$.

Analysis of simple elastic wave can be made by analyzing the resulting active from an instantaneous small disturbance induced at some point in a stationary fluid. The wavefront will fan out spherically as shown in Figure 2.13 (a) with velocity c. At time δt and succeeding intervals $2\delta t$, $3\delta t$. . . , etc. Next consider a similar disturbance in a fluid which moves at uniform velocity V from left to right. When V is less than c, the pressure wave would have moved faster than the fluid particles. If the pressure wave is observed from a fixed position, it would have travelled towards left spherically relative to the fluid which at the same time will be moving in the downstream direction with velocity V. With V less than speed of sound, flow is subsonic. When $V = c$, upstream portion of the sound wave remains stationary and is unable to move upstream against the flow as shown in Figure 2.13 (b). Now consider fluid is moving with speed V greater than the speed of sound as in Figure 2.13 (c). Fluid particles move faster than the pressure wave therefore if the spherical pressure wave diagram is constructed, it must be realized that their centre moves downstream faster than the rate at which they propagate radial relative to the stream. A circle is drawn form a conical tangent surface which is called the Mach cone. The half angle α at cone vertex is called the Mach angle. From the diagram, $\sin \alpha = \frac{c}{V} = \frac{1}{M}$. If the same continued emitting sound in supersonic flow outside the mach cone, there will be complete silent and the region of cone is called zone of action. This is contrary to what happens in that subsonic flow in which the disturbance spreads out in all directions in an asymmetric manner and there will be no zone of silence. Hence, it is obvious that sound is not heard from the plane moving at supersonic speed until the plane has passed.

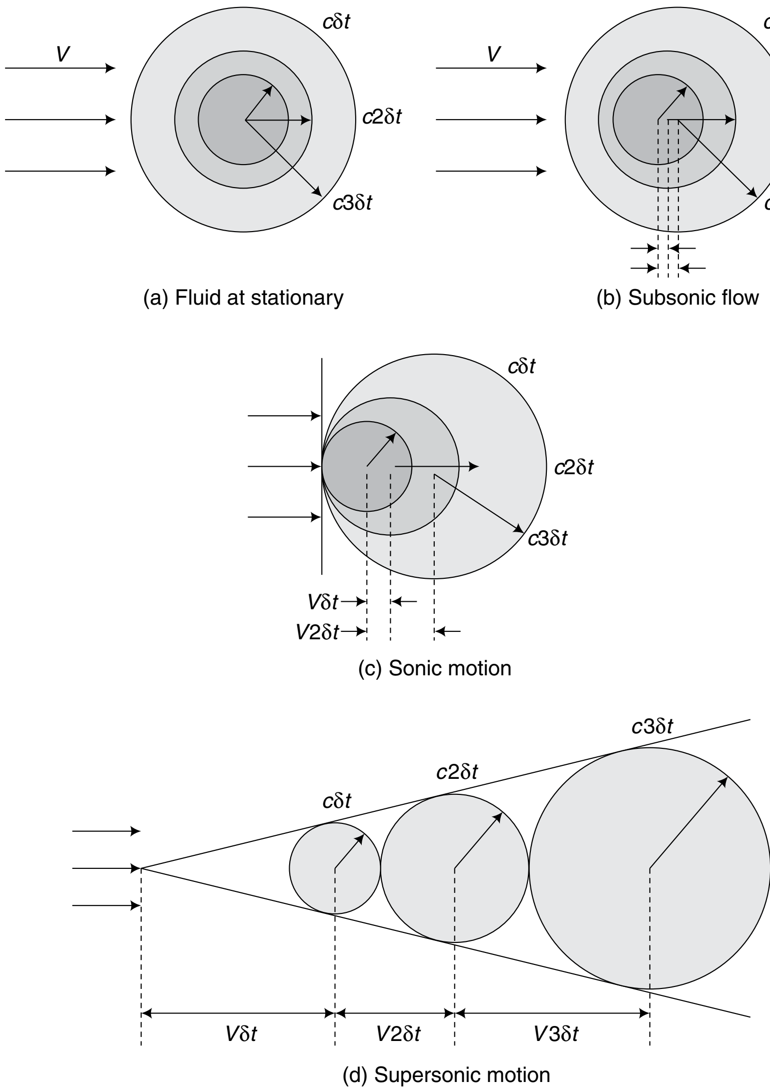


Fig. 2.13 Elastic waves at subsonic to supersonic flow.

Solution Velocity, $V = \frac{960 \times 1000}{3600} = 266.6$ m/s and temperature, $T = 273.4 + 21 = 294.4°K$.

Velocity of the sound, $c - \sqrt{\gamma RT} = \sqrt{1.4 \times 289 \times 294.4} = 345.12$ m/s

$$\text{Mach number} = \frac{V}{c} = \frac{266.6}{345.12} = 0.772$$

2.7.5 Calculate the Mach number at a point on a jet propelled aircraft, which is flying at 1100 km/hr at sea level where air temperature is 20°C. Take $\gamma = 1.4$, $R = 289$ J/kg°k.

Solution Speed of the aircraft = 1100 km/hr = 305.5 m/s

$$\text{Temperature} = 20°C = 273 + 20 = 293°K,\ R = 289 \text{ J/kg°k}$$

$$\sqrt{\gamma RT} = \sqrt{1.4 \times 289.4 \times 293} = 344.31 \text{ m/s}$$

$$\text{Mach number, } M = \frac{V}{c} = \frac{305.55}{344.31} = 0.89$$

2.7.6 An aeroplane is flying at a height of 15 km where the temperature is –50°C. The speed of the plane is constant $M = 2.0$, assume $\gamma = 1.4$ and $R = 287$ J/kg°k. Find the speed of the plane.

Solution Temperature, $T = -50°C = 223°K$,

Mach number, $M = 2.0,\ R = 287$ J/kg°K, $\gamma = 1.4$

$$a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 223} = 299.33 \text{ m/s.}$$

$$M = \frac{V}{c} = 2.0,\quad V = 598.66 \text{ m/s} = 55.2 \text{ km/hour}$$

2.7.7 For adiabatic flow of perfect gas at Mach number 2.0, determine the stagnation pressure, temperature and density ratios.

Solution

$$\frac{P_0}{P_1} = \left[1 + \frac{(\gamma - 1)}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}} = [1 + 0.8]^{3.5} = 8.24.$$

$$\frac{T_0}{T_1} = \left[1 + \frac{(\gamma - 1)}{2} M_1^2\right] = [1 + 0.8] = 1.08$$

$$\frac{\rho_0}{\rho_1} = \left(\frac{P_0}{P_1}\right)^{\frac{1}{8}} = 7.8^{\frac{1}{1.4}} = 4.325$$

2.7.8 A supersonic wind tunnel consists of a large reservoir containing gas under high pressure which is discharged through a convergent-divergent nozzle to a test section of constant cross-sectional area. The cross-sectional area of the throat of nozzle is 60 mm^2 and the Mach number in the test sector is 4. Calculate the cross-sectional area of the last sector assuming $\gamma = 1.4$.

Solution From the equation,

$$\frac{A}{A_t} = \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1} M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{A}{A_t} = \frac{1}{4}\left[\frac{2 + 6.4}{2.4}\right]^{\frac{2.41}{0.8}} = \frac{1}{4}[8.4]^3 = 148.176$$

Area of test section = 148.17 × 60 = 8890.56 mm^2

2.7.9 A jet propelled rocket flies at an altitude of 30,000 metres in standard atmosphere. The nozzle of the rocket gives a thrust of 5500 N with chamber pressure and temperature at 1250 kPa absolute and 1500°C respectively. What are the thrust and the exit areas, velocity and temperature? Assume pressure at 30,000 metres be 120 kPa.

Solution $$\frac{P}{P_0} = \frac{1250}{120} = 10.416$$

$$\frac{P}{P_0} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}} = [1 + 0.2\, M^2]^{3.5} = 10.416$$

$$\lfloor 1 + 0.2\, M^2 \rfloor = 1.45 \text{ and } M^2 = 4.7$$

Solving, $M = 2.17$

$$\frac{T_0}{T_1} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}} = [1 + 0.94] = 1.94. \text{ Hence, } T_1 = \frac{1500}{1.94} = 773.19°\text{C}$$

$$V = M_0 c = M_0 \sqrt{\gamma RT} = 2.17\sqrt{1.4 \times 289 \times 773.19} = 1566.89 \text{ m/sec.}$$

To determine the throat and exit areas, thrust must be taken into account. $5500 = \dot{m}V = (\rho_0 A_0 V)V$

$$\rho_0 = \frac{P_0}{RT_0}\ \frac{1250}{289 \times 773.19} = 4.13 \times 10^{-3} \text{ kg/m}^3.$$

$$A = \frac{5500}{5.94 \times 10^{-3} \times 1556.89^2} = 0.5420 \text{ m}^2 = 5420 \text{ cm}^2.$$

Using the relation

$$\frac{A}{A_{Th}} = \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1} M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{1}{2.17}\left[\frac{2 + 1.884}{2.4}\right]^3 = \frac{3.869}{2.17} = 1.786$$

$$A_{Th} = \frac{5420}{1.786} = 3034.7 \text{ cm}^2.$$

Since $\dfrac{T_{Th}}{T_{exit}} = 1.678,\ T_{exit} = \dfrac{598}{1.678} = 357.44°K$

$$c_{exit} = \sqrt{\gamma RT} = \sqrt{1.4 \times 289 \times 357.44} = 308.78$$

$$M_{exit} = \frac{V}{c} = \frac{950.65}{380.78} = 2.49$$

$$\frac{A}{A_{Th}} = \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{1}{2.499}\left[\frac{2 + 1.884}{2.4}\right]^3 = \frac{3.869}{2.17} = 1.576$$

$$A_{Th} = 1.476 \times 8.036 = 1\ 1.868\ \text{cm}^2$$

$$V_{max} = \sqrt{\frac{2\gamma}{\gamma - 1}RT_0} = \sqrt{\frac{2 \times 1.4}{0.4} \times 289 \times 598} = 1099.88\ \text{m/sec}$$

2.7.11 A De Laval nozzle is to have an exit Mach number of 1.5, with exit diameter, 0.2 m. Find the ratio of exit area/thrust area, exit pressure and temperature if $P_0 = 2$ bar, $T_0 = 20°C$

Solution Area, $A = \dfrac{\pi}{4}D^2 = \dfrac{\pi}{4}0.2^2 = 0.0314\ \text{m}^2$.

Area ratio, $$\frac{A}{A_{Th}} = \frac{1}{M}\left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{1}{1.5}[0.833 + 0.375]^3 = 1.169$$

Hence, $$A_{Th} = \frac{0.0314}{1.169} = 0.0268\ \text{m}^2.$$

Stagnation pressure, $P_0 = 2$ bar and stagnation temperature, $T_0 = 20°C = 263.4 + 20 = 293.4°K$. Using the relationship:

$$\frac{P_0}{P} = \left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}} = \left[1 + 0.2\ M_1^2\right]^{3.5} = 3.67$$

$$\frac{T_0}{T} = \left[1 + \frac{(\gamma - 1)}{2}M_1^2\right] = \left[1 + 0.2 \times M_1^2\right] = 1.45$$

Hence, $$P = \frac{2.0}{3.67} = 0.546\ \text{bar}$$

$$T = \frac{293.4}{1.45} = 202.06°K$$

2.7.12 Determine the velocity of a bullet fired in the air where the Mach angle observed is 30°. Given that the temperature of the air is 22°C, density is 1.2 kg/m^3. Take $\gamma = 1.4$ and $R = 287$ J/kg/K.

Solution $T = 295.4°\text{K}$ $R = 289$ J/kg°K

The velocity is given by $a = \sqrt{\gamma RT} = \sqrt{1.4 \times 289 \times 295.4} = 345.76$ m/s

For the Mach cone, $\sin \alpha = 1/M = 0.5$ and hence $M = 2.0 = V/c$

Velocity of the flow, $V = 2.0 \times 345.76 = 691.4$ m/s

2.7.13 A case of Mach angle $\frac{\pi}{6}$ radians is observed for a fighter aircraft at an altitude where the temperature is 280° Kelvin. If this Mach angle is estimated within ± 5°. What is the velocity of the fighter?

Solution The velocity of sound, $c = \sqrt{\gamma RT} = \sqrt{1.4 \times 289 \times 280} = 336.58$ m/s

$\frac{1}{M} = \frac{V}{c} = \sin \alpha = \sin \frac{\pi}{6} = 0.5$ and hence Mach number, $M = 2$ and $V = 673.16$ m/s

Since $V \sin \alpha$ is a constant, differentiating with respect to α

$$V \cos \alpha + \frac{dV}{d\alpha} = 0$$

At $\alpha = \frac{\pi}{6}$ radians, $V = 673.16$ m/s, $\frac{dV}{d\alpha} = 673.16 \times 0.866 = 582.95$ m/s

For the variation of $\alpha = \pm \frac{\pi}{6} \times 0.05 = \pm 0.0261$ radian

The error in $V = \pm 582.95 \times 0.0261 = \pm 15.21$ m/s

2.7.14 An observer on the ground hears the sonic boom of a plane 15 m when the plane is 20 m ahead of him. Estimate the speed of the plane for the ambient temperature 22°C.

Solution $\alpha = \frac{15}{20} = 0.75$ and $\alpha = 36.86°$

$\sin \alpha = 0.598 = V/c = \frac{1}{M}$ and hence $M = 1.67$.

Plane will fly at a corresponding speed to a Mach number of 1.67.

$T = 295.4°\text{K}$ $R = 289$ J/kg°K

The velocity is given by $a = \sqrt{\gamma RT} = \sqrt{1.4 \times 289 \times 295.4} = 345.76$

Hence, the velocity of the flight, $V = 3\ 45.76 \times 1.67 = 577.42$ m/s

2.7.15 A rocket travels at velocity greater than the speed of sound. If the shock wave angles 45°. Find the velocity of the projectile and the Mach number at the Mach line when the shock 704 kg/cm^2 absolute and temperature 10°C.